

The Chrysene Formalism Revisited: A Twelve-Axiom Foundation for Molecular Characterization, Tensorial Noncommutative Geometry, and Operator Response

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Abstract

We present a revised twelve-axiom formulation of the Chrysene Formalism that rigorously separates its molecular, tensorial, noncommutative, spectral, and physical layers. The molecular reduct is restricted to a finite four-ring polyhex comparison class whose canonical reference point group is conjugate in $O(3)$ to the standard C_{2h} realization and whose two primary attachment directions are intrinsically inequivalent. The 3, 6, 9, 12 $\leftrightarrow$ 2, 5, 8, 11 dual cipher is realized as a uniform one-step perimeter displacement.

A local structural tensor $\Psi_{\mathfrak{C}}$ generates a $\mathbb{Z}^3$ -indexed tensor network. Its quasi-local observable algebra and lattice-translation action produce the crossed-product sequence algebra $\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$, whose noncommutativity follows from translation of localized operations rather than from geometric orthogonality. The dual cipher defines a second structural tensor, a native $\mathbb{Z}_2 \times \mathbb{Z}_2$ four-state alphabet, and a commutant representation of the primary sequence algebra.

The sequence algebra is equipped with the spectral triple $(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}})$. An explicit faithful tracial representation and finite-filtration Dirac operator give bounded commutators and compact resolvent on the infinite $\mathbb{Z}^3$ architecture without compactifying the lattice or requiring translation invariance of $D_{\mathfrak{C}}$. Physical transfer, polarization response, and admissibility geometry are introduced as distinct subsequent structures.

An exhaustive enumeration of the seven free four-cell polyhexes selects a unique canonical carbon framework with the required C_{2h} point-group type, and its graph is independently identified with chrysene. An explicit extension of the standard numbered chrysene reference node satisfies all twelve axioms, proving non-vacuity. Consequently, every full model has canonical carbon-framework reduct polyhex-congruent to chrysene, while the standard 3, 6, 9, 12 $\leftrightarrow$ 2, 5, 8, 11 locant frame is retained on the reference and standard-locant realization class rather than derived uniquely from the axioms.

The $N_y = N_x + 2$ relation, a structurally induced $GF(4)$ field law, directional transfer and optical anisotropy, spectral-path metric reconstruction, real spectral structure, spectral dimension, higher-algebraic braiding, emergent assembly chirality, and dynamical scission remain theorem-level problems.

Keywords: Chrysene Formalism; chrysene; tensor space; tensor network; noncommutative geometry; spectral triple; Connes distance; C^* -algebra; crossed product; $\mathbb{Z}^3$ fibration; dual cipher; commutant; molecular graph; operator Hamiltonian; linear response.

1 Introduction

The Chrysene Formalism was introduced as a discrete mathematical architecture motivated by substituted chrysene and was subsequently developed through tensorial descriptions, operator algebras, noncommutative state spaces, finite structural encoding, spectral methods, discrete transport, and boundary constructions

[Sch26a, Sch26b, Sch26c]. Several concepts recur throughout that program: the 3, 6, 9, 12-substituted chrysene architecture, the geometric basis tensor Ψ_C , a three-dimensional discrete fibration, a noncommutative observable algebra, the adjacent 2, 5, 8, 11 dual cipher, and a spectral description based upon C^* -algebras and Dirac-type operators.

A previous restatement of the formalism improved its logical

discipline by separating molecular geometry from Hilbert-space orthogonality, directional anisotropy from chirality, four-state cardinality from $GF(4)$ field structure, and local constraint violation from global metric disconnection. Those distinctions are retained here. However, that restatement also weakened several structures that were central to the earlier program: the molecular reduct was allowed to be an arbitrary finite embedded graph, the three-dimensional tensor space was reduced to a primarily planar translation lattice, the 2, 5, 8, 11 dual cipher ceased to be part of the axiomatic structure, and the noncommutative-geometric role of a spectral triple was no longer explicit.

The present work therefore undertakes a second, more comprehensive restatement. Its objective is not to return to the stronger inferential claims of the original formulation, but to restore those structures that properly belong to the foundation while preserving the logical separations introduced by the first revision.

The resulting architecture is based upon five principles.

First, the molecular reduct is no longer arbitrary. It is restricted to a precisely defined class of four-ring polyhex molecular graphs. Within that class the carbon framework must possess C_{2h} symmetry, the distinguished 3, 6, 9, 12 and 2, 5, 8, 11 perimeter coordinates must support the primary and dual-cipher structures, and the two primary attachment directions must remain inequivalent under the carbon-framework stabilizer. This formulation permits molecular characterization without defining the molecular graph to be chrysene by fiat.

Second, the term *Chrysene Tensor Space* is given a literal tensorial interpretation. The embedded molecular graph remains the primitive chemical object, while its attachment interfaces determine finite-dimensional tensor factors. A distinguished structural tensor

$$\Psi_{\mathfrak{C}} \in \mathbb{T}_{\mathfrak{C}}$$

occupies each cell of a discrete three-dimensional network. Molecular bridges and stacking contacts correspond to contractions of tensor interfaces.

Third, the spatial base is restored as

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3,$$

with the discrete fibration

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \xrightarrow{\pi_{xy}} \mathbb{Z}^2.$$

The x - and y -directions remain governed by exact bridge-closure relations, while the third coordinate supplies the stacking fiber. The resulting construction therefore retains both the planar closure problem and the three-dimensional sequence space.

Fourth, noncommutativity is reconstructed from sequence dependence. For the quasi-local tensor algebra $\mathfrak{Q}_{\mathfrak{C}}$, lattice translation acts by

$$\alpha : \mathbb{Z}^3 \longrightarrow \text{Aut}(\mathfrak{Q}_{\mathfrak{C}}).$$

The structural sequence algebra is

$$\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3,$$

and its covariance relation

$$u_{\mathbf{m}} P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}$$

provides an explicit source of noncommutativity whenever localization is changed by translation. Thus the formalism does not infer operator noncommutativity from molecular orthogonality or anisotropy.

Fifth, the operator algebra is promoted to a noncommutative metric geometry through the spectral triple

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}).$$

The Dirac operator defines the spectral Lipschitz seminorm and Connes-type distance, while the physical node-bridge Hamiltonians remain distinct operators on separately specified physical representations. This distinction permits spectral geometry and electronic dynamics to interact in later models without identifying them axiomatically.

The 2, 5, 8, 11 dual cipher is restored simultaneously. The primary structural tensor $\Psi_{\mathfrak{C}}$ and its dual $\Psi'_{\mathfrak{C}}$ are related by an explicitly defined one-step locant transformation. Together with the binary stacking registry, this produces the native four-state alphabet

$$\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z.$$

At the operator level the dual representation generates the commutant of the primary von Neumann algebra. The four-state structure is therefore foundational, whereas a multiplicative law making it a field is not.

These restorations do not reinstate the stronger implication chains of the earlier formalism. In particular,

geometric anisotropy $\nRightarrow$ transfer anisotropy,

$$|\Sigma_{\mathfrak{C}}| = 4 \nRightarrow \Sigma_{\mathfrak{C}} \cong GF(4) \text{ as a field,}$$

$$d_C^*(x, y) = +\infty \nRightarrow \text{autonomous physical scission,}$$

without further hypotheses.

The central inferential convention is therefore

axiomatic foundation + explicit realization hypotheses $\implies$ derived consequences.	(1)
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The principal objectives of this article are:

1. to formulate a twelve-axiom foundation whose molecular reduct is compatible with a rigorous chrysene-characterization problem;
2. to restore the tensorial and $\mathbb{Z}^3$ -fibrated architecture of the Chrysene Tensor Space;

3. to derive the noncommutative sequence algebra from ordered translation and localization operations;
4. to restore the 3, 6, 9, 12 $\leftrightarrow$ 2, 5, 8, 11 dual cipher and its commutant realization;
5. to endow the sequence algebra with a spectral triple and noncommutative metric structure;
6. to retain physically distinct Hamiltonian, transfer, response, and admissibility layers; and
7. to separate molecular characterization from the subsequent theorem program involving $N + 2$ closure, $GF(4)$, directional transfer, optical response, metric reconstruction, higher algebra, chirality, and dynamical boundaries.

Section 2 fixes the logical levels and notation. Sections 3–8 state the twelve axioms. Section 9 gives the prerequisite architecture and molecular-characterization theorem. Section 11 states the deferred theorem program. Section 12 provides a concordance with the earlier formalism and the final sections discuss the scope and implications of the restatement.

2 Mathematical Setting, Logical Levels, and Conventions

The revised formalism distinguishes molecular characterization, axiomatic structure, realization-specific extension data, and derived consequences. This distinction is necessary because the present axioms constrain the molecular reduct more strongly than the previous restatement while continuing to leave many physical and algebraic properties theorem dependent.

2.1 Logical Levels of the Formalism

Let

$$\mathcal{A} := \{A_1, \dots, A_{12}\} \quad (2)$$

denote the complete axiom set, and let

$$\text{Mod}(\mathcal{A}) \quad (3)$$

denote the class of full structures satisfying all twelve axioms.

Definition 1 (Axiomatic statement). *A statement P is axiomatic when it is contained in $\mathcal{A}$ or follows solely from definitions necessary to interpret the axioms. We write*

$$\mathcal{A} \models P \quad (4)$$

when P holds in every full model.

Definition 2 (Molecular-reduct statement). *A statement is called a molecular-reduct statement when it concerns only*

$$\text{Red}_{\text{mol}}(\mathfrak{R})$$

and the graph, embedding, locants, symmetry stabilizer, or attachment data inherited from Axioms A_1 – A_3 .

Definition 3 (Realization or extension hypothesis). *Let*

$$\mathfrak{R} \in \text{Ext}_{12}(\mathfrak{C}_{\text{chr}})$$

be a specified full realization over the chrysene molecular reduct. A statement

$$H(\mathfrak{R})$$

is a realization hypothesis when it supplies additional geometric, chemical, registry, spectral, Hamiltonian, response, or dynamical information not required by the twelve axioms.

Definition 4 (Conjugated π -Framework Hypothesis). *Let*

$$\mathfrak{R} \in \text{Ext}_{12}(\mathfrak{C}_{\text{chr}})$$

be a chemically instantiated chrysene realization.

We say that $\mathfrak{R}$ satisfies the Conjugated π -Framework Hypothesis, retaining the notation

$$H_{\partial\pi}(\mathfrak{R}), \quad (5)$$

when the carbon sites and carbon–carbon bonds associated with the external perimeter

$$\partial G_{\text{chr}}$$

participate in the ordinary delocalized π -electronic framework of the sp^2 -conjugated benzenoid realization.

The statement $H_{\partial\pi}(\mathfrak{R})$ is a chemical realization hypothesis and is not an axiom of the Chrysene Formalism.

Remark 1 (Relation to the historical perimeter-conductivity condition). *The hypothesis $H_{\partial\pi}$ is the chemically qualified successor to the closed conjugated-perimeter or perimeter-conductivity condition used in earlier formulations of the Chrysene Formalism.*

It asserts participation of the relevant carbon framework in the ordinary delocalized π system of a conjugated sp^2 benzenoid. It does not assert the existence of a separate electrically conducting perimeter loop, a uniform molecular ring current, equal bond order around the perimeter, or a measured electrical conductivity.

Any quantitative transport statement requires a separately specified electronic Hamiltonian, state, coupling model, and transport observable within Axiom A_{10} or a subsequent physical realization.

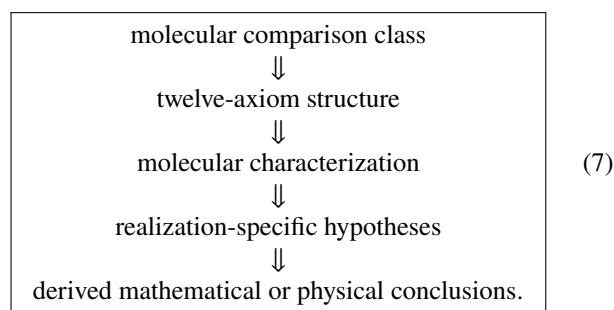
Definition 5 (Derived realization theorem). A statement $P_{\mathfrak{R}}$ is a derived realization theorem when

$$\mathcal{A} \cup \mathcal{H}_{\mathfrak{R}} \models P_{\mathfrak{R}} \quad (6)$$

for an explicitly stated hypothesis family $\mathcal{H}_{\mathfrak{R}}$.

Definition 6 (Empirical instantiation claim). A statement identifying an abstract model parameter with a measured or computed property of a synthesized molecular or supramolecular system is an empirical instantiation claim. Such a statement requires independent experimental or computational support and is not converted into an axiom merely by its use in motivating the formalism.

The logical hierarchy is therefore



3 Embedded Molecular Geometry and Local Tensor Structure

The revised Chrysene Formalism begins with an embedded molecular graph rather than with an abstract tensor taken as primitive. Tensorial structure is introduced only after the molecular interfaces have been defined. This order permits the molecular graph, its Euclidean embedding, and its tensor representation to remain mathematically distinct.

The present section therefore performs two tasks. First, it restricts the molecular comparison class sufficiently to support a later characterization theorem for the chrysene molecular reduct. Second, it constructs the finite-dimensional interface tensor space associated with the distinguished attachment directions. The resulting tensor space will become the local fiber of the three-dimensional network defined in Section 4.

The terminology “Chrysene Tensor Space” and “basis tensor” is retained from the earlier formalism [Sch26a, Sch26b, Sch26c],

but is given a more precise meaning below.

3.1 The Four-Ring Benzenoid Comparison Class

We first specify the molecular class within which the later chrysene-characterization theorem is to be formulated.

Definition 7 (Four-ring polyhex comparison class). Let $\mathcal{H}$ denote the abstract hexagonal tiling graph.

A finite graph G belongs to the four-ring polyhex comparison class $\mathcal{B}_4$ if there exists an edge-connected union of exactly four hexagonal cells of $\mathcal{H}$ whose induced carbon-skeleton graph is isomorphic to G .

Thus

$$\mathcal{B}_4 := \left\{ G : G \cong G(S) \right. \\ \left. \text{for some edge-connected four-cell polyhex } S \right\}. \quad (8)$$

The corresponding set of graph-isomorphism classes is denoted

$$\boxed{\mathcal{B}_4 / \cong}. \quad (9)$$

Accordingly,

$$G \in \mathcal{B}_4$$

means that G is an actual four-ring polyhex carbon-skeleton graph, whereas

$$[G]_{\cong} \in \mathcal{B}_4 / \cong$$

denotes its graph-isomorphism class.

Remark 2 (Role of the comparison class). Membership in $\mathcal{B}_4$ does not identify a candidate with chrysene. The class contains the competing four-ring polyhex topologies admitted by the stated combinatorial definition. The purpose of introducing $\mathcal{B}_4$ is to give a precise finite comparison domain for the molecular-characterization theorem developed later.

Remark 3 (Combinatorial status of the comparison class). Membership in $\mathcal{B}_4$ is a graph-theoretic condition and does not assert that every member corresponds to an isolable neutral closed-shell hydrocarbon of formula $C_{18}H_{12}$. The exhaustive classification contains carbon-skeleton graphs with different vertex counts and condensation patterns. Some correspond to familiar polycyclic aromatic hydrocarbon skeletons, whereas others serve only as members of the finite combinatorial comparison class required for molecular characterization.

Accordingly,

$$G \in \mathcal{B}_4$$

does not by itself imply chemical stability, a particular molecular formula, closed-shell electronic structure, synthetic accessibility, or experimental isolation.

Lemma 1 (Bipartiteness of the four-ring polyhex class). Every graph

$$G \in \mathcal{B}_4$$

is bipartite.

Equivalently, there exists a decomposition

$$V(G) = V_A(G) \sqcup V_B(G) \quad (10)$$

such that every edge of G has one endpoint in $V_A(G)$ and one endpoint in $V_B(G)$.

Proof. The abstract hexagonal tiling graph $\mathcal{H}$ is bipartite. Every $G \in \mathcal{B}_4$ is, by Definition 7, the carbon-skeleton graph induced by a finite union of cells of $\mathcal{H}$. The restriction of a bipartition of $\mathcal{H}$ to the vertices of G therefore gives a bipartition of G . $\square$

Remark 4 (Balanced bipartition is a stronger property). Bipartiteness alone does not imply

$$|V_A(G)| = |V_B(G)|.$$

Accordingly, balanced bipartiteness is not imposed as an additional axiom on the entire comparison class $\mathcal{B}_4$.

For the reference chrysene molecular reduct, however, the verified carbon-skeleton bipartition is balanced:

$$|V_A(G_{\text{chr}})| = |V_B(G_{\text{chr}})|. \quad (11)$$

This molecular fact may therefore be used in subsequent Chrysene realizations without being reintroduced as an independent axiom.

3.2 Embedded Labeled Benzenoid Molecular Node

Let $\mathcal{L}_{\text{loc}}$, $\mathcal{L}_{\text{chem}}$, and $\mathcal{L}_{\text{att}}$ be sets of locant, chemical, and attachment labels, respectively. Assume

$$\{2, 3, 5, 6, 8, 9, 11, 12\} \subseteq \mathcal{L}_{\text{loc}}. \quad (12)$$

Let

$$\overline{\mathcal{L}}_{\text{att}} := \mathcal{L}_{\text{att}} \sqcup \{\emptyset\}. \quad (13)$$

If attachment labels carry orientation-dependent data, assume an orthogonal action

$$\rho : O(3) \longrightarrow \text{Aut}(\overline{\mathcal{L}}_{\text{att}}) \quad (14)$$

with

$$\rho_R(\emptyset) = \emptyset \quad (R \in O(3)). \quad (15)$$

Axiom 1 (Embedded Labeled Benzenoid Molecular Node (A1)). The molecular reduct of the Chrysene Formalism is a quadruple

$$\mathfrak{C} = (V_{\mathfrak{C}}, E_{\mathfrak{C}}, \iota, \lambda) \quad (16)$$

satisfying the following conditions.

1. The molecular graph

$$G_{\mathfrak{C}} = (V_{\mathfrak{C}}, E_{\mathfrak{C}}) \quad (17)$$

belongs to the four-ring polyhex comparison class $\mathcal{B}_4$.

2. The molecular embedding

$$\iota : V_{\mathfrak{C}} \longrightarrow \mathbb{R}^3 \quad (18)$$

is injective.

3. The labeling datum

$$\lambda = (\lambda_{\text{loc}}, \lambda_{\text{chem}}, \lambda_{\text{att}}) \quad (19)$$

consists of maps

$$\lambda_{\text{loc}} : V_{\mathfrak{C}} \longrightarrow \mathcal{L}_{\text{loc}} \cup \{\emptyset\}, \quad (20)$$

$$\lambda_{\text{chem}} : V_{\mathfrak{C}} \sqcup E_{\mathfrak{C}} \longrightarrow \mathcal{L}_{\text{chem}}, \quad (21)$$

$$\lambda_{\text{att}} : V_{\mathfrak{C}} \longrightarrow \overline{\mathcal{L}}_{\text{att}}. \quad (22)$$

4. For each

$$j \in \{2, 3, 5, 6, 8, 9, 11, 12\}, \quad (23)$$

there exists a unique vertex $v_j \in V_{\mathfrak{C}}$ such that

$$\lambda_{\text{loc}}(v_j) = j. \quad (24)$$

5. The primary attachment set is

$$A_{\mathfrak{C}}^{\text{pri}} := \{v_3, v_6, v_9, v_{12}\}. \quad (25)$$

Its vertices carry non-null attachment data:

$$\lambda_{\text{att}}(v) \neq \emptyset \quad \left(v \in A_{\mathfrak{C}}^{\text{pri}}\right). \quad (26)$$

6. The distinguished dual-cipher locant set is

$$A_{\mathfrak{C}}^{\text{dual}} := \{v_2, v_5, v_8, v_{11}\}. \quad (27)$$

Its algebraic role is not imposed by Axiom A_1 ; it is developed in Axiom A_8 .

For every distinguished locant define

$$\mathbf{r}_j := \iota(v_j). \quad (28)$$

Definition 8 (Embedded labeled four-ring node class). *Define*

$$\mathfrak{N}_4 := \{\mathfrak{C} : \mathfrak{C} \text{ satisfies Axiom 1}\}. \quad (29)$$

Thus

$$\mathcal{B}_4$$

is the comparison class of abstract carbon-skeleton graphs, whereas

$$\mathfrak{N}_4$$

is the class of embedded labeled molecular nodes built over those graphs.

For $\mathfrak{C} \in \mathfrak{N}_4$, one has

$$G_{\mathfrak{C}} \in \mathcal{B}_4. \quad (30)$$

Remark 5 (The molecular node is not defined to be chrysene). *Axiom 1 restricts the molecular reduct to a well-defined four-ring polyhex comparison class and specifies the locant data required by the formalism. It does not assert that $G_{\mathfrak{C}}$ is the chrysene graph. Identification of the surviving molecular isomorphism class is reserved for the characterization theorem of Section 10.*

3.3 Symmetry Stabilizer and Chirality

For $g \in E(3)$, write

$$g(\mathbf{r}) = R_g \mathbf{r} + \mathbf{t}_g, \quad R_g \in O(3). \quad (31)$$

Definition 9 (Polyhex reference embedding). *For each $G \in \mathcal{B}_4$, let*

$$\iota_{\text{px}} : V(G) \longrightarrow \mathbb{R}^2$$

denote the embedding induced by a representative union of four regular hexagonal cells of the standard hexagonal tiling.

Two such reference embeddings are identified when related by a Euclidean isometry of the tiling.

The corresponding reference polyhex symmetry group is

$$\Gamma_G^{\text{px}} := \{g \in \text{Isom}(\mathbb{R}^2) : g(\iota_{\text{px}}(G)) = \iota_{\text{px}}(G)\}. \quad (32)$$

Its three-dimensional molecular extension is obtained by adjoining reflection through the molecular plane.

Definition 10 (Orthogonal equivalence of molecular point groups). *Let $G, H \leq O(3)$. We write*

$$G \sim_{O(3)} H \quad (33)$$

if there exists $Q \in O(3)$ such that

$$Q G Q^{-1} = H. \quad (34)$$

Fix once and for all a standard orthogonal realization

$$C_{2h}^{\text{std}} \leq O(3) \quad (35)$$

of the Schoenflies point group C_{2h} .

Thus

$$G \sim_{O(3)} C_{2h}^{\text{std}}$$

means equivalence as an embedded orthogonal point group, not merely abstract group isomorphism.

In particular, although C_{2h} and C_{2v} are abstractly isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, they are not identified by $\sim_{O(3)}$.

Axiom 2 (Symmetry Stabilizer (A2)). *Let $\mathfrak{C}$ be an embedded node satisfying Axiom 1.*

For $g \in E(3)$, write

$$g(\mathbf{r}) = R_g \mathbf{r} + \mathbf{t}_g, \quad R_g \in O(3). \quad (36)$$

The full labeled symmetry stabilizer of $\mathfrak{C}$ is

$$\Gamma_{\mathfrak{C}} = \left\{ g \in E(3) : \begin{array}{l} \exists \varphi_g \in \text{Aut}(G_{\mathfrak{C}}) \text{ such that} \\ g \circ \iota = \iota \circ \varphi_g, \\ \lambda_{\text{chem}} \circ \varphi_g^{\#} = \lambda_{\text{chem}}, \\ \lambda_{\text{att}} \circ \varphi_g = \rho_{R_g} \circ \lambda_{\text{att}}, \\ \varphi_g(A_{\mathfrak{C}}^{\text{pri}}) = A_{\mathfrak{C}}^{\text{pri}} \end{array} \right\}. \quad (37)$$

Here $\varphi_g^{\#}$ is the induced bijection on $V_{\mathfrak{C}} \sqcup E_{\mathfrak{C}}$.

The associated full labeled molecular point group is

$$\Pi_{\mathfrak{C}} := \{R_g : g \in \Gamma_{\mathfrak{C}}\} \leq O(3). \quad (38)$$

To distinguish symmetry of the carbon framework from symmetry reductions caused specifically by attachment labels or substituent identity, define the embedded carbon-framework stabilizer

$$\Gamma_{\mathfrak{C}}^{\text{emb}} := \left\{ g \in E(3) : \begin{array}{l} \exists \varphi_g \in \text{Aut}(G_{\mathfrak{C}}) \text{ such that} \\ g \circ \iota = \iota \circ \varphi_g, \\ \lambda_{\text{chem}} \circ \varphi_g^{\#} = \lambda_{\text{chem}} \end{array} \right\}. \quad (39)$$

Its associated embedded carbon-framework point group is

$$\Pi_{\mathfrak{C}}^{\text{emb}} := \{R_g : g \in \Gamma_{\mathfrak{C}}^{\text{emb}}\} \leq O(3). \quad (40)$$

Because the full labeled stabilizer preserves all data required of the embedded carbon framework,

$$\Gamma_{\mathfrak{C}} \leq \Gamma_{\mathfrak{C}}^{\text{emb}}. \quad (41)$$

For $G_{\mathfrak{C}} \in \mathcal{B}_4$, let

$$\iota_{\text{px}} : V_{\mathfrak{C}} \longrightarrow \mathbb{R}^2 \quad (42)$$

denote the canonical regular-polyhex reference embedding specified in Appendix A, and define its planar realization in three-dimensional Euclidean space by

$$\widehat{\iota}_{\text{px}}(v) := (\iota_{\text{px}}(v), 0) \in \mathbb{R}^3. \quad (43)$$

The reference carbon-framework stabilizer is defined purely from the canonical carbon-skeleton geometry:

$$\Gamma_{\mathfrak{C}}^{\text{core}} := \left\{ g \in E(3) : \begin{array}{l} \exists \varphi_g \in \text{Aut}(G_{\mathfrak{C}}) \text{ such that} \\ g \circ \widehat{\iota}_{\text{px}} = \widehat{\iota}_{\text{px}} \circ \varphi_g \end{array} \right\}. \quad (44)$$

Its associated reference carbon-framework point group is

$$\Pi_{\mathfrak{C}}^{\text{core}} := \{ R_g : g \in \Gamma_{\mathfrak{C}}^{\text{core}} \} \leq O(3). \quad (45)$$

The molecular reduct of the Chrysene Formalism is required to satisfy

$$\Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}}. \quad (46)$$

The requirement (46) is a statement about the canonical regular-polyhex carbon-framework geometry. It is independent of chemical coloring, substituent identity, attachment labels, or realization-specific distortions.

Chemical and attachment labels remain relevant to $\Gamma_{\mathfrak{C}}^{\text{emb}}$ and $\Gamma_{\mathfrak{C}}$, but they do not alter the reference carbon-framework symmetry used in the finite molecular classification.

The requirement (46) is therefore a property of the canonical regular-polyhex reference geometry and is independent of accidental distortions of a particular physical realization. If the physical carbon-framework embedding ι is Euclidean-congruent to the canonical reference embedding $\widehat{\iota}_{\text{px}}$, then $\Gamma_{\mathfrak{C}}^{\text{emb}}$ and $\Gamma_{\mathfrak{C}}^{\text{core}}$ are conjugate as Euclidean symmetry groups. If substituents, environment, strain, stacking, or another realization-specific perturbation lowers the symmetry of the physical embedding, then $\Pi_{\mathfrak{C}}^{\text{emb}}$ may be a proper subgroup of the reference C_{2h} group without altering the molecular-classification result.

Thus Axiom A₂ distinguishes three levels of symmetry:

$$\begin{array}{l} \text{canonical carbon-framework symmetry} \\ \Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}} \\ \text{embedded carbon-framework symmetry} \\ \Pi_{\mathfrak{C}}^{\text{emb}} \\ \text{fully labeled molecular symmetry} \\ \Pi_{\mathfrak{C}} \end{array} \quad (47)$$

The symmetry of a substituted realization depends additionally upon the chemical identity and orientation of its attachment data.

If symmetry-related primary sites carry chemically identical substituents with local attachment data transforming covariantly under the relevant carbon-framework operations, the fully decorated realization may retain some or all of the reference C_{2h} symmetry.

By contrast, chemically distinct substituents, inequivalent local orientations, or attachment patterns that destroy inversion or another reference symmetry may lower $\Pi_{\mathfrak{C}}$ without changing the identity of the underlying chrysene carbon framework. Only the first is used in the finite molecular-characterization theorem.

Definition 11 (Chirality). The embedded molecular node $\mathfrak{C}$ is called chiral if

$$\Pi_{\mathfrak{C}} \subseteq SO(3), \quad (48)$$

and achiral if

$$\exists R \in \Pi_{\mathfrak{C}} \quad \det R = -1. \quad (49)$$

Proposition 2 (C_{2h} implies achirality). If $\Pi_{\mathfrak{C}}$ is conjugate in $O(3)$ to the standard point group C_{2h} , then $\mathfrak{C}$ is achiral.

Proof. The point group C_{2h} contains inversion, represented by $-I_3$, and

$$\det(-I_3) = -1.$$

Determinant is invariant under conjugation in $O(3)$. Hence every orthogonal realization conjugate to C_{2h} contains an improper symmetry. $\square$

3.4 Directional Attachment Geometry

Define the ordered primary attachment vertices by

$$v_x^- := v_3, \quad v_x^+ := v_9, \quad (50)$$

$$v_y^- := v_{12}, \quad v_y^+ := v_6. \quad (51)$$

Axiom 3 (Directional Attachment and Anisotropy (A3)). Let $\mathfrak{C}$ satisfy Axioms 1–2.

Define the ordered primary attachment vertices by

$$v_x^- := v_3, \quad v_x^+ := v_9, \quad (52)$$

$$v_y^- := v_{12}, \quad v_y^+ := v_6. \quad (53)$$

Recall

$$\mathbf{r}_j := \iota(v_j).$$

The directional displacement vectors of the particular em-

bedded realization are

$$\mathbf{d}_x := \mathbf{r}_9 - \mathbf{r}_3, \quad (54)$$

$$\mathbf{d}_y := \mathbf{r}_6 - \mathbf{r}_{12}. \quad (55)$$

They are required to be nonzero and linearly independent:

$$\boxed{\mathbf{d}_x \neq \mathbf{0}, \quad \mathbf{d}_y \neq \mathbf{0}, \quad \mathbf{d}_x \times \mathbf{d}_y \neq \mathbf{0}.} \quad (56)$$

No Euclidean orthogonality condition

$$\mathbf{d}_x \cdot \mathbf{d}_y = 0 \quad (57)$$

is imposed.

Each primary attachment vertex carries a unit outward tangent

$$\tau : A_{\mathfrak{C}}^{\text{pri}} \longrightarrow S^2, \quad \tau(v_j) = \mathbf{t}_j, \quad (58)$$

satisfying covariance under the full labeled symmetry stabilizer:

$$\mathbf{t}_{\varphi_g(v)} = R_g \mathbf{t}_v, \quad g \in \Gamma_{\mathfrak{C}}, \quad v \in A_{\mathfrak{C}}^{\text{pri}}. \quad (59)$$

Define the embedded directional data

$$\mathcal{D}_x^{\text{emb}} := (v_3, v_9; \mathbf{d}_x, \mathbf{t}_3, \mathbf{t}_9), \quad (60)$$

$$\mathcal{D}_y^{\text{emb}} := (v_{12}, v_6; \mathbf{d}_y, \mathbf{t}_{12}, \mathbf{t}_6). \quad (61)$$

We write

$$\mathcal{D}_x^{\text{emb}} \sim_{\Gamma_{\mathfrak{C}}} \mathcal{D}_y^{\text{emb}} \quad (62)$$

if and only if there exists $g \in \Gamma_{\mathfrak{C}}$, with induced graph automorphism φ_g , such that

$$\varphi_g(v_3) = v_{12}, \quad \varphi_g(v_9) = v_6, \quad (63)$$

$$R_g \mathbf{d}_x = \mathbf{d}_y, \quad R_g \mathbf{t}_3 = \mathbf{t}_{12}, \quad R_g \mathbf{t}_9 = \mathbf{t}_6. \quad (64)$$

The embedded labeled realization is required to satisfy

$$\boxed{\mathcal{D}_x^{\text{emb}} \not\sim_{\Gamma_{\mathfrak{C}}} \mathcal{D}_y^{\text{emb}}.} \quad (65)$$

Independently, let

$$\widehat{\iota}_{\text{px}} : V_{\mathfrak{C}} \longrightarrow \mathbb{R}^3$$

be the canonical regular-polyhex reference embedding of Axiom 2, and define

$$\widehat{\mathbf{r}}_j := \widehat{\iota}_{\text{px}}(v_j). \quad (66)$$

The corresponding reference directional vectors are

$$\widehat{\mathbf{d}}_x := \widehat{\mathbf{r}}_9 - \widehat{\mathbf{r}}_3, \quad (67)$$

$$\widehat{\mathbf{d}}_y := \widehat{\mathbf{r}}_6 - \widehat{\mathbf{r}}_{12}. \quad (68)$$

They are required to satisfy

$$\boxed{\widehat{\mathbf{d}}_x \times \widehat{\mathbf{d}}_y \neq \mathbf{0}.} \quad (69)$$

Define the reference carbon-framework directional data

$$\widehat{\mathcal{D}}_x := (v_3, v_9; \widehat{\mathbf{d}}_x), \quad (70)$$

$$\widehat{\mathcal{D}}_y := (v_{12}, v_6; \widehat{\mathbf{d}}_y). \quad (71)$$

We write

$$\widehat{\mathcal{D}}_x \sim_{\Gamma_{\mathfrak{C}}^{\text{core}}} \widehat{\mathcal{D}}_y \quad (72)$$

if and only if there exists $g \in \Gamma_{\mathfrak{C}}^{\text{core}}$, with induced graph automorphism φ_g , such that

$$\varphi_g(v_3) = v_{12}, \quad \varphi_g(v_9) = v_6, \quad (73)$$

$$R_g \widehat{\mathbf{d}}_x = \widehat{\mathbf{d}}_y. \quad (74)$$

For the canonical molecular reduct we require

$$\boxed{\widehat{\mathcal{D}}_x \not\sim_{\Gamma_{\mathfrak{C}}^{\text{core}}} \widehat{\mathcal{D}}_y.} \quad (75)$$

The conditions (65) and (75) belong to different geometric levels. No subgroup relation between $\Gamma_{\mathfrak{C}}$ and $\Gamma_{\mathfrak{C}}^{\text{core}}$ is required to infer one from the other.

Thus canonical carbon-framework anisotropy and anisotropy of a particular embedded labeled realization are stated independently.

No electronic, spectral, transfer, or optical inequality follows from either directional inequivalence alone.

3.5 Planar Interface Tensor Space

We now introduce tensorial structure only after the molecular interfaces have been defined.

Definition 12 (Primary interface spaces). Associate with the four primary attachment vertices nonzero finite-dimensional complex vector spaces

$$V_3, \quad V_9, \quad V_{12}, \quad V_6, \quad (76)$$

and let

$$K_{\mathfrak{C}} \quad (77)$$

be a nonzero finite-dimensional internal structural-state space.

The planar interface tensor space is

$$\boxed{\mathbb{T}_{\mathfrak{C}}^{xy} := V_3 \otimes V_9 \otimes V_{12} \otimes V_6 \otimes K_{\mathfrak{C}}.} \quad (78)$$

The four factors V_3, V_9, V_{12}, V_6 represent attachment interfaces,

not Hilbert-space quantum channels. The internal factor $K_{\mathbb{C}}$ will later carry the discrete registry and cipher information specified by Axiom A₈.

Remark 6 (Meaning of “tensor”). Equation (78) uses the ordinary algebraic tensor product of finite-dimensional complex vector spaces. The term tensor space therefore has a literal linear-algebraic meaning in the revised formalism. The molecular graph $G_{\mathbb{C}}$ is not itself identified with a tensor. Rather, its labeled attachment structure determines the interface factors from which a local structural tensor is subsequently constructed.

Remark 7 (Structural tensor versus quantum state). An element of $\mathbb{T}_{\mathbb{C}}^{xy}$ is a structural tensor representation. It is not automatically a vector of the GNS Hilbert space $\mathcal{H}_{\omega}$, nor an electronic wavefunction, nor a response eigenstate. Any map between these spaces must be defined separately.

4 Directional Closure and the $\mathbb{Z}^3$ -Fibrated Tensor Network

The local molecular node becomes a spatially extended object by directional bridge formation in the x - and y -directions and by discrete stacking along a third direction. The x/y geometry is governed by exact closure equations. The third direction supplies the discrete fiber required for the three-dimensional Chrysene Tensor Space.

The resulting architecture is not identified with $\mathbb{Z}^3$ itself. Rather, $\mathbb{Z}^3$ is the discrete base indexing copies of the local tensor space.

4.1 Directional Bridge Families

Definition 13 (Embedded bridge realization). Let $\mathcal{L}_{\mathcal{B}}$ be a specified set of bridge-terminal labels. An embedded bridge realization is a tuple

$$\mathcal{B} = (V_{\mathcal{B}}, E_{\mathcal{B}}, \iota_{\mathcal{B}}, p_{\mathcal{B}}^-, p_{\mathcal{B}}^+, \lambda_{\mathcal{B}}) \quad (79)$$

such that $(V_{\mathcal{B}}, E_{\mathcal{B}})$ is a finite connected graph,

$$\iota_{\mathcal{B}} : V_{\mathcal{B}} \longrightarrow \mathbb{R}^3 \quad (80)$$

is injective,

$$p_{\mathcal{B}}^- \neq p_{\mathcal{B}}^+$$

are distinguished terminal vertices, and

$$\lambda_{\mathcal{B}} : \{p_{\mathcal{B}}^-, p_{\mathcal{B}}^+\} \longrightarrow \mathcal{L}_{\mathcal{B}} \quad (81)$$

is the terminal-label map.

Its oriented terminal displacement is

$$\mathbf{B}(\mathcal{B}) := \iota_{\mathcal{B}}(p_{\mathcal{B}}^+) - \iota_{\mathcal{B}}(p_{\mathcal{B}}^-). \quad (82)$$

Definition 14 (Bridge-interface compatibility). For $\alpha \in \{x, y\}$, define the ordered molecular interface datum

$$\mathcal{I}_{\alpha}(\mathbb{C}) := (\lambda_{\text{att}}(v_{\alpha}^+), \lambda_{\text{att}}(v_{\alpha}^-); \mathbf{t}_{v_{\alpha}^+}, \mathbf{t}_{v_{\alpha}^-}). \quad (83)$$

For each direction, fix an explicitly specified compatibility relation

$$\text{Comp}_{\alpha} \subseteq \mathcal{L}_{\mathcal{B}}^2 \times \mathcal{L}_{\text{att}}^2 \times (S^2)^2. \quad (84)$$

An embedded bridge $\mathcal{B}$ is compatible with the ordered α -interface when

$$(\lambda_{\mathcal{B}}(p_{\mathcal{B}}^-), \lambda_{\mathcal{B}}(p_{\mathcal{B}}^+); \lambda_{\text{att}}(v_{\alpha}^+), \lambda_{\text{att}}(v_{\alpha}^-); \mathbf{t}_{v_{\alpha}^+}, \mathbf{t}_{v_{\alpha}^-}) \in \text{Comp}_{\alpha}. \quad (85)$$

Define

$$\mathbb{B}_{\alpha}(\mathbb{C}) := \left\{ \mathcal{B} : \mathcal{B} \text{ satisfies Definition 13 and Equation (85)} \right\}. \quad (86)$$

The relation Comp_{α} is structural compatibility data. It does not by itself assert chemical synthesizability, bond formation, or energetic stability.

Axiom 4 (Directional Bridge Generation (A4)). For every $\alpha \in \{x, y\}$ there exists a nonempty set

$$\mathcal{N}_{\alpha} \subseteq \mathbb{Z}_{\geq 0} \quad (87)$$

and a bridge-generation map

$$B_{\alpha} : \mathcal{N}_{\alpha} \longrightarrow \mathbb{B}_{\alpha}(\mathbb{C}), \quad N \longmapsto B_{\alpha}(N). \quad (88)$$

Define

$$\mathbf{B}_{\alpha}(N) := \mathbf{B}(B_{\alpha}(N)). \quad (89)$$

For each direction there is also an N -independent terminal displacement

$$\mathbf{c}_{\alpha} \in \mathbb{R}^3, \quad (90)$$

so that the total connector displacement is

$$\mathbf{C}_{\alpha}(N) := \mathbf{c}_{\alpha} + \mathbf{B}_{\alpha}(N). \quad (91)$$

No equality between the two directional bridge families is imposed.

Definition 15 (Homogeneous bridge family). A bridge family in direction α is homogeneous if there exists a nonzero vector $\mathbf{b}_\alpha$ such that

$$\mathbf{B}_\alpha(N) = N\mathbf{b}_\alpha \quad (N \in \mathcal{N}_\alpha). \quad (92)$$

4.2 Exact Directional Closure

Axiom 5 (Directional Translational Closure (A5)). For each $\alpha \in \{x, y\}$, let

$$\mathbf{a}_\alpha \in \mathbb{R}^3 \quad (93)$$

be a distinguished directional translation generator intended to carry a reference attachment site on one molecular node to the corresponding attachment site on a translated copy. At the level of Axiom A5, $\mathbf{a}_x$ and $\mathbf{a}_y$ are closure data. No nearest-neighbor or minimal-length property is assumed. Axiom A6 subsequently incorporates them into the three-dimensional translation lattice.

For $N \in \mathcal{N}_\alpha$, define the closure residual

$$\rho_\alpha(N) := \mathbf{a}_\alpha - \mathbf{d}_\alpha - \mathbf{c}_\alpha - \mathbf{B}_\alpha(N). \quad (94)$$

Exact closure in direction α occurs at $N_\alpha^* \in \mathcal{N}_\alpha$ when

$$\rho_\alpha(N_\alpha^*) = \mathbf{0}. \quad (95)$$

Equivalently,

$$\mathbf{a}_\alpha = \mathbf{d}_\alpha + \mathbf{c}_\alpha + \mathbf{B}_\alpha(N_\alpha^*). \quad (96)$$

A regular network requires distinguished exact-closure counts

$$N_x^* \in \mathcal{N}_x, \quad N_y^* \in \mathcal{N}_y \quad (97)$$

satisfying

$$\rho_x(N_x^*) = \rho_y(N_y^*) = \mathbf{0}. \quad (98)$$

Definition 16 (Closure defect). For $N \in \mathcal{N}_\alpha$, define

$$\varepsilon_\alpha(N) := \|\rho_\alpha(N)\|_2. \quad (99)$$

Exact closure is stronger than merely having a small closure defect.

4.3 The Discrete Three-Dimensional Fibration

Let

$$\mathbf{e}_x = (1, 0, 0), \quad \mathbf{e}_y = (0, 1, 0), \quad \mathbf{e}_z = (0, 0, 1) \quad (100)$$

denote the standard basis of $\mathbb{Z}^3$.

Axiom 6 (Discrete $\mathbb{Z}^3$ -Fibrated Tensor Network (A6)). There exists a third translation vector

$$\mathbf{a}_z \in \mathbb{R}^3 \quad (101)$$

such that

$$\det(\mathbf{a}_x \quad \mathbf{a}_y \quad \mathbf{a}_z) \neq 0. \quad (102)$$

The full translation lattice is

$$\Lambda_{\mathfrak{G}} := \{m\mathbf{a}_x + n\mathbf{a}_y + p\mathbf{a}_z : (m, n, p) \in \mathbb{Z}^3\}. \quad (103)$$

The corresponding index space is

$$X_{\mathfrak{G}} := \mathbb{Z}^3. \quad (104)$$

For

$$\mathbf{k} = (m, n, p) \in X_{\mathfrak{G}}$$

and $v \in V_{\mathfrak{G}}$, define

$$\iota_{\mathfrak{R}}(\mathbf{k}, v) := \iota(v) + m\mathbf{a}_x + n\mathbf{a}_y + p\mathbf{a}_z. \quad (105)$$

The translation vectors are required to be chosen so that $\iota_{\mathfrak{R}}$ is injective.

The molecular vertex set of the periodic network is

$$V_{\mathfrak{R}} := X_{\mathfrak{G}} \times V_{\mathfrak{G}}. \quad (106)$$

Every lattice cell carries a translated copy of the molecular graph.

For $\alpha \in \{x, y\}$, the exact-closure connector joins

$$(\mathbf{k}, v_\alpha^+) \text{ to } (\mathbf{k} + \mathbf{e}_\alpha, v_\alpha^-). \quad (107)$$

Adjacent cells in the z -direction carry a discrete stacking relation

$$\mathbf{k} \sim_z \mathbf{k} + \mathbf{e}_z. \quad (108)$$

Each such relation is assigned a registry label

$$\eta_{\mathbf{k}} \in \mathcal{R}_z, \quad \mathcal{R}_z := \mathbb{Z}_2. \quad (109)$$

The registry label is not identified with molecular chirality. Let $\mathfrak{R}_{\text{und}}$ denote the underlying periodic network obtained by forgetting the registry labels on the z -edges.

The lattice translations act on this underlying network by

$$U : \mathbb{Z}^3 \longrightarrow \text{Aut}(\mathfrak{R}_{\text{und}}), \quad U_{\ell}(\mathbf{k}, v) := (\mathbf{k} + \ell, v). \quad (110)$$

They satisfy

$$U_0 = I, \quad U_{\mathbf{k}} U_{\ell} = U_{\mathbf{k} + \ell}. \quad (111)$$

Let

$$\eta : E_z \longrightarrow \mathbb{Z}_2$$

denote the registry field on the oriented z -edges, with

$$\eta_{\mathbf{k}}$$

assigned to the edge $\mathbf{k} \sim_z \mathbf{k} + \mathbf{e}_z$.

Translation transports the registry field according to

$$(\ell \cdot \eta)_{\mathbf{k}+\ell} := \eta_{\mathbf{k}}. \quad (112)$$

Thus U_ℓ is an isomorphism

$$(\mathfrak{R}_{\text{und}}, \eta) \longrightarrow (\mathfrak{R}_{\text{und}}, \ell \cdot \eta). \quad (113)$$

Only when

$$\ell \cdot \eta = \eta$$

is U_ℓ an automorphism of the fixed registry-decorated network.

tensor space is

$$\mathbb{T}_{\mathfrak{C}} := V_3 \otimes V_9 \otimes V_{12} \otimes V_6 \otimes V_{z^-} \otimes V_{z^+} \otimes K_{\mathfrak{C}}. \quad (119)$$

A nonzero distinguished element

$$\Psi_{\mathfrak{C}} \in \mathbb{T}_{\mathfrak{C}} \quad (120)$$

is called the local Chrysene structural tensor.

Remark 8 (Legacy terminology: geometric basis tensor). The symbol $\Psi_{\mathfrak{C}}$ is the rigorous replacement for the earlier “geometric basis tensor” Ψ_C . The word basis is retained as historical terminology for the distinguished local generator of the tensor network; it does not mean that $\Psi_{\mathfrak{C}}$ is, by itself, a linear basis for $\mathbb{T}_{\mathfrak{C}}$.

4.4 Base-Fiber Decomposition

Define the planar index projection

$$\pi_{xy} : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2, \quad \pi_{xy}(m, n, p) = (m, n). \quad (114)$$

Proposition 3 (Discrete fiber structure). *The sequence*

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \xrightarrow{\pi_{xy}} \mathbb{Z}^2 \quad (115)$$

is a discrete fibration in the elementary set-theoretic and group-theoretic sense. For every $(m, n) \in \mathbb{Z}^2$,

$$\pi_{xy}^{-1}(m, n) = \{(m, n, p) : p \in \mathbb{Z}\} \cong \mathbb{Z}. \quad (116)$$

Proof. The map π_{xy} is a surjective group homomorphism and

$$\ker \pi_{xy} = \{(0, 0, p) : p \in \mathbb{Z}\} \cong \mathbb{Z}. \quad (117)$$

Every fiber is a coset of this kernel. $\square$

Thus the earlier three-dimensional discrete construction is retained, while the x/y closure equations and the z -fiber remain mathematically distinguishable.

4.5 The Local Chrysene Structural Tensor

Choose nonzero finite-dimensional complex vector spaces

$$V_{z^-}, \quad V_{z^+} \quad (118)$$

associated with the two stacking interfaces of a local cell.

Definition 17 (Local Chrysene tensor space). *The full local*

4.6 Tensor-Space Fibration

For each $\mathbf{k} \in X_{\mathfrak{C}}$, let

$$\mathbb{T}_{\mathbf{k}} \cong \mathbb{T}_{\mathfrak{C}} \quad (121)$$

be a copy of the local tensor space.

Definition 18 (Tensorial geometric skeleton). *Define*

$$I_{\mathfrak{C}}^{\text{ten}} := \bigsqcup_{\mathbf{k} \in \mathbb{Z}^3} \{\mathbf{k}\} \times \mathbb{T}_{\mathbf{k}}, \quad (122)$$

with projection

$$p_{\text{ten}} : I_{\mathfrak{C}}^{\text{ten}} \longrightarrow X_{\mathfrak{C}}, \quad p_{\text{ten}}(\mathbf{k}, \Psi) = \mathbf{k}. \quad (123)$$

Every fiber satisfies

$$p_{\text{ten}}^{-1}(\mathbf{k}) \cong \mathbb{T}_{\mathfrak{C}}. \quad (124)$$

The translation-invariant local tensor section is

$$\Psi : X_{\mathfrak{C}} \longrightarrow I_{\mathfrak{C}}^{\text{ten}}, \quad \Psi(\mathbf{k}) = (\mathbf{k}, \Psi_{\mathfrak{C}}). \quad (125)$$

The object $I_{\mathfrak{C}}^{\text{ten}}$ is the tensorial geometric skeleton of the Chrysene Tensor Space. Its noncommutative operator completion is introduced later in Axioms A_7 – A_9 .

4.7 Network Contractions

The geometric connectors now acquire a tensor-network interpretation.

Definition 19 (Directional contraction maps). *Associated with the distinguished exact bridge counts N_x^*, N_y^* are*

nonzero bilinear contraction maps

$$C_x : V_9 \otimes V_3 \longrightarrow \mathbb{C}, \quad (126)$$

$$C_y : V_6 \otimes V_{12} \longrightarrow \mathbb{C}. \quad (127)$$

For each fiber registry state $\eta \in \mathbb{Z}_2$, there is a stacking contraction

$$C_z^{(\eta)} : V_{z^+} \otimes V_{z^-} \longrightarrow \mathbb{C}. \quad (128)$$

Hence the correspondence between the molecular-network and tensor-network descriptions is

$$\begin{array}{lll} C9 \leftrightarrow C3 & \longleftrightarrow & C_x, \\ C6 \leftrightarrow C12 & \longleftrightarrow & C_y, \\ z^+ \leftrightarrow z^- & \longleftrightarrow & C_z^{(\eta)}. \end{array} \quad (129)$$

Definition 20 (Finite tensor-network realization). Let $\Lambda \subseteq \mathbb{Z}^3$ be finite. Before contraction, the regional tensor is

$$\tilde{\Psi}_\Lambda := \bigotimes_{\mathbf{k} \in \Lambda} \Psi_{\mathbb{C}, \mathbf{k}}. \quad (130)$$

Contract every x -, y -, and z -interface whose two incident cells both belong to Λ . The resulting boundary tensor is

$$\Psi_\Lambda := \text{Contr}_{E_{\text{int}}(\Lambda)} \left(\tilde{\Psi}_\Lambda \right). \quad (131)$$

Uncontracted tensor factors correspond precisely to interfaces meeting the boundary of Λ .

Proposition 4 (Translation covariance of the tensor network). Suppose the same local tensor $\Psi_{\mathbb{C}}$ and the same direction-dependent contraction rules are assigned to every translated cell, with registry labels translated together with their associated z -edges. Then, for every finite $\Lambda \subseteq \mathbb{Z}^3$ and every $\ell \in \mathbb{Z}^3$, the tensor networks on Λ and $\Lambda + \ell$ are canonically isomorphic.

Proof. Translation sends each local cell to a cell carrying an isomorphic copy of $\mathbb{T}_{\mathbb{C}}$, preserves the orientation class of every x -, y -, and z -edge, and translates the corresponding registry data. Therefore it sends every contraction map in $E_{\text{int}}(\Lambda)$ to the corresponding contraction map in $E_{\text{int}}(\Lambda + \ell)$. The resulting contracted boundary tensor is preserved up to the canonical relabeling of its boundary factors. $\square$

Remark 9 (Tensor network does not yet imply noncommutativity). The tensor product and contraction operations introduced above are not, by themselves, the source of

noncommutativity. Noncommutativity enters when localized operators and translations act in an ordered fashion on the tensor network. The resulting sequence algebra is constructed in Axiom A7.

4.8 Planar Translation and Commensurate Registry

The planar translation lattice is the subgroup

$$\Lambda_{xy} := \{m\mathbf{a}_x + n\mathbf{a}_y : (m, n) \in \mathbb{Z}^2\} \subset \Lambda_{\mathbb{C}}. \quad (132)$$

A realization is *rectangular* when

$$\mathbf{a}_x \cdot \mathbf{a}_y = 0, \quad (133)$$

and *square* when, additionally,

$$\|\mathbf{a}_x\|_2 = \|\mathbf{a}_y\|_2. \quad (134)$$

Neither condition fixes the directional bridge counts.

Definition 21 (Axially aligned homogeneous direction). A direction $\alpha \in \{x, y\}$ is called *axially aligned and homogeneous* if

$$\mathbf{b}_\alpha = \ell_\alpha \mathbf{e}_\alpha, \quad \ell_\alpha > 0, \quad \|\mathbf{e}_\alpha\|_2 = 1, \quad (135)$$

and

$$\mathbf{a}_\alpha \parallel \mathbf{e}_\alpha. \quad (136)$$

Define

$$L_\alpha := \mathbf{a}_\alpha \cdot \mathbf{e}_\alpha, \quad (137)$$

$$s_\alpha := (\mathbf{d}_\alpha + \mathbf{c}_\alpha) \cdot \mathbf{e}_\alpha. \quad (138)$$

Proposition 5 (Scalar directional closure). Under exact closure and the hypotheses of Definition 21,

$$L_\alpha = s_\alpha + N_\alpha^* \ell_\alpha. \quad (139)$$

Proof. Substitute

$$\mathbf{B}_\alpha(N_\alpha^*) = N_\alpha^* \ell_\alpha \mathbf{e}_\alpha$$

into Equation (96) and take the Euclidean inner product with $\mathbf{e}_\alpha$. $\square$

Definition 22 (Directional registry coordinate). Define

$$\delta_\alpha := \frac{s_\alpha}{\ell_\alpha}, \quad K_\alpha := \frac{L_\alpha}{\ell_\alpha}. \quad (140)$$

Then exact closure gives

$$K_\alpha = N_\alpha^\star + \delta_\alpha. \quad (141)$$

Remark 10 (Deferred count relations). *Neither*

$$\delta_x - \delta_y = 2$$

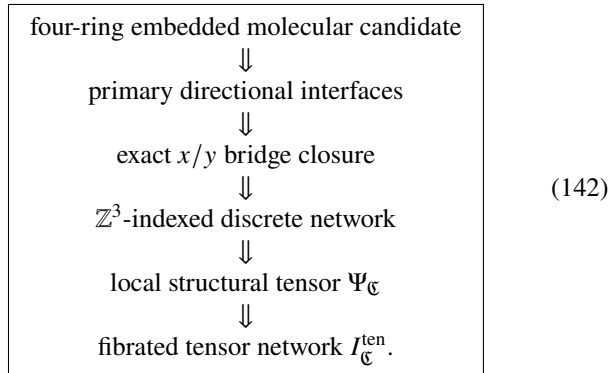
nor

$$N_y^\star = N_x^\star + 2$$

is imposed by Axioms A_1 – A_6 . Such relations require additional commensurability or molecular-registry hypotheses and remain part of the subsequent theorem program.

4.9 Geometric Hierarchy After Axioms A_1 – A_6

The first six axioms now define the sequence



At this stage the construction is tensorial and three-dimensional but is not yet a noncommutative geometry. The next stage equips the tensor network with localized observables, ordered translation and cipher operations, a noncommutative C^* -algebra, and ultimately a spectral triple.

5 Tensor Sequence Algebra and Noncommutative Action

Axioms A_1 – A_6 define a three-dimensional tensor network whose local structural tensor $\Psi_{\mathfrak{C}}$ is replicated over the discrete base

$$X_{\mathfrak{C}} = \mathbb{Z}^3.$$

The purpose of the present section is to pass from this tensorial geometric skeleton to an operator algebra that retains the ordering of localized structural operations and lattice translations.

The distinction is essential. A tensor product alone is not a noncommutative geometry, and the fact that two molecular attachment directions are geometrically inequivalent does not

imply that their associated operators fail to commute. Noncommutativity will instead arise from the covariance of localized tensor observables under the nontrivial translation action of the $\mathbb{Z}^3$ -indexed network.

In particular, the order

$$\text{localized operation} \longrightarrow \text{translation}$$

need not produce the same operator as

$$\text{translation} \longrightarrow \text{localized operation}.$$

This order dependence constitutes the sequence structure underlying the noncommutative algebra of the revised Chrysene Formalism.

5.1 Local Tensor State Spaces

Recall from Definition 17 that the local structural tensor space is

$$\mathbb{T}_{\mathfrak{C}} = V_3 \otimes V_9 \otimes V_{12} \otimes V_6 \otimes V_{z^-} \otimes V_{z^+} \otimes K_{\mathfrak{C}}. \quad (143)$$

Choose once and for all a positive-definite Hermitian inner product on each finite-dimensional tensor factor and equip $\mathbb{T}_{\mathfrak{C}}$ with the induced tensor-product inner product.

Definition 23 (Local tensor Hilbert space). *The finite-dimensional Hilbert space underlying one tensor-network cell is*

$$\mathcal{K}_{\mathfrak{C}} := \mathbb{T}_{\mathfrak{C}}, \quad (144)$$

equipped with the preceding Hermitian structure. For every $\mathbf{k} \in \mathbb{Z}^3$, let

$$\mathcal{K}_{\mathbf{k}} \cong \mathcal{K}_{\mathfrak{C}} \quad (145)$$

be the corresponding translated copy.

We require

$$\dim_{\mathbb{C}} \mathcal{K}_{\mathfrak{C}} \geq 2. \quad (146)$$

Remark 11 (Structural versus electronic Hilbert spaces). *The space $\mathcal{K}_{\mathfrak{C}}$ is the Hilbert realization of the local structural tensor degrees of freedom. It is not identified with the electronic Hamiltonian space introduced later, nor with the GNS Hilbert space until a representation of the observable algebra has been chosen. The notation therefore preserves the distinction*

$$\begin{aligned} &\text{tensor structure} \neq \text{algebraic representation} \\ &\neq \text{electronic dynamics.} \end{aligned} \quad (147)$$

5.2 Finite-Region Tensor Algebras

Fix a lexicographic ordering of $\mathbb{Z}^3$. For every finite set

$$\Lambda \in \mathbb{Z}^3,$$

define

$$\mathcal{K}_\Lambda := \bigotimes_{\mathbf{k} \in \Lambda} \mathcal{K}_{\mathbf{k}}, \quad (148)$$

where the tensor factors are written in the fixed order.

Definition 24 (Finite-region tensor observable algebra). *The algebra of bounded structural observables supported in Λ is*

$$\mathfrak{Q}_\Lambda := B(\mathcal{K}_\Lambda). \quad (149)$$

Because $\mathcal{K}_\Lambda$ is finite dimensional, $\mathfrak{Q}_\Lambda$ is a finite-dimensional unital C^* -algebra.

If

$$\Lambda \subseteq \Lambda',$$

the canonical inclusion is

$$J_{\Lambda'}^\Lambda : \mathfrak{Q}_\Lambda \longrightarrow \mathfrak{Q}_{\Lambda'}, \quad A \longmapsto A \otimes I_{\Lambda' \setminus \Lambda}. \quad (150)$$

These inclusions satisfy

$$J_{\Lambda'}^\Lambda = \text{id}, \quad (151)$$

$$J_{\Lambda''}^{\Lambda'} \circ J_{\Lambda'}^\Lambda = J_{\Lambda''}^\Lambda \quad (\Lambda \subseteq \Lambda' \subseteq \Lambda''). \quad (152)$$

Definition 25 (Local tensor algebra). *The algebraic local observable algebra is the directed union*

$$\mathfrak{Q}_{\text{loc}} := \bigcup_{\Lambda \in \mathbb{Z}^3} \mathfrak{Q}_\Lambda, \quad (153)$$

where the algebras are identified through the embeddings $J_{\Lambda'}^\Lambda$.

Definition 26 (Quasi-local tensor algebra). *The quasi-local Chrysene tensor algebra is the norm completion*

$$\mathfrak{Q}_{\mathfrak{C}} := \overline{\mathfrak{Q}_{\text{loc}}}^{\|\cdot\|}. \quad (154)$$

Equivalently,

$$\mathfrak{Q}_{\mathfrak{C}} \cong \varinjlim_{\Lambda \in \mathbb{Z}^3} \mathfrak{Q}_\Lambda. \quad (155)$$

Remark 12 (Role of tensor contractions). *The contraction maps of Definition 19 specify how the local tensors are geometrically connected. The algebra $\mathfrak{Q}_{\mathfrak{C}}$ instead records bounded operations on the local structural degrees of freedom. Bridge and stacking interactions will enter the physical Hamiltonian later; they are not implemented by quotienting the quasi-local algebra by the tensor-contraction relations.*

5.3 Localized Operators

For

$$X \in B(\mathcal{K}_{\mathfrak{C}})$$

and $\mathbf{k} \in \mathbb{Z}^3$, write

$$\iota_{\mathbf{k}}(X) \in \mathfrak{Q}_{\mathfrak{C}} \quad (156)$$

for the canonical copy of X acting on the $\mathbf{k}$ -th tensor factor and as the identity elsewhere.

Thus, on any finite region $\Lambda \ni \mathbf{k}$,

$$\iota_{\mathbf{k}}(X) = I \otimes \cdots \otimes X \otimes \cdots \otimes I. \quad (157)$$

Lemma 6 (Disjoint local operators commute). *If*

$$\mathbf{k} \neq \ell,$$

then for every

$$X, Y \in B(\mathcal{K}_{\mathfrak{C}})$$

one has

$$[\iota_{\mathbf{k}}(X), \iota_{\ell}(Y)] = 0. \quad (158)$$

Proof. Choose a finite region Λ containing $\mathbf{k}$ and ℓ . The two operators act nontrivially on distinct tensor factors of $\mathcal{K}_\Lambda$. Hence

$$(X \otimes I)(I \otimes Y) = X \otimes Y = (I \otimes Y)(X \otimes I).$$

The equality is preserved by the canonical embeddings into $\mathfrak{Q}_{\mathfrak{C}}$. $\square$

This lemma is useful because it shows that global noncommutativity need not arise from assigning noncommuting observables to spatially disjoint cells.

5.4 Translation Automorphisms

The geometric translation action of Axiom 6 induces a corresponding action on the tensor observables.

For

$$\mathbf{m} \in \mathbb{Z}^3$$

and finite $\Lambda \in \mathbb{Z}^3$, let

$$W_{\mathbf{m}}^\Lambda : \mathcal{K}_\Lambda \longrightarrow \mathcal{K}_{\Lambda+\mathbf{m}} \quad (159)$$

be the canonical unitary obtained by translating each tensor factor

$$\mathcal{K}_{\mathbf{k}} \longrightarrow \mathcal{K}_{\mathbf{k}+\mathbf{m}}.$$

For $A \in \mathfrak{Q}_\Lambda$, define

$$\alpha_{\mathbf{m}}(A) := W_{\mathbf{m}}^\Lambda A \left(W_{\mathbf{m}}^\Lambda \right)^\dagger. \quad (160)$$

Proposition 7 (Translation action on the quasi-local algebra). *The maps $\alpha_{\mathbf{m}}$ extend uniquely to $*$ -automorphisms of $\mathfrak{Q}_{\mathfrak{C}}$ and define a group action*

$$\alpha : \mathbb{Z}^3 \longrightarrow \text{Aut}(\mathfrak{Q}_{\mathfrak{C}}) \quad (161)$$

satisfying

$$\alpha_0 = \text{id}, \quad (162)$$

$$\alpha_{\mathbf{m}} \circ \alpha_{\mathbf{n}} = \alpha_{\mathbf{m}+\mathbf{n}}. \quad (163)$$

For every localized operator,

$$\alpha_{\mathbf{m}}(\iota_{\mathbf{k}}(X)) = \iota_{\mathbf{k}+\mathbf{m}}(X). \quad (164)$$

Proof. The maps $W_{\mathbf{m}}^{\Lambda}$ are unitary, so conjugation defines a $*$ -isomorphism

$$\mathfrak{Q}_{\Lambda} \longrightarrow \mathfrak{Q}_{\Lambda+\mathbf{m}}.$$

These maps are compatible with the canonical inclusions $J_{\Lambda}^{\Lambda'}$. They therefore define an isometric $*$ -automorphism of $\mathfrak{Q}_{\text{loc}}$, which extends uniquely by continuity to its norm completion $\mathfrak{Q}_{\mathfrak{C}}$.

Composition of the factor translations gives

$$W_{\mathbf{m}}^{\Lambda+\mathbf{n}} W_{\mathbf{n}}^{\Lambda} = W_{\mathbf{m}+\mathbf{n}}^{\Lambda},$$

from which the group law follows. Equation (164) is immediate from the definition. $\square$

5.5 The Noncommutative Sequence Algebra

We now adjoin the translation operations themselves to the quasi-local observable algebra.

Axiom 7 (Quasi-Local Tensor Algebra and Noncommutative Sequence Action (A7)). *The structural observable algebra of the Chrysene Formalism is the crossed-product C^* -algebra*

$$\mathfrak{A}_{\mathfrak{C}} := \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3. \quad (165)$$

The notation

$$\mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$$

may be interpreted as the full crossed product.

Because $\mathbb{Z}^3$ is a discrete amenable group, the canonical quotient from the full crossed product to the reduced crossed product is an isomorphism:

$$\mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3 \cong \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha,r} \mathbb{Z}^3. \quad (166)$$

Accordingly, no distinction between the two completions affects the axiomatic construction. When a regular represen-

tation or canonical conditional expectation is used explicitly, as in Appendix B, the reduced notation is employed. It is generated by $\mathfrak{Q}_{\mathfrak{C}}$ and a family of unitaries

$$\{u_{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}^3\} \quad (167)$$

satisfying

$$u_0 = I, \quad (168)$$

$$u_{\mathbf{m}} u_{\mathbf{n}} = u_{\mathbf{m}+\mathbf{n}}, \quad (169)$$

$$u_{\mathbf{m}}^* = u_{-\mathbf{m}}, \quad (170)$$

$$u_{\mathbf{m}} A u_{\mathbf{m}}^* = \alpha_{\mathbf{m}}(A), \quad A \in \mathfrak{Q}_{\mathfrak{C}}. \quad (171)$$

The realization is additionally equipped with a state

$$\omega : \mathfrak{A}_{\mathfrak{C}} \longrightarrow \mathbb{C} \quad (172)$$

satisfying

$$\omega(A^* A) \geq 0, \quad (173)$$

$$\omega(I) = 1. \quad (174)$$

Remark 13 (Algebraic crossed-product multiplication). *On the dense algebraic crossed product, a typical element is a finite sum*

$$\sum_{\mathbf{m}} A_{\mathbf{m}} u_{\mathbf{m}}, \quad A_{\mathbf{m}} \in \mathfrak{Q}_{\mathfrak{C}}. \quad (175)$$

The covariance relation implies

$$(A_{\mathbf{m}} u_{\mathbf{m}})(B_{\mathbf{n}} u_{\mathbf{n}}) = A_{\mathbf{m}} \alpha_{\mathbf{m}}(B_{\mathbf{n}}) u_{\mathbf{m}+\mathbf{n}}. \quad (176)$$

Similarly,

$$(A_{\mathbf{m}} u_{\mathbf{m}})^* = \alpha_{-\mathbf{m}}(A_{\mathbf{m}}^*) u_{-\mathbf{m}}. \quad (177)$$

Thus the translation history is retained algebraically in the multiplication law.

5.6 Sequence Dependence

Definition 27 (Structural operation word). *A structural operation word is a finite ordered tuple*

$$w = (G_1, \dots, G_r), \quad (178)$$

where each G_j is either a localized tensor operator in $\mathfrak{Q}_{\text{loc}}$ or a translation unitary $u_{\mathbf{m}}$.

If the operations are applied in the listed order, define the associated sequence operator by

$$\text{Op}(w) := G_r G_{r-1} \cdots G_1. \quad (179)$$

Definition 28 (Sequence dependence). *Two words w and w' containing the same elementary operations but in different order are called sequence-distinguishable when*

$$\text{Op}(w) \neq \text{Op}(w'). \quad (180)$$

The tensor network is called operator-sequence dependent if at least one such pair exists.

Remark 14 (Sequence order is not yet physical time). *Definition 28 concerns ordered algebraic composition. It does not, by itself, identify the ordering parameter with physical time. A temporal interpretation requires a dynamical law, introduced later through the Hamiltonian or another specified evolution.*

5.7 Noncommutativity from Translation of Local Structure

The following theorem gives the principal mathematical reason for calling $\mathfrak{A}_{\mathfrak{G}}$ a noncommutative sequence algebra.

Theorem 8 (Sequence-induced noncommutativity). *Under Axioms A₆–A₇, the crossed-product algebra $\mathfrak{A}_{\mathfrak{G}}$ is noncommutative.*

More precisely, there exist a localized projection

$$P_{\mathbf{k}} \in \mathfrak{Q}_{\mathfrak{G}}$$

and a nonzero translation

$$\mathbf{m} \in \mathbb{Z}^3$$

such that

$$\boxed{[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0}. \quad (181)$$

Proof. By Equation (146), choose orthonormal vectors $e_0, e_1 \in \mathcal{K}_{\mathfrak{G}}$. Let

$$P := |e_0\rangle\langle e_0| \in B(\mathcal{K}_{\mathfrak{G}}) \quad (182)$$

and define

$$P_{\mathbf{k}} := \iota_{\mathbf{k}}(P). \quad (183)$$

Choose any $\mathbf{m} \neq \mathbf{0}$. Then

$$\mathbf{k} + \mathbf{m} \neq \mathbf{k}.$$

By Equation (164),

$$\alpha_{\mathbf{m}}(P_{\mathbf{k}}) = P_{\mathbf{k}+\mathbf{m}}. \quad (184)$$

The two projections are distinct. Indeed, on the two-site tensor factor

$$\mathcal{K}_{\mathbf{k}} \otimes \mathcal{K}_{\mathbf{k}+\mathbf{m}},$$

one has

$$P_{\mathbf{k}}(e_0 \otimes e_1) = e_0 \otimes e_1, \quad (185)$$

$$P_{\mathbf{k}+\mathbf{m}}(e_0 \otimes e_1) = 0. \quad (186)$$

Hence

$$P_{\mathbf{k}+\mathbf{m}} \neq P_{\mathbf{k}}.$$

Using the covariance relation,

$$u_{\mathbf{m}}P_{\mathbf{k}} = \alpha_{\mathbf{m}}(P_{\mathbf{k}})u_{\mathbf{m}} = P_{\mathbf{k}+\mathbf{m}}u_{\mathbf{m}}. \quad (187)$$

Therefore

$$[u_{\mathbf{m}}, P_{\mathbf{k}}] = u_{\mathbf{m}}P_{\mathbf{k}} - P_{\mathbf{k}}u_{\mathbf{m}} \quad (188)$$

$$= (P_{\mathbf{k}+\mathbf{m}} - P_{\mathbf{k}})u_{\mathbf{m}}. \quad (189)$$

The factor in parentheses is nonzero and $u_{\mathbf{m}}$ is unitary. Consequently,

$$[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0.$$

Thus $\mathfrak{A}_{\mathfrak{G}}$ is noncommutative. $\square$

Corollary 1 (Order of localization and translation matters). *Let*

$$w_1 := (P_{\mathbf{k}}, u_{\mathbf{m}}), \quad (190)$$

$$w_2 := (u_{\mathbf{m}}, P_{\mathbf{k}}), \quad (191)$$

where $\mathbf{m} \neq \mathbf{0}$. Then

$$\boxed{\text{Op}(w_1) \neq \text{Op}(w_2)}. \quad (192)$$

Hence the Chrysene tensor network is operator-sequence dependent.

Proof. By Definition 27,

$$\text{Op}(w_1) = u_{\mathbf{m}}P_{\mathbf{k}},$$

whereas

$$\text{Op}(w_2) = P_{\mathbf{k}}u_{\mathbf{m}}.$$

Their difference is the nonzero commutator of Theorem 8. $\square$

5.8 Local Commutativity and Global Noncommutativity

The preceding result does not require spatially separated local projections to fail to commute.

Let

$$\mathfrak{P}_{\mathfrak{G}} := C^*\left(\{P_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^3\}\right) \subseteq \mathfrak{Q}_{\mathfrak{G}}. \quad (193)$$

Proposition 9 (Commutative local sector with noncommutative sequence extension). *The algebra $\mathfrak{P}_{\mathfrak{C}}$ is commutative, while the C^* -algebra generated by $\mathfrak{P}_{\mathfrak{C}}$ and the translation unitaries $\{u_{\mathbf{m}}\}$ is noncommutative.*

Proof. For $\mathbf{k} \neq \ell$, Lemma 6 gives

$$P_{\mathbf{k}}P_{\ell} = P_{\ell}P_{\mathbf{k}}.$$

Each generator also commutes with itself, so $\mathfrak{P}_{\mathfrak{C}}$ is commutative.

Theorem 8 gives

$$[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0$$

for $\mathbf{m} \neq 0$. Hence the algebra generated jointly by the two families is noncommutative. $\square$

Remark 15 (Origin of noncommutativity). *Proposition 9 isolates the source of noncommutativity used in the revised formalism. The essential relation is not*

$$\mathbf{d}_x \cdot \mathbf{d}_y = 0 \implies [A_x, A_y] \neq 0.$$

Rather, it is the covariant sequence relation

$$u_{\mathbf{m}}P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}}u_{\mathbf{m}}. \quad (194)$$

Noncommutativity therefore arises because translation changes the localization of a structural observable.

5.9 GNS Representation of the Sequence Algebra

Axiom 7 includes a state

$$\omega : \mathfrak{A}_{\mathfrak{C}} \rightarrow \mathbb{C}.$$

The representation of this algebra does not require a new axiom.

Proposition 10 (GNS representation). *Associated with $(\mathfrak{A}_{\mathfrak{C}}, \omega)$ there exists a cyclic $*$ -representation*

$$(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega}) \quad (195)$$

satisfying

$$\omega(A) = \langle \Omega_{\omega}, \pi_{\omega}(A)\Omega_{\omega} \rangle_{\mathcal{H}_{\omega}} \quad (A \in \mathfrak{A}_{\mathfrak{C}}). \quad (196)$$

Proof. This is the Gelfand–Naimark–Segal construction applied to the state ω on the unital C^* -algebra $\mathfrak{A}_{\mathfrak{C}}$. $\square$

Definition 29 (Von Neumann sequence algebra). *The von Neumann algebra generated by the GNS representation is*

$$\mathfrak{M}_{\omega} := \pi_{\omega}(\mathfrak{A}_{\mathfrak{C}})''. \quad (197)$$

5.10 Distinction from Noncocommutativity

Remark 16 (Noncommutativity versus noncocommutativity). *The result*

$$AB \neq BA$$

concerns multiplication in the observable algebra. It is distinct from a statement of noncocommutativity

$$\Delta^{\text{op}} \neq \Delta,$$

which presupposes the existence of a coproduct Δ and therefore additional Hopf-algebraic structure.

No coproduct, universal R -matrix, or braided monoidal structure is assumed in Axiom A₇. Such structures, if later introduced, require independent definitions and proofs.

5.11 Status of the Tensor-Space Construction

The first seven axioms now establish the chain

$$\begin{array}{c} \text{embedded molecular node} \\ \Downarrow \\ \text{local structural tensor } \Psi_{\mathfrak{C}} \\ \Downarrow \\ \mathbb{Z}^3\text{-fibrated tensor network} \\ \Downarrow \\ \text{quasi-local tensor algebra } \mathfrak{Q}_{\mathfrak{C}} \\ \Downarrow \\ \text{translation action } \alpha : \mathbb{Z}^3 \rightarrow \text{Aut}(\mathfrak{Q}_{\mathfrak{C}}) \\ \Downarrow \\ \text{sequence algebra } \mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3 \\ \Downarrow \\ \exists A, B \in \mathfrak{A}_{\mathfrak{C}} \text{ with } [A, B] \neq 0. \end{array} \quad (198)$$

Thus the noncommutative character of the revised Chrysene Formalism is no longer inferred from Euclidean orthogonality or molecular anisotropy. It is generated explicitly by the ordered action of translations on localized tensor observables.

The next stage introduces the primary and dual structural tensors, the

$$\{3, 6, 9, 12\} \longleftrightarrow \{2, 5, 8, 11\}$$

dual-cipher transformation, and the corresponding commutant relation inside $\mathfrak{M}_{\omega}$.

6 Dual Tensor, Cipher Encoding, and Commutant Duality

Axiom A₇ constructs the noncommutative sequence algebra from localized tensor observables and the ordered action of the $\mathbb{Z}^3$ -translation group. We now introduce the second structural

ingredient of the original Chrysene Formalism: the dual-cipher relation between the primary

$$\{3, 6, 9, 12\}$$

attachment pattern and the adjacent

$$\{2, 5, 8, 11\}$$

pattern.

The revised construction separates three notions that must not be conflated:

$$\boxed{\begin{array}{c} \text{geometric dual-cipher transformation} \\ \neq \\ \text{finite four-state encoding} \\ \neq \\ \text{operator-algebraic commutant duality.} \end{array}} \quad (199)$$

Axiom A_8 will connect these structures explicitly. It will not, however, identify a four-element encoding alphabet with $GF(4)$ without an independently specified multiplicative law.

6.1 The Primary–Dual Locant Transformation

Let

$$S_{\text{pri}} := \{3, 6, 9, 12\}, \quad (200)$$

$$S_{\text{dual}} := \{2, 5, 8, 11\}. \quad (201)$$

Recall from Axiom 1 that the associated distinguished vertex sets are

$$A_{\mathfrak{C}}^{\text{pri}} = \{v_3, v_6, v_9, v_{12}\}, \quad (202)$$

$$A_{\mathfrak{C}}^{\text{dual}} = \{v_2, v_5, v_8, v_{11}\}. \quad (203)$$

Definition 30 (One-step dual-cipher shift). *The primary-to-dual cipher shift is the bijection*

$$\boxed{\delta_{\text{dc}} : S_{\text{pri}} \longrightarrow S_{\text{dual}}} \quad (204)$$

defined by

$$\delta_{\text{dc}}(3) = 2, \quad \delta_{\text{dc}}(6) = 5, \quad (205)$$

$$\delta_{\text{dc}}(9) = 8, \quad \delta_{\text{dc}}(12) = 11. \quad (206)$$

At the vertex level,

$$\delta_{\text{dc}}^V(v_j) := v_{\delta_{\text{dc}}(j)}. \quad (207)$$

Definition 31 (Cipher involution). *Define*

$$\sigma_{\text{dc}} : S_{\text{pri}} \sqcup S_{\text{dual}} \longrightarrow S_{\text{pri}} \sqcup S_{\text{dual}} \quad (208)$$

by exchanging the paired coordinates

$$(3, 2), \quad (6, 5), \quad (9, 8), \quad (12, 11). \quad (209)$$

Then

$$\boxed{\sigma_{\text{dc}}^2 = \text{id}.} \quad (210)$$

Thus the cipher transformation carries a canonical $\mathbb{Z}_2$ -action.

Remark 17 (The cipher shift is not a molecular symmetry). *The map δ_{dc} is a transformation of the distinguished locant data. It is not assumed to belong to the molecular symmetry stabilizer $\Gamma_{\mathfrak{C}}$. In particular, the notation does not assert the existence of a Euclidean isometry carrying the primary substituted realization onto the dual realization. Historically, the one-coordinate perimeter shift was represented as a discrete angular phase change. In the revised formalism the primitive object is the explicitly defined locant bijection (204); any realization as a physical rotation is an additional geometric statement.*

6.2 Combinatorial Complementarity

Lemma 11 (Disjoint primary and dual locant supports). *The two distinguished attachment-site sets satisfy*

$$\boxed{A_{\mathfrak{C}}^{\text{pri}} \cap A_{\mathfrak{C}}^{\text{dual}} = \emptyset.} \quad (211)$$

Proof. The eight locants

$$2, 3, 5, 6, 8, 9, 11, 12$$

are pairwise distinct, and λ_{loc} is a function. Hence no vertex can simultaneously carry two different locant labels. Therefore the vertices appearing in the two distinguished sets are disjoint. $\square$

Remark 18 (Combinatorial versus steric disjointness). *Lemma 11 is a graph-theoretic statement. It does not, by itself, establish that three-dimensional substituent exclusion volumes are disjoint. A stronger assertion such as*

$$V_{\text{sub}}(\Psi_{\mathfrak{C}}) \cap V_{\text{sub}}(\Psi'_{\mathfrak{C}}) = \emptyset \quad (212)$$

requires an explicit definition of the exclusion-volume functional and verification for the molecular realization. Such steric non-overlap may therefore be included in a later molecular instantiation theorem without being silently assumed here.

6.3 The Dual Structural Tensor

Associate nonzero finite-dimensional complex interface spaces

$$V_2, \quad V_5, \quad V_8, \quad V_{11} \quad (213)$$

with the four dual locants.

We require the paired primary and dual interfaces to have matching dimensions:

$$\dim V_3 = \dim V_2, \quad \dim V_6 = \dim V_5, \quad (214)$$

$$\dim V_9 = \dim V_8, \quad \dim V_{12} = \dim V_{11}. \quad (215)$$

Choose unitary interface identifications

$$D_3 : V_3 \longrightarrow V_2, \quad D_6 : V_6 \longrightarrow V_5, \quad (216)$$

$$D_9 : V_9 \longrightarrow V_8, \quad D_{12} : V_{12} \longrightarrow V_{11}. \quad (217)$$

Definition 32 (Dual local tensor space). *The dual local structural tensor space is*

$$\mathbb{T}'_{\mathbb{C}} := V_2 \otimes V_8 \otimes V_{11} \otimes V_5 \otimes V_{z^-} \otimes V_{z^+} \otimes K_{\mathbb{C}}. \quad (218)$$

The ordering of the first four factors is chosen to correspond factorwise to

$$(3, 9, 12, 6) \longmapsto (2, 8, 11, 5).$$

Definition 33 (Geometric dual-cipher tensor map). *Define*

$$\mathcal{D}_{\text{dc}} := D_3 \otimes D_9 \otimes D_{12} \otimes D_6 \otimes I_{z^-} \otimes I_{z^+} \otimes I_{K_{\mathbb{C}}} \quad (219)$$

as a unitary isomorphism

$$\mathcal{D}_{\text{dc}} : \mathbb{T}_{\mathbb{C}} \longrightarrow \mathbb{T}'_{\mathbb{C}}. \quad (220)$$

The dual structural tensor is

$$\Psi'_{\mathbb{C}} := \mathcal{D}_{\text{dc}} \Psi_{\mathbb{C}}. \quad (221)$$

Remark 19 (Meaning of duality). *The adjective dual in $\Psi'_{\mathbb{C}}$ refers first to the cipher pairing (209). It does not mean that $\mathbb{T}'_{\mathbb{C}}$ is the algebraic dual vector space $\mathbb{T}_{\mathbb{C}}^*$. Nor does it imply a Hilbert-space orthogonality relation between $\Psi_{\mathbb{C}}$ and $\Psi'_{\mathbb{C}}$. The operator-algebraic duality is introduced separately below.*

6.4 Primary and Dual Tensor Sections

The two structural tensors define parallel tensor sections over the same $\mathbb{Z}^3$ base.

Definition 34 (Primary and dual tensor fibrations). *Let*

$$I_{\mathbb{C}}^{\text{pri}} := \bigsqcup_{\mathbf{k} \in \mathbb{Z}^3} \{\mathbf{k}\} \times \mathbb{T}_{\mathbb{C}}, \quad (222)$$

$$I_{\mathbb{C}}^{\text{dual}} := \bigsqcup_{\mathbf{k} \in \mathbb{Z}^3} \{\mathbf{k}\} \times \mathbb{T}'_{\mathbb{C}}. \quad (223)$$

Their distinguished translation-invariant sections are

$$\Psi^{\text{pri}}(\mathbf{k}) := (\mathbf{k}, \Psi_{\mathbb{C}}), \quad (224)$$

$$\Psi^{\text{dual}}(\mathbf{k}) := (\mathbf{k}, \Psi'_{\mathbb{C}}). \quad (225)$$

The fiberwise cipher isomorphism

$$\widehat{\mathcal{D}}_{\text{dc}} : I_{\mathbb{C}}^{\text{pri}} \longrightarrow I_{\mathbb{C}}^{\text{dual}} \quad (226)$$

is defined by

$$\widehat{\mathcal{D}}_{\text{dc}}(\mathbf{k}, \Psi) := (\mathbf{k}, \mathcal{D}_{\text{dc}} \Psi). \quad (227)$$

Thus the primary and dual tensor constructions occupy the same discrete base

$$X_{\mathbb{C}} = \mathbb{Z}^3,$$

but have distinct labeled local interface realizations.

6.5 The Native Four-State Encoding

The cipher involution supplies one binary coordinate. The z -registry of Axiom 6 supplies a second.

Define

$$\epsilon \in \mathbb{Z}_2 \quad (228)$$

by

$$\epsilon = \begin{cases} 0, & \text{primary tensor } \Psi_{\mathbb{C}}, \\ 1, & \text{dual tensor } \Psi'_{\mathbb{C}}. \end{cases} \quad (229)$$

Recall that

$$\eta \in \mathbb{Z}_2 \quad (230)$$

denotes the discrete z -registry state.

Definition 35 (Chrysene structural alphabet). *The local structural alphabet is*

$$\Sigma_{\mathbb{C}} := \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z. \quad (231)$$

Its elements are written

$$s_{\epsilon, \eta} := (\epsilon, \eta), \quad \epsilon, \eta \in \mathbb{Z}_2. \quad (232)$$

Consequently,

$$|\Sigma_{\mathbb{C}}| = 4. \quad (233)$$

Explicitly,

$$\Sigma_{\mathfrak{C}} = \{s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}\}. \quad (234)$$

Definition 36 (Additive structural law). *The canonical additive operation on $\Sigma_{\mathfrak{C}}$ is componentwise addition modulo 2:*

$$(\epsilon, \eta) \boxplus (\epsilon', \eta') := (\epsilon + \epsilon', \eta + \eta') \pmod{2}. \quad (235)$$

Proposition 12 (Additive encoding group). *The structural alphabet satisfies*

$$(\Sigma_{\mathfrak{C}}, \boxplus) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (236)$$

as an abelian group.

Proof. This follows directly from Definition 36. $\square$

Remark 20 (No $GF(4)$ field law yet). *The additive group*

$$\mathbb{Z}_2^2$$

is isomorphic to the additive group of $GF(4)$, but this fact does not specify the multiplicative structure of a field.

Thus

$$|\Sigma_{\mathfrak{C}}| = 4 \quad (237)$$

and even

$$(\Sigma_{\mathfrak{C}}, \boxplus) \cong (GF(4), +) \quad (238)$$

do not imply that $\Sigma_{\mathfrak{C}}$ has a distinguished multiplication making it a field. A structurally induced multiplication remains a theorem-level problem.

6.6 Local Cipher Flip in the Sequence Algebra

The finite alphabet may be represented internally within the local structural-state factor.

Definition 37 (Cipher and registry factors). *Axiom A₈ requires the internal factor of the local tensor space to admit a decomposition*

$$K_{\mathfrak{C}} \cong K_{\text{aux}} \otimes K_{\text{dc}} \otimes K_z, \quad (239)$$

where

$$K_{\text{dc}} \cong \mathbb{C}^2, \quad K_z \cong \mathbb{C}^2. \quad (240)$$

Choose orthonormal bases

$$\{|0\rangle_{\text{dc}}, |1\rangle_{\text{dc}}\} \subset K_{\text{dc}}, \quad (241)$$

$$\{|0\rangle_z, |1\rangle_z\} \subset K_z. \quad (242)$$

Define the cipher flip

$$F_{\text{dc}}|\epsilon\rangle_{\text{dc}} := |\epsilon + 1\rangle_{\text{dc}}, \quad \epsilon \in \mathbb{Z}_2. \quad (243)$$

Then

$$F_{\text{dc}}^2 = I. \quad (244)$$

Let

$$\Delta_0 := I_{\text{interfaces}} \otimes I_{\text{aux}} \otimes F_{\text{dc}} \otimes I_z \in B(\mathcal{K}_{\mathfrak{C}}). \quad (245)$$

For $\mathbf{k} \in \mathbb{Z}^3$, define its localized copy

$$\Delta_{\mathbf{k}} := \iota_{\mathbf{k}}(\Delta_0) \in \mathfrak{Q}_{\mathfrak{C}} \subset \mathfrak{A}_{\mathfrak{C}}. \quad (246)$$

Proposition 13 (Translation covariance of the cipher). *For every $\mathbf{k}, \mathbf{m} \in \mathbb{Z}^3$,*

$$u_{\mathbf{m}} \Delta_{\mathbf{k}} u_{\mathbf{m}}^* = \Delta_{\mathbf{k}+\mathbf{m}}. \quad (247)$$

Proof. This is Equation (164) applied to the local operator Δ_0 . $\square$

Corollary 2 (Cipher–translation sequence dependence). *If $\mathbf{m} \neq 0$, then*

$$[u_{\mathbf{m}}, \Delta_{\mathbf{k}}] \neq 0. \quad (248)$$

Proof. By Proposition 13,

$$u_{\mathbf{m}} \Delta_{\mathbf{k}} = \Delta_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}. \quad (249)$$

Because Δ_0 is non-scalar and its localized copies act on distinct tensor factors,

$$\Delta_{\mathbf{k}+\mathbf{m}} \neq \Delta_{\mathbf{k}}$$

when $\mathbf{m} \neq 0$. Hence

$$[u_{\mathbf{m}}, \Delta_{\mathbf{k}}] = (\Delta_{\mathbf{k}+\mathbf{m}} - \Delta_{\mathbf{k}}) u_{\mathbf{m}} \neq 0. \quad (250)$$

$\square$

Thus the dual cipher supplies a second explicit realization of the sequence-dependent noncommutativity established in Theorem 8.

6.7 Operator-Algebraic Commutant Duality

We now state the algebraic counterpart of the geometric dual tensor.

Recall the GNS representation

$$(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega}) \quad (251)$$

of the sequence algebra $\mathfrak{A}_{\mathfrak{C}}$ and its von Neumann closure

$$\mathfrak{M}_{\omega} := \pi_{\omega}(\mathfrak{A}_{\mathfrak{C}})''. \quad (252)$$

Definition 38 (Primary operator algebra). *The primary operator realization is*

$$\mathfrak{M}_{\text{pri}} := \mathfrak{M}_{\omega}. \quad (253)$$

The commutant is

$$\mathfrak{M}'_{\text{pri}} := \{B \in B(\mathcal{H}_{\omega}) : [A, B] = 0 \text{ for every } A \in \mathfrak{M}_{\text{pri}}\}. \quad (254)$$

Axiom 8 (Dual-Cipher Tensor Encoding and Commutant Duality (A8)). *The Chrysene Tensor Space is equipped with the following dual-cipher structure.*

1. *The primary and dual locant configurations are related by the involutive cipher pairing σ_{dc} of Definition 31.*
2. *The primary–dual cipher is perimeter realized. Let*

$$\partial G_{\mathfrak{C}}$$

denote the chosen oriented external perimeter cycle of the molecular graph. Then, for every

$$j \in \{3, 6, 9, 12\},$$

the primary locant v_j and its paired dual locant $v_{\delta_{\text{dc}}(j)}$ are adjacent on the perimeter:

$$\{v_j, v_{\delta_{\text{dc}}(j)}\} \in E(\partial G_{\mathfrak{C}}). \quad (255)$$

Moreover, the four pairings are required to have the same orientation relative to the chosen perimeter cycle. Thus the map

$$\delta_{\text{dc}} : \{3, 6, 9, 12\} \longrightarrow \{2, 5, 8, 11\} \quad (256)$$

is realized as a uniform one-step perimeter displacement rather than as an arbitrary relabeling of distinguished vertices.

This perimeter realization does not imply that δ_{dc} belongs to the molecular symmetry stabilizer $\Gamma_{\mathfrak{C}}$.

3. *The primary structural tensor $\Psi_{\mathfrak{C}}$ and dual structural tensor $\Psi'_{\mathfrak{C}}$ are related by the unitary fiber isomorphism*

$$\Psi'_{\mathfrak{C}} = \mathcal{D}_{\text{dc}} \Psi_{\mathfrak{C}}. \quad (257)$$

4. *The native local structural alphabet is*

$$\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z. \quad (258)$$

5. *The internal tensor factor admits the decomposition (239), and the cipher transformation is represented locally by the involution Δ_0 .*

6. *There exists a conjugate-linear isometric involution*

$$J_{\text{dc}} : \mathcal{H}_{\omega} \longrightarrow \mathcal{H}_{\omega}, \quad J_{\text{dc}}^2 = I, \quad (259)$$

satisfying

$$J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}} = \mathfrak{M}'_{\text{pri}}. \quad (260)$$

7. *Define the dual operator algebra by*

$$\mathfrak{M}_{\text{dual}} := J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}}. \quad (261)$$

Hence

$$\mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}. \quad (262)$$

6.8 The Opposite-Algebra Representation

The commutant realization can be stated equivalently through the opposite algebra.

Definition 39 (Dual representation). *For $A \in \mathfrak{A}_{\mathfrak{C}}$, define*

$$\pi_{\text{dual}}(A^{\text{op}}) := J_{\text{dc}} \pi_{\omega}(A^*) J_{\text{dc}}. \quad (263)$$

Then

$$\pi_{\text{dual}} : \mathfrak{A}_{\mathfrak{C}}^{\text{op}} \longrightarrow B(\mathcal{H}_{\omega}) \quad (264)$$

*is a *-representation of the opposite algebra.*

Proposition 14 (Axiom A8 realization of the dual commutant). *Under Axiom 8,*

$$\pi_{\text{dual}} \left(\mathfrak{A}_{\mathfrak{C}}^{\text{op}} \right)'' = \mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}. \quad (265)$$

Proof. Because $\pi_{\omega}(\mathfrak{A}_{\mathfrak{C}})'' = \mathfrak{M}_{\text{pri}}$, taking adjoints leaves the generated von Neumann algebra unchanged. Conjugation by J_{dc} therefore gives

$$\pi_{\text{dual}} \left(\mathfrak{A}_{\mathfrak{C}}^{\text{op}} \right)'' = J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}} \quad (266)$$

$$= \mathfrak{M}'_{\text{pri}} \quad (267)$$

by Equation (260). $\square$

This is the precise operator-algebraic meaning assigned in the revised formalism to the earlier statement that the dual tensor $\Psi'_{\mathfrak{C}}$ generates the commutant lattice.

6.9 Primary–Dual Commutativity

Theorem 15 (Primary–dual commutant theorem). *For every*

$$A \in \mathfrak{M}_{\text{pri}} \quad (268)$$

and

$$B \in \mathfrak{M}_{\text{dual}}, \quad (269)$$

one has

$$[A, B] = 0. \quad (270)$$

Proof. By Axiom 8,

$$\mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}.$$

The result is therefore the defining property of the commutant. $\square$

Corollary 3 (Mutual commutant recovery). *The primary algebra is recovered from the dual algebra by*

$$\mathfrak{M}'_{\text{dual}} = \mathfrak{M}_{\text{pri}}. \quad (271)$$

Proof. Using the bicommutant theorem,

$$\mathfrak{M}'_{\text{dual}} = \left(\mathfrak{M}'_{\text{pri}} \right)' = \mathfrak{M}''_{\text{pri}}. \quad (272)$$

Because $\mathfrak{M}_{\text{pri}}$ is already a von Neumann algebra,

$$\mathfrak{M}''_{\text{pri}} = \mathfrak{M}_{\text{pri}}.$$

$\square$

6.10 Compatibility with Global Noncommutativity

The preceding theorem does not make the Chrysene sequence algebra commutative.

Proposition 16 (Dual commutativity does not remove sequence noncommutativity). *The relations*

$$[\mathfrak{M}_{\text{pri}}, \mathfrak{M}_{\text{dual}}] = 0 \quad (273)$$

and

$$[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0 \quad (274)$$

are compatible.

Hence the existence of a commuting dual-cipher algebra does not imply that the global Chrysene sequence algebra is commutative.

Proof. The first relation states that every element of the commutant $\mathfrak{M}'_{\text{pri}}$ commutes with every element of $\mathfrak{M}_{\text{pri}}$. The second relation is an internal noncommutativity relation among operators represented within the primary sequence algebra itself. No contradiction occurs. $\square$

Thus the revised formalism admits simultaneously

$$\begin{array}{c} \text{primary–dual algebraic insulation} \\ \text{and} \\ \text{sequence-dependent global noncommutativity.} \end{array} \quad (275)$$

6.11 The Four-State Cipher as Structural Data

The structural content of Axiom A_8 may be summarized by the commutative diagram

$$\begin{array}{ccc} \Psi_{\mathfrak{C}} & \xrightarrow{\mathcal{D}_{\text{dc}}} & \Psi'_{\mathfrak{C}} \\ \downarrow & & \downarrow \\ \epsilon = 0 & \xrightarrow{+1 \pmod{2}} & \epsilon = 1 \end{array} \quad (276)$$

$\eta \in \mathbb{Z}_2$ independently records the z -registry
 $\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z$

At the operator level this becomes

$$\mathfrak{M}_{\text{pri}} \xleftrightarrow{J_{\text{dc}}} \mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}. \quad (277)$$

The first relation is tensorial and geometric; the second is operator-algebraic. Axiom A_8 requires their coexistence but does not identify them as the same mathematical map.

6.12 What Axiom A_8 Does Not Yet Assert

Axiom 8 restores the primary/dual tensor architecture and a native four-state alphabet. It does not imply any of the following statements without additional hypotheses:

$$\Psi_{\mathfrak{C}} \perp \Psi'_{\mathfrak{C}}, \quad (278)$$

$$V_{\text{sub}}(\Psi_{\mathfrak{C}}) \cap V_{\text{sub}}(\Psi'_{\mathfrak{C}}) = \emptyset, \quad (279)$$

$$\Sigma_{\mathfrak{C}} \cong GF(4) \quad \text{as a field,} \quad (280)$$

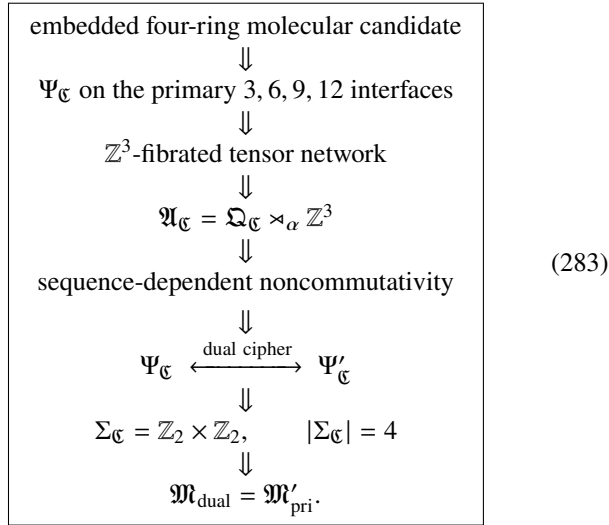
$$\mathcal{D}_x \perp \mathcal{D}_y \quad \text{in a quantum Hilbert space,} \quad (281)$$

$$\sigma_{\text{dc}} \in \Gamma_{\mathfrak{C}}. \quad (282)$$

The first two are geometric or molecular claims requiring explicitly defined inner products or exclusion-volume models. The third requires a structurally derived multiplication. The fourth would again conflate molecular and Hilbert-space orthogonality. The fifth would incorrectly identify the cipher transformation with a molecular point-group symmetry.

6.13 Axiomatic Hierarchy Through A_8

After Axioms A_1 – A_8 , the revised Chrysene Formalism has the structure



The next segment equips the resulting noncommutative tensor sequence algebra with metric spectral data through a discrete spectral triple

$$(\mathcal{A}_{\mathbb{C}}, \mathcal{H}_{\omega}, D_{\mathbb{C}}).$$

This provides the missing passage from a noncommutative operator algebra to a noncommutative geometry in the sense required by the spectral formulation.

7 Spectral Noncommutative Geometry of the Chrysene Tensor Space

Axioms A_1 – A_8 have constructed a tensorial, $\mathbb{Z}^3$ -fibrated sequence space together with the noncommutative C^* -algebra

$$\mathfrak{A}_{\mathbb{C}} = \mathfrak{Q}_{\mathbb{C}} \rtimes_{\alpha} \mathbb{Z}^3. \quad (284)$$

The noncommutativity of this algebra follows from the ordered action of translations on localized tensor observables rather than from Euclidean orthogonality of the molecular attachment directions.

A noncommutative algebra alone, however, does not yet constitute a noncommutative metric geometry. To recover geometric information spectrally, the algebra must be represented on a Hilbert space and equipped with a Dirac-type operator whose commutators with a dense subalgebra are bounded.

We therefore introduce a discrete spectral triple in the sense motivated by the noncommutative-geometric framework of Connes [Con94]. The construction also provides the rigorous setting in which spectral distance, directional spectral cost, and bounded metric fluctuations may subsequently be defined.

7.1 The Spectral Core of the Sequence Algebra

Recall the algebraic local tensor algebra

$$\mathfrak{Q}_{\text{loc}} = \bigcup_{\Lambda \in \mathbb{Z}^3} \mathfrak{Q}_{\Lambda}.$$

Definition 40 (Algebraic sequence core). *Define*

$$\mathfrak{A}_{\mathbb{C}}^{\text{alg}} := \left\{ \sum_{\mathbf{m} \in F} A_{\mathbf{m}} u_{\mathbf{m}} : \begin{array}{l} F \subseteq \mathbb{Z}^3, \\ A_{\mathbf{m}} \in \mathfrak{Q}_{\text{loc}} \end{array} \right\}. \quad (285)$$

This is a unital involutive algebra dense in $\mathfrak{A}_{\mathbb{C}}$.

Its multiplication is inherited from Axiom 7:

$$(A_{\mathbf{m}} u_{\mathbf{m}})(B_{\mathbf{n}} u_{\mathbf{n}}) = A_{\mathbf{m}} \alpha_{\mathbf{m}}(B_{\mathbf{n}}) u_{\mathbf{m}+\mathbf{n}}. \quad (286)$$

Definition 41 (Spectral core). *A spectral core is a norm-dense unital $*$ -subalgebra*

$$\mathfrak{A}_{\mathbb{C}}^{\circ} \subseteq \mathfrak{A}_{\mathbb{C}} \quad (287)$$

such that

$$\mathfrak{A}_{\mathbb{C}}^{\text{alg}} \subseteq \mathfrak{A}_{\mathbb{C}}^{\circ}. \quad (288)$$

Thus every finite-support tensor observable and every finite translation word belongs to the domain on which the Dirac commutator will be required to be bounded.

7.2 Faithful Spectral Representation

Axiom A_7 supplies a state

$$\omega : \mathfrak{A}_{\mathbb{C}} \longrightarrow \mathbb{C}$$

and the corresponding GNS representation

$$(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega}).$$

For the spectral realization we strengthen the state condition.

Definition 42 (Faithful spectral state). *A state ω is called a faithful spectral state when*

$$\omega(A^* A) = 0 \implies A = 0. \quad (289)$$

Lemma 17 (Faithfulness of the GNS representation). *If ω is faithful, then*

$$\pi_{\omega} : \mathfrak{A}_{\mathbb{C}} \longrightarrow B(\mathcal{H}_{\omega}) \quad (290)$$

is faithful.

Proof. If $\pi_\omega(A) = 0$, then

$$0 = \|\pi_\omega(A)\Omega_\omega\|^2 = \omega(A^*A).$$

Faithfulness of ω therefore gives $A = 0$. $\square$

7.3 The Discrete Spectral Triple

Axiom 9 (Discrete Spectral Triple and Noncommutative Metric Geometry (A9)). *The state ω of the sequence algebra is a faithful spectral state. The Chrysene Tensor Space is equipped with a densely defined self-adjoint operator*

$$D_{\mathfrak{C}} : \text{Dom}(D_{\mathfrak{C}}) \subset \mathcal{H}_\omega \longrightarrow \mathcal{H}_\omega \quad (291)$$

satisfying the following conditions.

1. For every

$$a \in \mathfrak{A}_{\mathfrak{C}}^\circ,$$

one has

$$\pi_\omega(a) \text{Dom}(D_{\mathfrak{C}}) \subseteq \text{Dom}(D_{\mathfrak{C}}). \quad (292)$$

2. For every $a \in \mathfrak{A}_{\mathfrak{C}}^\circ$, the operator initially defined on $\text{Dom}(D_{\mathfrak{C}})$ by

$$[D_{\mathfrak{C}}, \pi_\omega(a)] := D_{\mathfrak{C}}\pi_\omega(a) - \pi_\omega(a)D_{\mathfrak{C}} \quad (293)$$

extends to a bounded operator on $\mathcal{H}_\omega$.

3. The resolvent of $D_{\mathfrak{C}}$ is compact. Equivalently,

$$(I + D_{\mathfrak{C}}^2)^{-1/2} \in \mathcal{K}(\mathcal{H}_\omega), \quad (294)$$

where $\mathcal{K}(\mathcal{H}_\omega)$ denotes the compact operators.

Consequently,

$$(\mathfrak{A}_{\mathfrak{C}}^\circ, \mathcal{H}_\omega, D_{\mathfrak{C}}) \quad (295)$$

is a unital spectral triple.

Because $\mathfrak{A}_{\mathfrak{C}}$ is noncommutative by Theorem 8, Axiom 9 equips the Chrysene Tensor Space with a genuine noncommutative spectral geometry.

7.4 Compact Resolvent on the Infinite Lattice

The requirement of compact resolvent in Axiom 9 must be distinguished from translation invariance of the Dirac operator.

On the infinite lattice

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3,$$

a Dirac operator that is strictly translation invariant in the regular lattice representation generally has continuous Fourier-space

multiplicity and does not provide the compact-resolvent structure required by Axiom A9.

Accordingly, Axiom A9 does not require $D_{\mathfrak{C}}$ to commute with lattice translations. Instead, the appropriate compatibility condition is boundedness of the translation commutators.

Definition 43 (Position operators on $\mathbb{Z}^3$). *Let*

$$\mathcal{H}_{\text{lat}} := \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^4. \quad (296)$$

For

$$\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3,$$

let $\delta_{\mathbf{n}}$ denote the standard basis vector of $\ell^2(\mathbb{Z}^3)$.

Define the coordinate operators

$$X_j \delta_{\mathbf{n}} := n_j \delta_{\mathbf{n}}, \quad j = 1, 2, 3. \quad (297)$$

Let

$$\gamma_1, \gamma_2, \gamma_3 \in M_4(\mathbb{C})$$

be self-adjoint matrices satisfying the Euclidean Clifford relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I_4. \quad (298)$$

Definition 44 (Confining lattice Dirac operator). *Define*

$$D_{\text{lat}} := \sum_{j=1}^3 X_j \otimes \gamma_j \quad (299)$$

on the natural domain

$$\text{Dom}(D_{\text{lat}}) := \left\{ \psi : \sum_{\mathbf{n} \in \mathbb{Z}^3} |\mathbf{n}|^2 \|\psi(\mathbf{n})\|^2 < \infty \right\}. \quad (300)$$

Proposition 18 (Compact resolvent of the lattice Dirac operator). *The operator D_{lat} is self-adjoint and satisfies*

$$(I + D_{\text{lat}}^2)^{-1/2} \in \mathcal{K}(\mathcal{H}_{\text{lat}}). \quad (301)$$

Proof. For $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, set

$$\Gamma(\mathbf{n}) := \sum_{j=1}^3 n_j \gamma_j. \quad (302)$$

Each $\Gamma(\mathbf{n})$ is a finite-dimensional self-adjoint matrix. By the Clifford relations,

$$\Gamma(\mathbf{n})^2 = |\mathbf{n}|^2 I_4, \quad (303)$$

and hence

$$\|\Gamma(\mathbf{n})\| = |\mathbf{n}|. \quad (304)$$

Under the orthogonal decomposition

$$\mathcal{H}_{\text{lat}} = \bigoplus_{\mathbf{n} \in \mathbb{Z}^3} (\mathbb{C}\delta_{\mathbf{n}} \otimes \mathbb{C}^4), \quad (305)$$

the operator D_{lat} is the direct sum of the self-adjoint matrices $\Gamma(\mathbf{n})$.

The maximal domain of this direct-sum operator is therefore

$$\left\{ \psi : \sum_{\mathbf{n}} \|\Gamma(\mathbf{n})\psi(\mathbf{n})\|^2 < \infty \right\}, \quad (306)$$

which, because $\Gamma(\mathbf{n})^2 = |\mathbf{n}|^2 I_4$, is exactly the domain in Definition 44. Thus D_{lat} is self-adjoint.

The Clifford relations further give

$$D_{\text{lat}}^2 = (X_1^2 + X_2^2 + X_3^2) \otimes I_4. \quad (307)$$

Hence on the lattice basis,

$$(I + D_{\text{lat}}^2)^{-1/2} (\delta_{\mathbf{n}} \otimes \xi) = \frac{1}{\sqrt{1 + |\mathbf{n}|^2}} \delta_{\mathbf{n}} \otimes \xi. \quad (308)$$

For every bounded interval of $|\mathbf{n}|$ there are only finitely many lattice points, while

$$\frac{1}{\sqrt{1 + |\mathbf{n}|^2}} \rightarrow 0 \quad (|\mathbf{n}| \rightarrow \infty). \quad (309)$$

Therefore the resolvent factor is the norm limit of finite-rank operators and is compact. $\square$

Proposition 19 (Bounded translation commutators). *For $\mathbf{m} \in \mathbb{Z}^3$, let*

$$U_{\mathbf{m}} \delta_{\mathbf{n}} := \delta_{\mathbf{n}+\mathbf{m}}. \quad (310)$$

Then

$$[D_{\text{lat}}, U_{\mathbf{m}}] = \left(\sum_{j=1}^3 m_j \gamma_j \right) U_{\mathbf{m}}. \quad (311)$$

In particular,

$$\|[D_{\text{lat}}, U_{\mathbf{m}}]\| < \infty \quad (312)$$

for every $\mathbf{m} \in \mathbb{Z}^3$.

Proof. For each coordinate operator,

$$X_j U_{\mathbf{m}} \delta_{\mathbf{n}} = (n_j + m_j) \delta_{\mathbf{n}+\mathbf{m}}, \quad (313)$$

$$U_{\mathbf{m}} X_j \delta_{\mathbf{n}} = n_j \delta_{\mathbf{n}+\mathbf{m}}. \quad (314)$$

Hence

$$[X_j, U_{\mathbf{m}}] = m_j U_{\mathbf{m}}.$$

Summing over the Clifford components gives Equation (311). $\square$

Remark 21 (Confinement rather than translation invariance). *The operator D_{lat} is deliberately not translation invariant:*

$$U_{\mathbf{m}} D_{\text{lat}} U_{\mathbf{m}}^* \neq D_{\text{lat}} \quad (315)$$

for $\mathbf{m} \neq 0$.

Instead, translation changes the Dirac operator only by a bounded operator. This is precisely the behavior required for a compact-resolvent spectral geometry on the infinite $\mathbb{Z}^3$ lattice while retaining bounded translation commutators. Thus the coexistence

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3$$

and

$$(I + D_{\mathfrak{C}}^2)^{-1/2} \in \mathcal{K}(\mathcal{H}_{\omega})$$

does not require compactification of the molecular network.

Remark 22 (Extension to the full Chrysene Hilbert space). *The operator D_{lat} provides the canonical lattice component of a possible $D_{\mathfrak{C}}$.*

Because the full GNS representation also contains the quasi-local tensor degrees of freedom, a complete analytic realization must combine this lattice confinement with an internal operator having compact spectral growth and bounded commutators on the local tensor algebra.

One possible graded-product form is

$$D_{\mathfrak{C}} = D_{\text{int}} \otimes \sigma_1 + D_{\text{lat}} \otimes \sigma_2, \quad (316)$$

where D_{int} is a compact-resolvent operator for the quasi-local tensor sector and σ_1, σ_2 are anticommuting Pauli matrices.

7.5 The Dirac Lipschitz Seminorm

Definition 45 (Spectral Lipschitz seminorm). *For*

$$a \in \mathfrak{A}_{\mathfrak{C}}^{\circ},$$

define

$$L_D(a) := \|[D_{\mathfrak{C}}, \pi_{\omega}(a)]\|. \quad (317)$$

Proposition 20 (Leibniz inequality). *For $a, b \in \mathfrak{A}_{\mathfrak{C}}^{\circ}$,*

$$L_D(ab) \leq L_D(a)\|b\| + \|a\|L_D(b). \quad (318)$$

Proof. On $\text{Dom}(D_{\mathfrak{C}})$,

$$[D_{\mathfrak{C}}, \pi_{\omega}(ab)] = [D_{\mathfrak{C}}, \pi_{\omega}(a)]\pi_{\omega}(b) \quad (319)$$

$$+ \pi_{\omega}(a)[D_{\mathfrak{C}}, \pi_{\omega}(b)]. \quad (320)$$

Because a faithful $*$ -representation of a C^* -algebra is isometric,

$$\|\pi_{\omega}(a)\| = \|a\|.$$

Taking operator norms therefore gives the stated Leibniz inequality. $\square$

Proposition 21 (Adjoint symmetry). *For every $a \in \mathfrak{A}_{\mathfrak{C}}^{\circ}$,*

$$L_D(a^*) = L_D(a). \quad (321)$$

Proof. Because $D_{\mathfrak{C}}$ is self-adjoint,

$$[D_{\mathfrak{C}}, \pi_{\omega}(a^*)] = -[D_{\mathfrak{C}}, \pi_{\omega}(a)]^*.$$

The two operators therefore have equal norm. $\square$

7.6 Spectral Cost of Discrete Translation

Write

$$U_{\mathbf{m}} := \pi_{\omega}(u_{\mathbf{m}}), \quad \mathbf{m} \in \mathbb{Z}^3. \quad (322)$$

Because the translation generators belong to $\mathfrak{A}_{\mathfrak{C}}^{\circ}$, their Dirac commutators are bounded.

Definition 46 (Spectral translation seminorm). *Define*

$$\ell_D(\mathbf{m}) := \|[D_{\mathfrak{C}}, U_{\mathbf{m}}]\|, \quad \mathbf{m} \in \mathbb{Z}^3. \quad (323)$$

Let

$$c_x := \ell_D(e_x), \quad (324)$$

$$c_y := \ell_D(e_y), \quad (325)$$

$$c_z := \ell_D(e_z). \quad (326)$$

These numbers measure the Dirac-commutator cost of the three elementary sequence translations.

Proposition 22 (Translation seminorm properties). *For all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^3$,*

$$\ell_D(\mathbf{0}) = 0, \quad (327)$$

$$\ell_D(-\mathbf{m}) = \ell_D(\mathbf{m}), \quad (328)$$

$$\ell_D(\mathbf{m} + \mathbf{n}) \leq \ell_D(\mathbf{m}) + \ell_D(\mathbf{n}). \quad (329)$$

Proof. The first identity is immediate from $U_{\mathbf{0}} = I$. Since $U_{-\mathbf{m}} = U_{\mathbf{m}}^*$,

$$[D_{\mathfrak{C}}, U_{\mathbf{m}}^*] = -U_{\mathbf{m}}^*[D_{\mathfrak{C}}, U_{\mathbf{m}}]U_{\mathbf{m}}^*, \quad (330)$$

and unitary invariance of the operator norm gives (328). Finally,

$$[D_{\mathfrak{C}}, U_{\mathbf{m}+\mathbf{n}}] = [D_{\mathfrak{C}}, U_{\mathbf{m}}U_{\mathbf{n}}] \quad (331)$$

$$= [D_{\mathfrak{C}}, U_{\mathbf{m}}]U_{\mathbf{n}} + U_{\mathbf{m}}[D_{\mathfrak{C}}, U_{\mathbf{n}}]. \quad (332)$$

Taking norms gives (329). $\square$

Corollary 4 (Coordinate bound). *For*

$$\mathbf{m} = (m_x, m_y, m_z) \in \mathbb{Z}^3,$$

one has

$$\ell_D(\mathbf{m}) \leq |m_x|c_x + |m_y|c_y + |m_z|c_z. \quad (333)$$

Proof. Write

$$\mathbf{m} = m_x e_x + m_y e_y + m_z e_z$$

and repeatedly apply Proposition 22. $\square$

Remark 23 (Spectral anisotropy is not imposed). *The quantities*

$$c_x, \quad c_y, \quad c_z$$

need not be equal. However, neither

$$c_x \neq c_y$$

nor any prescribed inequality among them is imposed by the present axiom.

Thus molecular directional anisotropy and spectral anisotropy remain distinct statements.

7.7 Connes Spectral Distance

Let

$$S(\mathfrak{A}_{\mathfrak{C}}) \quad (334)$$

denote the state space of the C^* -algebra $\mathfrak{A}_{\mathfrak{C}}$.

Definition 47 (Spectral distance). *For states*

$$\varphi, \psi \in S(\mathfrak{A}_{\mathfrak{C}}),$$

define

$$d_D(\varphi, \psi) := \sup \left\{ |\varphi(a) - \psi(a)| : \begin{array}{l} a = a^* \in \mathfrak{A}_{\mathfrak{C}}^{\circ}, \\ L_D(a) \leq 1 \end{array} \right\}. \quad (335)$$

The value $+\infty$ is permitted.

Proposition 23 (Extended pseudometric property). *The function d_D is an extended pseudometric on $S(\mathfrak{A}_{\mathbb{C}})$. In particular,*

$$d_D(\varphi, \psi) \geq 0, \quad (336)$$

$$d_D(\varphi, \varphi) = 0, \quad (337)$$

$$d_D(\varphi, \psi) = d_D(\psi, \varphi), \quad (338)$$

$$d_D(\varphi, \chi) \leq d_D(\varphi, \psi) + d_D(\psi, \chi). \quad (339)$$

Proof. Non-negativity and symmetry follow from the absolute value. The diagonal vanishes identically. For every admissible a ,

$$|\varphi(a) - \chi(a)| \leq |\varphi(a) - \psi(a)| + |\psi(a) - \chi(a)|. \quad (340)$$

Taking the supremum over the same Lipschitz ball gives the triangle inequality, with the usual arithmetic in $[0, \infty]$. $\square$

Definition 48 (Metric-nondegenerate spectral realization). *The spectral triple is called metric-nondegenerate when*

$$L_D(a) = 0 \implies a \in \mathbb{C}I \quad (341)$$

for $a = a^* \in \mathfrak{A}_{\mathbb{C}}^{\circ}$.

Proposition 24 (Separation of states). *If the spectral realization is metric-nondegenerate, then*

$$d_D(\varphi, \psi) = 0 \implies \varphi = \psi. \quad (342)$$

Hence d_D is an extended metric on the state space.

Proof. Assume $\varphi \neq \psi$. Because $\mathfrak{A}_{\mathbb{C}}^{\circ}$ is norm dense in $\mathfrak{A}_{\mathbb{C}}$, there exists a self-adjoint $a \in \mathfrak{A}_{\mathbb{C}}^{\circ}$ such that

$$\varphi(a) \neq \psi(a).$$

The element a cannot be scalar. Metric nondegeneracy therefore gives

$$L_D(a) > 0.$$

Scaling a by $L_D(a)^{-1}$ produces an element of the Lipschitz ball for which the two states have distinct values. Hence

$$d_D(\varphi, \psi) > 0.$$

$\square$

7.8 Structural States and Spectral Distance

The abstract state space of the spectral triple is larger than the finite or discrete set of structural configurations used in the admissibility theory.

Definition 49 (Structural-state embedding). *Let $\mathcal{X}_{\text{str}}$ be a specified structural state set. A spectral realization of structural states is an injective map*

$$J_{\text{str}} : \mathcal{X}_{\text{str}} \longrightarrow S(\mathfrak{A}_{\mathbb{C}}), \quad (343)$$

assigning to every structural configuration a state of the sequence algebra.

Whenever such a realization has been specified, define

$$d_D^{\text{str}}(x, y) := d_D(J_{\text{str}}(x), J_{\text{str}}(y)). \quad (344)$$

Remark 24 (Spectral and path distances are distinct). *The spectral distance d_D^{str} and the admissible path distance d_C^* introduced later in Axiom A₁₂ are defined by different mathematical constructions.*

Accordingly,

$$d_D^{\text{str}}(x, y) = d_C^*(x, y) \quad (345)$$

is not an axiom.

Equality, bi-Lipschitz equivalence, or any other precise relation between the two metrics requires a separately proved compatibility theorem for a specified Dirac operator and structural-state embedding.

7.9 Algebraic One-Forms

The spectral triple also supplies a canonical differential calculus.

Definition 50 (Represented one-forms). *The space of represented algebraic one-forms is*

$$\Omega_D^1(\mathfrak{A}_{\mathbb{C}}) := \left\{ \sum_{j=1}^N \pi_{\omega}(a_j) [D_{\mathbb{C}}, \pi_{\omega}(b_j)] : \begin{array}{l} N < \infty, \\ a_j, b_j \in \mathfrak{A}_{\mathbb{C}}^{\circ} \end{array} \right\}. \quad (346)$$

Every element of Ω_D^1 is bounded because the represented algebra elements and the Dirac commutators appearing in its definition are bounded.

7.10 Inner Metric Fluctuations

Definition 51 (Bounded inner fluctuation). *Let*

$$A \in \Omega_D^1(\mathfrak{A}_{\mathbb{C}}^{\circ}) \quad (347)$$

satisfy

$$A^* = A. \quad (348)$$

Define the fluctuated Dirac operator by

$$D_{\mathfrak{C},A} := D_{\mathfrak{C}} + A \quad (349)$$

on $\text{Dom}(D_{\mathfrak{C}})$.

Proposition 25 (Stability under bounded inner fluctuations). *If $A = A^*$ is a bounded represented one-form, then $D_{\mathfrak{C},A}$ is self-adjoint on $\text{Dom}(D_{\mathfrak{C}})$.*

Moreover,

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C},A}) \quad (350)$$

again has bounded commutators and compact resolvent.

Proof. Since A is bounded and self-adjoint, the bounded perturbation theorem gives self-adjointness of $D_{\mathfrak{C}} + A$ on $\text{Dom}(D_{\mathfrak{C}})$.

For $a \in \mathfrak{A}_{\mathfrak{C}}^{\circ}$,

$$[D_{\mathfrak{C},A}, \pi_{\omega}(a)] = [D_{\mathfrak{C}}, \pi_{\omega}(a)] + [A, \pi_{\omega}(a)], \quad (351)$$

and both terms are bounded.

Compactness of the resolvent is preserved under bounded perturbation. Indeed, for any z belonging to the common resolvent set, the resolvent identity gives

$$\begin{aligned} (D_{\mathfrak{C},A} - zI)^{-1} - (D_{\mathfrak{C}} - zI)^{-1} \\ = -(D_{\mathfrak{C},A} - zI)^{-1} A (D_{\mathfrak{C}} - zI)^{-1}. \end{aligned} \quad (352)$$

The second resolvent is compact and the remaining factors are bounded, so the right-hand side is compact. Hence the fluctuated resolvent is compact. $\square$

Remark 25 (Interpretation of inner fluctuations). *The replacement*

$$D_{\mathfrak{C}} \mapsto D_{\mathfrak{C}} + A$$

is an algebraic metric fluctuation of the spectral geometry. No specific molecular force, electromagnetic field, or physical deformation is identified with A without an additional model specifying that correspondence.

7.11 Cipher Conjugation and Real Spectral Structure

Axiom A_8 introduced the conjugate-linear isometric involution

$$J_{\text{dc}}$$

satisfying

$$J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}} = \mathfrak{M}'_{\text{pri}}.$$

This resembles one component of the real-structure machinery of noncommutative geometry, but the two concepts are not

identified axiomatically.

Remark 26 (Cipher conjugation is not automatically a real structure). *The operator J_{dc} is introduced by Axiom A_8 to implement primary–dual commutant duality.*

For J_{dc} to serve additionally as the real structure of a real spectral triple, further conditions would be required, including appropriate sign relations with $D_{\mathfrak{C}}$ and an order-one condition of the form

$$[[D_{\mathfrak{C}}, \pi_{\omega}(a)], J_{\text{dc}} \pi_{\omega}(b) J_{\text{dc}}^{-1}] = 0 \quad (353)$$

for a, b in an appropriate spectral core.

These conditions are not consequences of commutant duality alone and are therefore left to the subsequent theorem program.

7.12 The Dirac Operator Is Not the Electronic Hamiltonian

Remark 27 (Geometric versus physical generators). *The Dirac operator*

$$D_{\mathfrak{C}}$$

encodes the metric spectral geometry of the noncommutative sequence space.

The electronic Hamiltonians introduced in Axiom A_{10} ,

$$H_{\alpha}^{(N)},$$

encode physical dynamics of specified node–bridge–node motifs.

No identity

$$D_{\mathfrak{C}} = H_{\alpha}^{(N)} \quad (354)$$

is assumed.

Likewise, boundedness of

$$[D_{\mathfrak{C}}, \pi_{\omega}(a)]$$

is a spectral-geometric condition and does not imply a dynamical commutation relation

$$[H_{\alpha}^{(N)}, \pi_{\omega}(a)] = 0.$$

A relationship between $D_{\mathfrak{C}}$ and a physical Hamiltonian requires an explicit representation theorem or model-dependent identification.

7.13 Spectral Dimension

The discrete base of the tensor network is $\mathbb{Z}^3$, but its algebraic spectral dimension is not fixed solely by this combinatorial fact.

Definition 52 (Spectral dimension). *When the relevant trace ideals are defined, set*

$$d_{\text{sp}} := \inf \left\{ p > 0 : \text{Tr} \left(I + D_{\mathfrak{C}}^2 \right)^{-p/2} < \infty \right\}. \quad (355)$$

If the set is empty, define

$$d_{\text{sp}} = +\infty.$$

Remark 28 (Three lattice directions do not prove spectral dimension three). *The relation*

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3$$

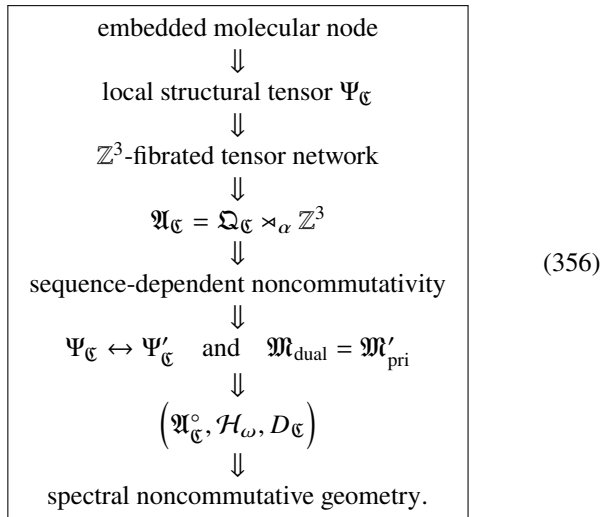
does not by itself imply

$$d_{\text{sp}} = 3.$$

Such a statement depends upon the eigenvalue growth of the particular Dirac operator and therefore belongs to the spectral theorem program.

7.14 What Axiom A_9 Establishes

Axioms A_1 – A_9 now produce the hierarchy



Thus the terminology *noncommutative Chrysene Tensor Space* has acquired three separate mathematical components:

1. a tensor-network structure over $\mathbb{Z}^3$;
2. a genuinely noncommutative sequence algebra generated by ordered local and translation operations;
3. a spectral triple furnishing metric information through the Dirac commutator.

The next segment introduces the physical Hamiltonian and effective directional transfer operators on top of this spectral-geometric foundation.

8 Physical Operators, Transfer, Response, and Admissibility

Axioms A_1 – A_9 define the molecular, tensorial, sequence-algebraic, dual-cipher, and spectral-geometric structure of the revised Chrysene Formalism. We now introduce the physical operator layer.

Three distinctions remain essential.

First, the spectral Dirac operator

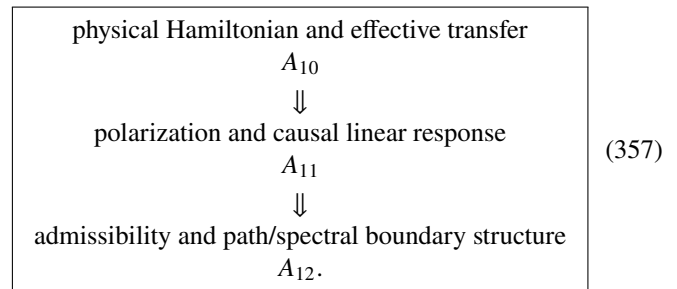
$$D_{\mathfrak{C}}$$

encodes noncommutative metric geometry, whereas a physical Hamiltonian encodes dynamical evolution of a specified electronic or node–bridge–node realization.

Second, the existence of distinct molecular or network directions does not imply inequivalent physical Hamiltonians or transfer operators.

Third, the metric geometry generated by $D_{\mathfrak{C}}$ is distinct from the constrained path geometry used to describe admissible structural transitions.

The final three axioms therefore have the logical organization



8.1 Directional Physical Realizations

Let

$$\alpha \in \{x, y\}, \quad N \in \mathcal{N}_{\alpha}.$$

The corresponding node–bridge–node motif of Axioms A_4 – A_6 contains a left terminal node, a bridge, and a right terminal node.

Chemical realization hypotheses. The physical operator layer may depend upon chemical information not contained in the graph-theoretic or spectral axioms. In particular, when electronic propagation through the aromatic carbon framework is invoked, the realization shall explicitly state whether it assumes the Conjugated π -Framework Hypothesis

$$H_{\partial\pi}(\mathfrak{R}).$$

Neither C_{2h} symmetry, bipartiteness, nor the existence of the tensor network implies this electronic hypothesis.

Definition 53 (Motif observable algebra). *For each pair (α, N) , let*

$$\mathfrak{A}_{\alpha, N}^{\text{mot}} \subseteq \mathfrak{A}_{\mathbb{C}}^{\circ} \quad (358)$$

be a unital involutive subalgebra containing the structural observables assigned to the corresponding finite node–bridge–node motif.

Definition 54 (Physical motif representation). *A physical realization of the motif consists of a complex Hilbert space*

$$\mathcal{H}_{\alpha}^{(N)} = \mathcal{H}_{L, \alpha} \oplus \mathcal{H}_{B, \alpha}^{(N)} \oplus \mathcal{H}_{R, \alpha} \quad (359)$$

and a unital $*$ -representation

$$\varpi_{\alpha, N} : \mathfrak{A}_{\alpha, N}^{\text{mot}} \longrightarrow B\left(\mathcal{H}_{\alpha}^{(N)}\right). \quad (360)$$

Remark 29 (Representation spaces remain distinct). *The physical motif space $\mathcal{H}_{\alpha}^{(N)}$ need not coincide with the GNS space $\mathcal{H}_{\omega}$. Likewise, the representation $\varpi_{\alpha, N}$ need not be the restriction of π_{ω} unless a unitary equivalence or explicit embedding has been supplied.*

Thus

$$\mathcal{H}_{\omega} \neq \mathcal{H}_{\alpha}^{(N)}$$

is permitted, and no identification is made implicitly.

The diagonal blocks are self-adjoint,

$$H_{L, \alpha} = H_{L, \alpha}^{\dagger}, \quad (362)$$

$$H_{B, \alpha}^{(N)} = \left(H_{B, \alpha}^{(N)}\right)^{\dagger}, \quad (363)$$

$$H_{R, \alpha} = H_{R, \alpha}^{\dagger}, \quad (364)$$

and the contact operators have domains

$$W_{L, \alpha}^{(N)} : \mathcal{H}_{L, \alpha} \longrightarrow \mathcal{H}_{B, \alpha}^{(N)}, \quad (365)$$

$$W_{R, \alpha}^{(N)} : \mathcal{H}_{R, \alpha} \longrightarrow \mathcal{H}_{B, \alpha}^{(N)}, \quad (366)$$

$$\mathcal{J}_{\alpha}^{(N)} : \mathcal{H}_{R, \alpha} \longrightarrow \mathcal{H}_{L, \alpha}. \quad (367)$$

The operator $\mathcal{J}_{\alpha}^{(N)}$ represents direct terminal-to-terminal coupling not mediated by the bridge. It may vanish only under an explicitly specified physical model.

For

$$z \in \rho\left(H_{B, \alpha}^{(N)}\right), \quad (368)$$

define the bridge resolvent

$$R_{B, \alpha}^{(N)}(z) := \left(zI - H_{B, \alpha}^{(N)}\right)^{-1}. \quad (369)$$

The effective directional transfer operator is

$$\mathcal{T}_{\alpha}^{(N)}(z) := \mathcal{J}_{\alpha}^{(N)} + \left(W_{L, \alpha}^{(N)}\right)^{\dagger} R_{B, \alpha}^{(N)}(z) W_{R, \alpha}^{(N)}. \quad (370)$$

Thus

$$\mathcal{T}_{\alpha}^{(N)}(z) : \mathcal{H}_{R, \alpha} \longrightarrow \mathcal{H}_{L, \alpha}. \quad (371)$$

No equality or inequality between the x - and y -directional Hamiltonians or transfer operators is imposed by Axiom A₁₀.

8.2 Directional Hamiltonian and Effective Transfer

Axiom 10 (Directional Hamiltonian and Effective Transfer (A₁₀)). *For every*

$$\alpha \in \{x, y\}, \quad N \in \mathcal{N}_{\alpha},$$

the physical motif realization carries a bounded self-adjoint Hamiltonian

$$H_{\alpha}^{(N)} = \begin{pmatrix} H_{L, \alpha} & \left(W_{L, \alpha}^{(N)}\right)^{\dagger} & \mathcal{J}_{\alpha}^{(N)} \\ W_{L, \alpha}^{(N)} & H_{B, \alpha}^{(N)} & W_{R, \alpha}^{(N)} \\ \left(\mathcal{J}_{\alpha}^{(N)}\right)^{\dagger} & \left(W_{R, \alpha}^{(N)}\right)^{\dagger} & H_{R, \alpha} \end{pmatrix}. \quad (361)$$

8.3 Schur-Complement Interpretation

Let

$$\mathcal{H}_{T, \alpha} := \mathcal{H}_{L, \alpha} \oplus \mathcal{H}_{R, \alpha} \quad (372)$$

be the terminal Hilbert space.

Reordering the decomposition as

$$\mathcal{H}_{T, \alpha} \oplus \mathcal{H}_{B, \alpha}^{(N)},$$

define

$$H_{T, \alpha}^{(N)} := \begin{pmatrix} H_{L, \alpha} & \mathcal{J}_{\alpha}^{(N)} \\ \left(\mathcal{J}_{\alpha}^{(N)}\right)^{\dagger} & H_{R, \alpha} \end{pmatrix} \quad (373)$$

and

$$W_{\alpha}^{(N)} := \begin{pmatrix} W_{L, \alpha}^{(N)} & W_{R, \alpha}^{(N)} \end{pmatrix}. \quad (374)$$

Definition 55 (Energy-dependent terminal Hamiltonian). For

$$z \in \rho \left(H_{B,\alpha}^{(N)} \right),$$

define

$$H_{\text{eff},\alpha}^{(N)}(z) := H_{T,\alpha}^{(N)} + W_{\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{\alpha}^{(N)}. \quad (375)$$

In block form,

$$\begin{aligned} \Sigma_{L,\alpha}^{(N)}(z) &:= W_{L,\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{L,\alpha}^{(N)}, \\ \Sigma_{R,\alpha}^{(N)}(z) &:= W_{R,\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{R,\alpha}^{(N)}. \end{aligned} \quad (376)$$

$$H_{\text{eff},\alpha}^{(N)}(z) = \begin{pmatrix} H_{L,\alpha} + \Sigma_{L,\alpha}^{(N)}(z) & \mathcal{T}_{\alpha}^{(N)}(z) \\ \mathcal{T}_{\alpha}^{(N)}(\bar{z})^{\dagger} & H_{R,\alpha} + \Sigma_{R,\alpha}^{(N)}(z) \end{pmatrix} \quad (377)$$

Indeed, the upper-right block is

$$\mathcal{J}_{\alpha}^{(N)} + W_{L,\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{R,\alpha}^{(N)} = \mathcal{T}_{\alpha}^{(N)}(z), \quad (378)$$

whereas the lower-left block is

$$\mathcal{J}_{\alpha}^{(N)\dagger} + W_{R,\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{L,\alpha}^{(N)} = \mathcal{T}_{\alpha}^{(N)}(\bar{z})^{\dagger}. \quad (379)$$

The last equality follows from self-adjointness of $H_{B,\alpha}^{(N)}$, which gives the resolvent identity

$$R_{B,\alpha}^{(N)}(\bar{z})^{\dagger} = R_{B,\alpha}^{(N)}(z). \quad (380)$$

For real

$$z \in \rho \left(H_{B,\alpha}^{(N)} \right),$$

one has

$$\bar{z} = z$$

and therefore

$$\mathcal{T}_{\alpha}^{(N)}(\bar{z})^{\dagger} = \mathcal{T}_{\alpha}^{(N)}(z)^{\dagger}. \quad (381)$$

For non-real z , the energy-dependent effective operator

$$H_{\text{eff},\alpha}^{(N)}(z)$$

need not be self-adjoint.

Remark 30 (Transfer as an off-diagonal Schur block). The operator

$$\mathcal{T}_{\alpha}^{(N)}(z)$$

is the upper-right effective terminal coupling obtained after elimination of the bridge degrees of freedom.

The corresponding lower-left block is

$$\mathcal{T}_{\alpha}^{(N)}(\bar{z})^{\dagger}.$$

Only for real z in the bridge resolvent set does this reduce to the ordinary adjoint

$$\mathcal{T}_{\alpha}^{(N)}(z)^{\dagger}.$$

The transfer operator is not a geometric attachment vector and is not determined solely by the bridge count N .

8.4 Bridge-Mediated and Translation-Covariant Models

Definition 56 (Strictly bridge-mediated model). A directional physical realization is called strictly bridge mediated when

$$\mathcal{J}_{\alpha}^{(N)} = 0. \quad (382)$$

Definition 57 (Translation-covariant terminal dynamics). Suppose the terminal molecular nodes are related by the geometric translation of Axiom A₆. The physical terminal realization is called translation covariant if there exists a unitary

$$V_{\alpha} : \mathcal{H}_{L,\alpha} \longrightarrow \mathcal{H}_{R,\alpha} \quad (383)$$

such that

$$H_{R,\alpha} = V_{\alpha} H_{L,\alpha} V_{\alpha}^{\dagger}. \quad (384)$$

Remark 31 (Geometric translation is not enough). Translation covariance of the molecular graph does not by itself prove Equation (384). The electronic environment, boundary conditions, and physical model must also transform compatibly.

8.5 Analyticity and Resolvent Control

Proposition 26 (Analyticity of directional transfer). For fixed (α, N) , the map

$$z \longmapsto \mathcal{T}_{\alpha}^{(N)}(z) \quad (385)$$

is operator-valued analytic on

$$\rho \left(H_{B,\alpha}^{(N)} \right).$$

Moreover,

$$\left\| \mathcal{T}_\alpha^{(N)}(z) - \mathcal{J}_\alpha^{(N)} \right\| \leq \frac{\|W_{L,\alpha}^{(N)}\| \|W_{R,\alpha}^{(N)}\|}{\text{dist}\left(z, \sigma(H_{B,\alpha}^{(N)})\right)}. \quad (386)$$

Proof. The resolvent of a self-adjoint operator is analytic on its resolvent set and satisfies

$$\|R_{B,\alpha}^{(N)}(z)\| = \frac{1}{\text{dist}\left(z, \sigma(H_{B,\alpha}^{(N)})\right)}.$$

The result follows from Equation (370) and submultiplicativity of the operator norm. $\square$

8.6 Directional Transfer Comparison

Because the two transfer operators can have different domains and codomains, direct equality need not even be a well-formed statement.

Definition 58 (Unitary comparison of transfer operators).

Let

$$N_x \in \mathcal{N}_x, \quad N_y \in \mathcal{N}_y,$$

and let z belong to both bridge resolvent sets.

We write

$$\mathcal{T}_x^{(N_x)}(z) \simeq_u \mathcal{T}_y^{(N_y)}(z) \quad (387)$$

if there exist unitaries

$$U_L : \mathcal{H}_{L,x} \longrightarrow \mathcal{H}_{L,y}, \quad (388)$$

$$U_R : \mathcal{H}_{R,x} \longrightarrow \mathcal{H}_{R,y} \quad (389)$$

such that

$$U_L \mathcal{T}_x^{(N_x)}(z) = \mathcal{T}_y^{(N_y)}(z) U_R. \quad (390)$$

Remark 32 (Transfer anisotropy remains derived). Neither

$$\mathcal{D}_x \not\sim_{\Gamma_\mathfrak{C}} \mathcal{D}_y$$

nor

$$N_x^\star \neq N_y^\star$$

implies

$$\mathcal{T}_x \not\sim_u \mathcal{T}_y.$$

A transfer-anisotropy theorem must compare the actual bridge Hamiltonians, contacts, and direct couplings.

8.7 Transfer Moments

If

$$|z| > \|H_{B,\alpha}^{(N)}\|,$$

then

$$R_{B,\alpha}^{(N)}(z) = \sum_{k=0}^{\infty} \frac{\left(H_{B,\alpha}^{(N)}\right)^k}{z^{k+1}}. \quad (391)$$

Definition 59 (Directional transfer moments). Define

$$M_{k,\alpha}^{(N)} := \left(W_{L,\alpha}^{(N)}\right)^\dagger \left(H_{B,\alpha}^{(N)}\right)^k W_{R,\alpha}^{(N)}. \quad (392)$$

Then

$$\mathcal{T}_\alpha^{(N)}(z) = \mathcal{J}_\alpha^{(N)} + \sum_{k=0}^{\infty} \frac{M_{k,\alpha}^{(N)}}{z^{k+1}}. \quad (393)$$

The comparison of the two moment sequences is deferred to the directional-transfer theorem program.

8.8 Dirac Geometry and Physical Hamiltonians

Proposition 27 (Logical independence of $D_\mathfrak{C}$ and $H_\alpha^{(N)}$). Axiom A_9 does not determine the physical Hamiltonian $H_\alpha^{(N)}$, and Axiom A_{10} does not determine the Dirac operator $D_\mathfrak{C}$.

Proof. The operator $D_\mathfrak{C}$ is defined on $\mathcal{H}_\omega$ through bounded commutators with the spectral core and compact resolvent. The physical Hamiltonian $H_\alpha^{(N)}$ is defined on the independently specified motif space $\mathcal{H}_\alpha^{(N)}$ through its block decomposition and physical contact operators. Neither axiom supplies an identification of the two Hilbert spaces or operators. $\square$

Remark 33 (Possible future compatibility). A later model may provide an intertwiner

$$\mathcal{U}_{\alpha,N} : \mathcal{H}_\omega^{\text{loc}} \longrightarrow \mathcal{H}_\alpha^{(N)}$$

or a functional relation between $D_\mathfrak{C}$ and $H_\alpha^{(N)}$. Such a relation is additional structure rather than part of the foundational axioms.

8.9 Polarization and Causal Linear Response

We now introduce the response of a physical realization to a weak external electric field.

Definition 60 (Response representation). *A response realization consists of a Hilbert space $\mathcal{H}_{\text{resp}}$, a unital $*$ -representation*

$$\varpi_{\text{resp}} : \mathfrak{A}_{\text{resp}} \longrightarrow B(\mathcal{H}_{\text{resp}}) \quad (394)$$

of a specified physical observable subalgebra, and a self-adjoint Hamiltonian

$$H_{\text{resp}} = H_{\text{resp}}^\dagger. \quad (395)$$

Let

$$|0\rangle \in \mathcal{H}_{\text{resp}}, \quad \langle 0|0\rangle = 1, \quad (396)$$

satisfy

$$H_{\text{resp}}|0\rangle = E_0|0\rangle. \quad (397)$$

Axiom 11 (Polarization and Linear Response (A11)). *The response realization carries bounded self-adjoint dipole operators*

$$\mu_i = \mu_i^\dagger, \quad i \in \{x, y, z\}. \quad (398)$$

A weak applied electric field

$$\mathbf{E}(t) = (E_x(t), E_y(t), E_z(t))$$

couples according to the perturbation convention

$$V_E(t) := - \sum_j E_j(t) \mu_j. \quad (399)$$

The Heisenberg dipole operator is

$$\mu_i(t) := e^{iH_{\text{resp}}t/\hbar} \mu_i e^{-iH_{\text{resp}}t/\hbar}. \quad (400)$$

We use the Fourier–Laplace convention

$$e^{izt}, \quad \text{Im } z > 0,$$

so that the causal integral converges in the upper half-plane; this is the convention used throughout the following response formulas. For

$$z \in \mathbb{C}, \quad \text{Im } z > 0,$$

define the causal polarizability tensor by

$$\alpha_{ij}(z) := \frac{i}{\hbar} \int_0^\infty e^{izt} \langle 0 | [\mu_i(t), \mu_j] | 0 \rangle dt. \quad (401)$$

When the boundary value exists in the stated topology,

$$\alpha_{ij}(\omega) := \lim_{\eta \downarrow 0} \alpha_{ij}(\omega + i\eta). \quad (402)$$

No inequality among tensor components is imposed axiomatically.

8.10 Finite-Dimensional Spectral Representation

Proposition 28 (Spectral representation of polarizability). *Assume $\mathcal{H}_{\text{resp}}$ is finite dimensional and let*

$$\{|n\rangle\}_{n \in I}$$

be an orthonormal eigenbasis satisfying

$$H_{\text{resp}}|n\rangle = E_n|n\rangle.$$

Then, for $\text{Im } z > 0$,

$$\alpha_{ij}(z) = \sum_{n \neq 0} \left[\frac{\langle 0 | \mu_i | n \rangle \langle n | \mu_j | 0 \rangle}{E_n - E_0 - \hbar z} + \frac{\langle 0 | \mu_j | n \rangle \langle n | \mu_i | 0 \rangle}{E_n - E_0 + \hbar z} \right]. \quad (403)$$

Proof. Insert the finite resolution of the identity

$$I = \sum_n |n\rangle \langle n|$$

into each term of the commutator in Equation (401). Because the sum is finite, termwise integration is justified. The resulting elementary Laplace integrals give Equation (403). $\square$

8.11 Static Response and Inversion Symmetry

Definition 61 (Static polarizability). *Whenever the limit exists, define*

$$\alpha_{ij}^{(0)} := \lim_{\omega \rightarrow 0} \alpha_{ij}(\omega). \quad (404)$$

Proposition 29 (Inversion symmetry eliminates the permanent dipole). *Suppose there exists a unitary inversion operator $\mathcal{P}$ such that*

$$\mathcal{P} H_{\text{resp}} \mathcal{P}^\dagger = H_{\text{resp}}, \quad (405)$$

$$\mathcal{P} \mu_i \mathcal{P}^\dagger = -\mu_i, \quad (406)$$

and suppose the reference eigenstate is nondegenerate. Then

$$\langle 0 | \mu_i | 0 \rangle = 0 \quad (i = x, y, z). \quad (407)$$

Proof. Nondegeneracy implies

$$\mathcal{P}|0\rangle = e^{i\theta}|0\rangle$$

for some θ . Therefore

$$\langle 0|\mu_i|0\rangle = \langle 0|\mathcal{P}^\dagger \mathcal{P} \mu_i \mathcal{P}^\dagger \mathcal{P}|0\rangle \quad (408)$$

$$= -\langle 0|\mu_i|0\rangle. \quad (409)$$

Hence the expectation value vanishes. $\square$

Remark 34 (Centrosymmetry does not imply isotropic polarizability). Equation (407) does not imply

$$\alpha_{xx}(\omega) = \alpha_{yy}(\omega).$$

Directional response depends on transition energies and dipole matrix elements and therefore remains a theorem-level property.

8.12 Finite Response versus Bulk Dielectric Response

Let

$$\{\mathfrak{N}_M\}_{M \geq 1}$$

be an increasing family of finite network realizations with volumes $V_M > 0$ and finite-system polarizabilities $\alpha_{ij}^{(M)}(\omega)$.

Definition 62 (Bulk susceptibility). If the limit exists and is independent of the admissible exhaustion and boundary prescription, define the dimensionless susceptibility by

$$\chi_{ij}(\omega) := \lim_{M \rightarrow \infty} \frac{\alpha_{ij}^{(M)}(\omega)}{\varepsilon_0 V_M}. \quad (410)$$

The relative dielectric tensor is then

$$\varepsilon_{r,ij}(\omega) := \delta_{ij} + \chi_{ij}(\omega). \quad (411)$$

Remark 35 (No finite-fragment-to-bulk inference). A finite molecular polarizability does not become a bulk dielectric tensor by relabeling. Existence and boundary independence of Equation (410) must be established separately.

8.13 Admissible Structural States

The final axiom compares two kinds of geometry without identifying them:

1. the spectral geometry generated by $D_{\mathfrak{C}}$; and
2. the constrained transition geometry generated by admissible paths in a structural state graph.

Let

$$\mathcal{X} \quad (412)$$

be a nonempty structural state set.

Axiom 12 (Admissibility and Spectral/Path Boundary (A12)). The structural state set $\mathcal{X}$ is equipped with the following data.

1. An injective spectral realization

$$J_{\text{str}} : \mathcal{X} \longrightarrow S(\mathfrak{A}_{\mathfrak{C}}). \quad (413)$$

2. A symmetric transition relation

$$\mathcal{E}_0 \subseteq \mathcal{X} \times \mathcal{X}, \quad (414)$$

with

$$(x, y) \in \mathcal{E}_0 \iff (y, x) \in \mathcal{E}_0.$$

3. A symmetric positive edge-weight function

$$w : \mathcal{E}_0 \longrightarrow (0, \infty) \quad (415)$$

satisfying

$$w(x, y) = w(y, x). \quad (416)$$

4. A finite family of constraint functionals

$$C_j : \mathcal{X} \longrightarrow (-\infty, \infty], \quad j = 1, \dots, m, \quad (417)$$

with finite thresholds

$$\tau_j \in \mathbb{R}. \quad (418)$$

The admissible structural domain is

$$\Omega_{\text{adm}} := \{x \in \mathcal{X} : C_j(x) < \tau_j \text{ for every } j\}. \quad (419)$$

The induced admissible transition relation is

$$\mathcal{E}_{\text{adm}} := \{(x, y) \in \mathcal{E}_0 : x, y \in \Omega_{\text{adm}}\}. \quad (420)$$

For $x, y \in \Omega_{\text{adm}}$, let $\mathcal{P}_{\text{adm}}(x, y)$ denote the set of finite paths

$$\gamma = (x_0, \dots, x_r), \quad x_0 = x, \quad x_r = y, \quad (421)$$

with

$$(x_{k-1}, x_k) \in \mathcal{E}_{\text{adm}}.$$

Their weighted length is

$$L_w(\gamma) := \sum_{k=1}^r w(x_{k-1}, x_k). \quad (422)$$

The extended admissible path distance is

$$d_{\mathcal{C}}^*(x, y) := \begin{cases} \inf_{\gamma \in \mathcal{P}_{\text{adm}}(x, y)} L_w(\gamma), & \mathcal{P}_{\text{adm}}(x, y) \neq \emptyset, \\ +\infty, & \mathcal{P}_{\text{adm}}(x, y) = \emptyset. \end{cases} \quad (423)$$

The corresponding structural spectral distance is

$$d_D^{\text{str}}(x, y) := d_D(J_{\text{str}}(x), J_{\text{str}}(y)). \quad (424)$$

No equality, inequality, or finiteness equivalence between $d_{\mathcal{C}}^*$ and d_D^{str} is imposed axiomatically.

8.14 Constraint Boundary

Definition 63 (Constraint boundary). *Define*

$$\partial_C \Omega_{\text{adm}} := \left\{ x \in \mathcal{X} : \begin{array}{l} C_j(x) \leq \tau_j \text{ for all } j, \\ C_k(x) = \tau_k \text{ for at least one } k \end{array} \right\}. \quad (425)$$

Remark 36 (Not automatically a topological boundary). *The set $\partial_C \Omega_{\text{adm}}$ is defined by constraints. It need not coincide with a topological boundary unless an appropriate topology on $\mathcal{X}$ has independently been supplied.*

8.15 Path-Metric Properties

Proposition 30 (Extended pseudometric and metric criterion). *The function d_C^* is an extended pseudometric on Ω_{adm} . If, in addition,*

$$w_* := \inf_{(x,y) \in \mathcal{E}_{\text{adm}}} w(x,y) > 0, \quad (426)$$

then d_C^* is an extended metric.

Proof. Non-negativity and symmetry follow from positivity and symmetry of the edge weights. The trivial path gives

$$d_C^*(x, x) = 0.$$

The triangle inequality follows by concatenating admissible paths and taking infima.

Under Equation (426), every path joining distinct states contains at least one edge and therefore has length at least w_* . Hence

$$x \neq y \implies d_C^*(x, y) \geq w_* > 0.$$

□

Corollary 5 (Infinite distance and admissible connectivity). *For $x, y \in \Omega_{\text{adm}}$,*

$$d_C^*(x, y) = +\infty \quad (427)$$

if and only if x and y belong to different connected components of

$$\mathcal{G}_{\text{adm}} := (\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}}).$$

Remark 37 (Threshold violation does not imply global disconnection). *The removal of a state violating one or more*

constraints does not automatically imply that all surviving states separated by that vertex have infinite distance. If an admissible bypass exists, the remaining path distance is finite.

8.16 Spectral-Path Compatibility

Definition 64 (Exact spectral-path compatibility). *The realization is called exactly spectral-path compatible on a subset $Y \subseteq \Omega_{\text{adm}}$ when*

$$d_D^{\text{str}}(x, y) = d_C^*(x, y) \quad (x, y \in Y). \quad (428)$$

Definition 65 (Bi-Lipschitz spectral-path compatibility). *The two distances are called bi-Lipschitz compatible on Y if there exist constants*

$$0 < c_- \leq c_+ < \infty \quad (429)$$

such that, whenever $d_C^*(x, y) < \infty$,

$$c_- d_C^*(x, y) \leq d_D^{\text{str}}(x, y) \leq c_+ d_C^*(x, y). \quad (430)$$

Definition 66 (Disconnectivity compatibility). *The two geometries are called disconnectivity compatible when*

$$d_C^*(x, y) = +\infty \iff d_D^{\text{str}}(x, y) = +\infty \quad (431)$$

for all $x, y \in \Omega_{\text{adm}}$.

Remark 38 (Compatibility remains theorem-level). *Axiom A_{12} merely places both metrics on the same structural state domain. It does not assert exact compatibility, bi-Lipschitz compatibility, or disconnectivity compatibility. These become theorem targets for specifically constructed Dirac operators and admissibility graphs.*

8.17 Operator Realization of Admissibility

The constraint geometry does not itself dynamically remove states.

Definition 67 (Admissible and rejected Hilbert sectors). *Suppose a physical Hilbert space $\mathcal{H}_{\text{phys}}$ admits an orthogonal decomposition*

$$\mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{adm}} \oplus \mathcal{H}_{\text{rej}}. \quad (432)$$

Let

$$Q_{\text{adm}} : \mathcal{H}_{\text{phys}} \longrightarrow \mathcal{H}_{\text{adm}} \quad (433)$$

be the orthogonal projection and set

$$Q_{\text{rej}} := I - Q_{\text{adm}}. \quad (434)$$

Then

$$Q_{\text{adm}}^2 = Q_{\text{adm}} = Q_{\text{adm}}^\dagger, \quad (435)$$

$$Q_{\text{rej}}^2 = Q_{\text{rej}} = Q_{\text{rej}}^\dagger, \quad (436)$$

$$Q_{\text{adm}} Q_{\text{rej}} = 0. \quad (437)$$

Proposition 31 (Dynamical preservation criterion). *Let*

$$H_{\text{phys}} = H_{\text{phys}}^\dagger$$

be a self-adjoint physical Hamiltonian, possibly unbounded. Assume

$$Q_{\text{adm}} \text{Dom}(H_{\text{phys}}) \subseteq \text{Dom}(H_{\text{phys}}) \quad (438)$$

and

$$H_{\text{phys}} Q_{\text{adm}} \psi = Q_{\text{adm}} H_{\text{phys}} \psi \quad (\psi \in \text{Dom}(H_{\text{phys}})). \quad (439)$$

Then $\mathcal{H}_{\text{adm}}$ and $\mathcal{H}_{\text{rej}}$ are reducing subspaces of H_{phys} , and

$$e^{-itH_{\text{phys}}/\hbar} \mathcal{H}_{\text{adm}} \subseteq \mathcal{H}_{\text{adm}} \quad (440)$$

for every $t \in \mathbb{R}$.

If H_{phys} is bounded, conditions (438)–(439) reduce simply to

$$[H_{\text{phys}}, Q_{\text{adm}}] = 0. \quad (441)$$

Proof. Conditions (438)–(439) state that the orthogonal projection Q_{adm} reduces the self-adjoint operator H_{phys} . Hence Q_{adm} commutes with the spectral projections of H_{phys} and therefore with every bounded Borel function of H_{phys} . In particular,

$$Q_{\text{adm}} e^{-itH_{\text{phys}}/\hbar} = e^{-itH_{\text{phys}}/\hbar} Q_{\text{adm}},$$

which proves preservation of the admissible and rejected sectors. $\square$

Remark 39 (Projection is not physical scission). *The existence of Q_{rej} does not imply that the physical system spontaneously applies this projection. If*

$$[H_{\text{phys}}, Q_{\text{adm}}] \neq 0,$$

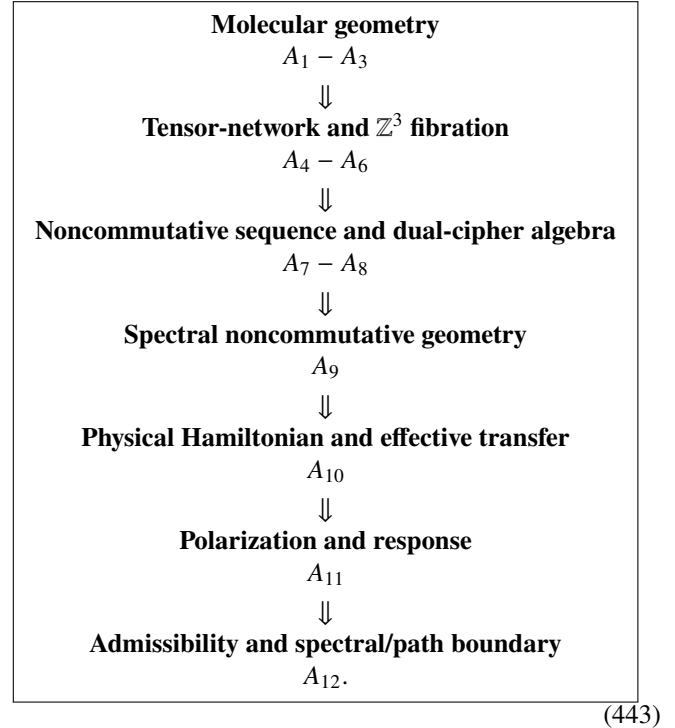
amplitude can couple between the two sectors. Physical

excision or scission therefore requires an additional unitary, dissipative, stochastic, measurement, or control mechanism. Accordingly,

$$C_j(x) \geq \tau_j \not\Rightarrow \text{automatic physical removal}. \quad (442)$$

8.18 Completion of the Twelve-Axiom Architecture

Axioms A_1 – A_{12} now define the sequence



The architecture deliberately preserves the logical separations

molecular anisotropy $\Rightarrow$ transfer anisotropy,
 transfer anisotropy $\Rightarrow$ optical anisotropy,
 spectral distance $\neq$ admissible path distance,
 constraint violation $\Rightarrow$ physical scission.

(444)

These implications may hold for a specified Chrysene realization only after the additional hypotheses required for their proofs have been supplied.

9 Model Existence and Molecular Characterization

The preceding sections define a twelve-axiom system whose structures extend from an embedded four-ring molecular node to a $\mathbb{Z}^3$ -fibrated tensor network, a noncommutative sequence

algebra, a dual-cipher commutant representation, a spectral triple, physical transfer operators, linear response, and constrained admissibility geometry.

We now address a distinct question:

Does the molecular reduct of a full twelve-axiom realization remain arbitrary within the four-ring polyhex class, or do the combined geometric requirements select a unique molecular architecture?

The answer must be formulated carefully. Uniqueness of the *molecular reduct* does not imply uniqueness of every Hamiltonian, state, bridge family, spectral triple, response tensor, or admissibility model built over that reduct.

9.1 Full Models and Molecular Reducts

Let

$$\mathcal{A} := \{A_1, \dots, A_{12}\} \quad (445)$$

denote the revised axiom set.

Definition 68 (Full model). A full model of the revised Chrysene Formalism is a collection

$$\mathfrak{R} \quad (446)$$

of all molecular, network, tensor, algebraic, spectral, physical, response, and admissibility data required by Axioms A_1 – A_{12} , satisfying every condition appearing in those axioms. The class of full models is denoted

$$\text{Mod}(\mathcal{A}). \quad (447)$$

Definition 69 (Molecular reduct). For

$$\mathfrak{R} \in \text{Mod}(\mathcal{A}),$$

define

$$\text{Red}_{\text{mol}} : \text{Mod}(\mathcal{A}) \longrightarrow \mathfrak{R}_4 \quad (448)$$

by

$$\text{Red}_{\text{mol}}(\mathfrak{R}) := \mathfrak{C}_{\mathfrak{R}}, \quad (449)$$

where $\mathfrak{C}_{\mathfrak{R}}$ is the embedded labeled molecular node appearing in Axiom A_1 .

Definition 70 (Canonical carbon-framework reduct). For $\mathfrak{C} \in \mathfrak{R}_4$, define its canonical carbon-framework realization by

$$\text{CanGraph}(\mathfrak{C}) := (G_{\mathfrak{C}}, \widehat{t}_{\text{px}}). \quad (450)$$

For a full model $\mathfrak{R}$, define

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) := \text{CanGraph}(\text{Red}_{\text{mol}}(\mathfrak{R})). \quad (451)$$

Definition 71 (Canonical polyhex congruence). Let

$$(G_1, \widehat{t}_1), \quad (G_2, \widehat{t}_2)$$

be canonical carbon-framework realizations.

We write

$$(G_1, \widehat{t}_1) \simeq_{\text{px}} (G_2, \widehat{t}_2) \quad (452)$$

when there exist a graph isomorphism

$$\varphi : G_1 \longrightarrow G_2$$

and a Euclidean isometry

$$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

such that

$$g \circ \widehat{t}_1 = \widehat{t}_2 \circ \varphi. \quad (453)$$

Define the reference canonical chrysene framework by

$$G_{\text{chr}}^{\text{px}} := (G_{\text{chr}}, \widehat{t}_{\text{px}, \text{chr}}). \quad (454)$$

Definition 72 (Canonical labeled polyhex reduct). Let

$$\mathfrak{R} \in \text{Mod}(\mathcal{A})$$

have embedded molecular reduct

$$\text{Red}_{\text{mol}}(\mathfrak{R}) = \mathfrak{C}_{\mathfrak{R}}.$$

Define its canonical labeled polyhex reduct by

$$\text{CanRed}_{\text{mol}}(\mathfrak{R}) := (G_{\mathfrak{C}_{\mathfrak{R}}}, \widehat{t}_{\text{px}}, \lambda_{\text{loc}}, \lambda_{\text{chem}}; A_{\mathfrak{C}_{\mathfrak{R}}}^{\text{pri}}, A_{\mathfrak{C}_{\mathfrak{R}}}^{\text{dual}}). \quad (455)$$

The attachment labels λ_{att} and the physical embedding ι are not included in $\text{CanRed}_{\text{mol}}$, because they may depend upon the particular molecular or supramolecular realization.

Let

$$\mathfrak{C}_{\text{chr}}^{\text{px}} \quad (456)$$

denote the corresponding canonical labeled polyhex reduct of the reference chrysene structure.

Definition 73 (Label-compatible polyhex congruence). Let $\mathfrak{C}_1^{\text{px}}$ and $\mathfrak{C}_2^{\text{px}}$ be canonical labeled polyhex reducts.

We write

$$\mathfrak{C}_1^{\text{px}} \simeq_{\text{px,lab}} \mathfrak{C}_2^{\text{px}} \quad (457)$$

when there exist a graph isomorphism

$$\varphi : G_1 \longrightarrow G_2$$

and a Euclidean isometry

$$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

such that

$$g \circ \widehat{t}_{\text{px},1} = \widehat{t}_{\text{px},2} \circ \varphi, \quad (458)$$

$$\lambda_{\text{loc},2} \circ \varphi = \lambda_{\text{loc},1}, \quad (459)$$

$$\lambda_{\text{chem},2} \circ \varphi^\# = \lambda_{\text{chem},1}. \quad (460)$$

Equivalently, the two objects represent the same canonically embedded, locant-preserving carbon-framework polyhex.

9.2 The Reference Chrysene Node

Definition 74 (Reference chrysene molecular node). *Let*

$$\boxed{\mathfrak{C}_{\text{chr}}} \quad (461)$$

denote the embedded labeled carbon-skeleton graph of chrysene equipped with its standard carbon connectivity and standard locant labeling.

The explicit graph adjacency, locant assignment, and embedding convention are fixed once and for all in Appendix A.

The distinguished primary and dual locant sets of $\mathfrak{C}_{\text{chr}}$ are

$$A_{\text{chr}}^{\text{pri}} = \{v_3, v_6, v_9, v_{12}\}, \quad (462)$$

$$A_{\text{chr}}^{\text{dual}} = \{v_2, v_5, v_8, v_{11}\}. \quad (463)$$

The primary-to-dual shift is

$$3 \mapsto 2, \quad 6 \mapsto 5, \quad 9 \mapsto 8, \quad 12 \mapsto 11. \quad (464)$$

9.3 Perimeter Compatibility of the Dual Cipher

Because $G_{\mathfrak{C}} \in \mathcal{B}_4$, its polyhex realization possesses a distinguished external perimeter cycle, denoted

$$\partial G_{\mathfrak{C}}. \quad (465)$$

Definition 75 (Perimeter-realized cipher). *The primary–dual cipher is called perimeter realized when*

$$v_j, v_{\delta_{\text{dc}}(j)} \in V(\partial G_{\mathfrak{C}}) \quad (466)$$

and

$$\boxed{\{v_j, v_{\delta_{\text{dc}}(j)}\} \in E(\partial G_{\mathfrak{C}})} \quad (467)$$

for every

$$j \in \{3, 6, 9, 12\}.$$

The orientation of $\partial G_{\mathfrak{C}}$ is chosen so that each dual locant is the same one-step perimeter neighbor of its primary partner.

Remark 40 (Why this condition is needed). *Without Definition 75, the labels 2, 3, 5, 6, 8, 9, 11, 12 could be assigned abstractly to unrelated graph vertices. Perimeter realization turns the dual cipher into a graph-theoretic property of the molecular reduct rather than a relabeling convention.*

9.4 The Chrysene Characterization Signature

Definition 76 (Chrysene characterization predicate). *For an embedded labeled node*

$$\mathfrak{C} \in \mathfrak{R}_4,$$

define

$$\text{ChrSig}(\mathfrak{C})$$

to mean that all of the following conditions hold.

(C1) *The molecular carbon-skeleton graph belongs to the four-ring polyhex comparison class:*

$$\boxed{G_{\mathfrak{C}} \in \mathcal{B}_4}. \quad (468)$$

(C2) *Its canonical carbon-framework point group satisfies*

$$\boxed{\Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}}}. \quad (469)$$

(C3) *The primary–dual cipher is realized as a uniform one-step displacement on an oriented external perimeter cycle*

$$\partial G_{\mathfrak{C}}.$$

Explicitly, for every

$$j \in \{3, 6, 9, 12\},$$

the paired primary and dual vertices satisfy

$$\boxed{\{v_j, v_{\delta_{\text{dc}}(j)}\} \in E(\partial G_{\mathfrak{C}})}, \quad (470)$$

and all four pairings have the same orientation relative to the chosen perimeter cycle.

(C4) *The two canonical primary displacement vectors are linearly independent:*

$$\boxed{\widehat{\mathbf{d}}_x \times \widehat{\mathbf{d}}_y \neq \mathbf{0}}. \quad (471)$$

(C5) The canonical directional data are inequivalent under the reference carbon-framework stabilizer:

$$\widehat{\mathcal{D}}_x \not\sim_{\Gamma_{\mathfrak{C}}^{\text{core}}} \widehat{\mathcal{D}}_y. \quad (472)$$

Proposition 32 (Axioms imply the characterization signature). *If*

$$\mathfrak{R} \in \text{Mod}(\mathcal{A}),$$

then its molecular reduct satisfies

$$\text{ChrSig}(\text{Red}_{\text{mol}}(\mathfrak{R})). \quad (473)$$

Proof. Condition (C1) is Axiom A_1 . Condition (C2) is the carbon-framework clause of Axiom A_2 . Conditions (C4)–(C5) are the strengthened directional requirements of Axiom A_3 . The primary and dual locant sets are part of Axiom A_1 , while their cipher pairing is part of Axiom A_8 . With perimeter realization included as a molecular compatibility condition of the dual cipher, condition (C3) follows. $\square$

9.5 Finite Molecular Classification

The uniqueness problem is now finite and purely molecular.

Definition 77 (Characterization node class). *Define*

$$\mathfrak{N}_4^{\text{CF}} := \{\mathfrak{C} \in \mathfrak{N}_4 : \text{ChrSig}(\mathfrak{C})\}. \quad (474)$$

Proposition 33 (Reduction to finite carbon-framework classification). *The carbon-framework uniqueness problem for the twelve-axiom system reduces to determination of the canonical polyhex-congruence classes*

$$\{[\text{CanGraph}(\mathfrak{C})]_{\approx_{\text{px}}} : \mathfrak{C} \in \mathfrak{N}_4^{\text{CF}}\}. \quad (475)$$

Theorem 34 (Finite carbon-framework classification). *The finite classification carried out in Appendix A establishes*

$$\{[\text{CanGraph}(\mathfrak{C})]_{\approx_{\text{px}}} : \mathfrak{C} \in \mathfrak{N}_4^{\text{CF}}\} = \{[G_{\text{chr}}^{\text{px}}]_{\approx_{\text{px}}}\}. \quad (476)$$

Proof. This is Theorem 55. $\square$

9.6 Extension of the Chrysene Reduct

Molecular uniqueness and existence of a full operator realization are logically separate.

Definition 78 (Extension class). *For an embedded labeled molecular node $\mathfrak{C}$, let*

$$\text{Ext}_{12}(\mathfrak{C}) \quad (477)$$

denote the class of all choices of bridge, tensor-network, sequence algebra, dual-cipher, spectral, Hamiltonian, response, and admissibility data extending $\mathfrak{C}$ to a full model of Axioms A_1 – A_{12} .

Theorem 35 (Existence of a full Chrysene extension). *The explicit construction of Appendix B establishes*

$$\text{Ext}_{12}(\mathfrak{C}_{\text{chr}}) \neq \emptyset. \quad (478)$$

Proof. This is Theorem 61. $\square$

Remark 41 (Status of the analytic extension condition). *The principal analytic issue in establishing non-vacuity is the existence of a faithful spectral realization satisfying Axiom A_9 , together with compatible realizations of Axioms A_{10} – A_{12} . Appendix B constructs such a realization explicitly. Therefore*

$$\text{Ext}_{12}(\mathfrak{C}_{\text{chr}}) \neq \emptyset, \quad (479)$$

and the analytic-extension condition (AE) is discharged in the present article.

9.7 The Chrysene Characterization Theorem

Theorem 36 (Chrysene Characterization and Existence Theorem). *The revised twelve-axiom Chrysene Formalism is non-vacuous. Moreover, every full model has the canonical carbon framework of chrysene.*

More precisely:

1.

$$\text{Mod}(\mathcal{A}) \neq \emptyset. \quad (480)$$

2. For every $\mathfrak{R} \in \text{Mod}(\mathcal{A})$,

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \approx_{\text{px}} G_{\text{chr}}^{\text{px}}. \quad (481)$$

3. Equivalently,

$$\begin{aligned} & \left\{ [\text{CanGraphRed}_{\text{mol}}(\mathfrak{R})]_{\approx_{\text{px}}} : \mathfrak{R} \in \text{Mod}(\mathcal{A}) \right\} \\ &= \left\{ [G_{\text{chr}}^{\text{px}}]_{\approx_{\text{px}}} \right\}. \end{aligned} \quad (482)$$

4. Every full model possesses a noncommutative sequence algebra

$$\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$$

with

$$\exists A, B \in \mathfrak{A}_{\mathfrak{C}} \quad [A, B] \neq 0.$$

5. Uniqueness of the canonical chrysene carbon framework does not imply uniqueness of the complete twelve-axiom realization.

Proof. By Theorem 35, the reference chrysene node admits a full twelve-axiom extension. Hence

$$\text{Mod}(\mathcal{A}) \neq \emptyset.$$

Let

$$\mathfrak{R} \in \text{Mod}(\mathcal{A})$$

be arbitrary. By Proposition 32, its molecular reduct belongs to $\mathfrak{R}_4^{\text{CF}}$.

The finite carbon-framework classification theorem then gives

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \simeq_{\text{px}} G_{\text{chr}}^{\text{px}}.$$

This proves parts 2 and 3.

Part 4 follows from Theorem 8.

Finally, the axioms permit distinct physical embeddings, locant realizations, attachment labels, bridge families, states, spectral triples, Hamiltonians, response operators, and admissibility structures over the same canonical carbon framework. Therefore canonical molecular-framework uniqueness does not imply categoricity of the complete model class. $\square$

Definition 79 (Standard-locant Chrysene realization class). Define

$$\text{Mod}_{\text{chr}}^{\text{std}}(\mathcal{A}) := \{ \mathfrak{R} \in \text{Mod}(\mathcal{A}) : \text{CanRed}_{\text{mol}}(\mathfrak{R}) \simeq_{\text{px}, \text{lab}} \mathfrak{C}_{\text{chr}}^{\text{px}} \}. \quad (483)$$

This subclass consists of full models whose canonical chrysene framework carries the standard locant convention used throughout the Chrysene Formalism, including the distinguished

$$\{3, 6, 9, 12\}$$

primary frame and

$$\{2, 5, 8, 11\}$$

dual frame.

Proposition 37 (Non-vacuity of the standard-locant subclass). The explicit realization $\mathfrak{R}_{\text{ex}}$ of Appendix B belongs

to

$$\text{Mod}_{\text{chr}}^{\text{std}}(\mathcal{A}).$$

Hence

$$\boxed{\text{Mod}_{\text{chr}}^{\text{std}}(\mathcal{A}) \neq \emptyset.} \quad (484)$$

Remark 42 (Role of the standard locant frame). The finite classification theorem derives the identity of the canonical carbon framework; it does not claim that chemical nomenclature is generated uniquely by the twelve axioms. The standard chrysene locant convention is instead fixed on the reference molecular realization and retained as the coordinate frame for the subsequent Chrysene theorem program.

Accordingly, later chrysene-specific realization theorems may be formulated over

$$\text{Ext}_{12}(\mathfrak{C}_{\text{chr}})$$

or equivalently over the appropriate standard-locant subclass, without converting an external nomenclature convention into an axiomatically derived theorem.

Remark 43 (Canonical uniqueness versus physical congruence). Theorem 36 does not assert that every realization-specific embedding ι is Euclidean-congruent to the ideal regular-polyhex embedding, nor does it assert that the twelve axioms uniquely generate the standard chemical locant convention.

Rather, it states that the underlying canonically embedded carbon framework is uniquely the chrysene framework.

The standard numbered chrysene realization is selected separately by Definition 79.

If a particular realization additionally satisfies

$$\iota \simeq \widehat{\iota}_{\text{px}}$$

by Euclidean congruence and belongs to the standard-locant realization class, then its embedded numbered molecular reduct may be compared directly with the reference embedded chrysene node.

Corollary 6 (Canonical carbon-framework uniqueness without full categoricity). The twelve-axiom system is categorical at the level of its canonical carbon framework:

$$\boxed{\left| \left\{ \left[\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \right]_{\simeq_{\text{px}}} : \mathfrak{R} \in \text{Mod}(\mathcal{A}) \right\} \right| = 1.} \quad (485)$$

The complete twelve-axiom theory need not be categorical, because distinct full realizations may share this same

canonical chrysene framework.

9.8 Why the Previous Abstract Countermodel Is Excluded

Proposition 38 (Exclusion of the four-cycle countermodel). *The four-vertex square-cycle structure used in the earlier non-vacuity construction is not a model of the present twelve-axiom system.*

Proof. The square-cycle molecular graph is not the carbon-skeleton graph of an edge-connected four-ring polyhex and therefore fails Axiom A_1 .

It also lacks the required eight distinguished locants

$$2, 3, 5, 6, 8, 9, 11, 12$$

and cannot realize the primary/dual perimeter cipher of Axiom A_8 .

Hence it is excluded before any operator-algebraic or response data are considered. $\square$

9.9 What Molecular Uniqueness Does Not Prove

Remark 44 (Logical scope of the characterization theorem). *Theorem 36 is a molecular characterization result. It does not, by itself, establish*

$$N_y^* = N_x^* + 2, \quad (486)$$

$$\mathcal{T}_x \neq_u \mathcal{T}_y, \quad (487)$$

$$\alpha_{xx} \neq \alpha_{yy}, \quad (488)$$

$$\Sigma_{\mathfrak{C}} \cong GF(4) \quad \text{as a field}, \quad (489)$$

$$d_D^{\text{str}} = d_C^*, \quad (490)$$

nor does it establish an autonomous physical scission mechanism.

Each statement remains a separate theorem-level problem over the the unique canonical chrysene carbon framework.

9.10 The Revised Meaning of “Chrysene Isomorphism”

The phrase *Chrysene Isomorphism Theorem* was used in the earlier formalism for a stronger collection of geometric, quantum, finite-field, optimal-transport, and boundary assertions [Sch26b, Sch26a].

The revised theorem makes the narrower and mathematically explicit statement that the canonical carbon framework, rather than every realization-specific labeling or embedding, is uniquely

chrysene.

$$\begin{array}{c} \text{twelve-axiom model } \mathfrak{R} \\ \Downarrow \\ \text{Red}_{\text{mol}}(\mathfrak{R}) \in \mathfrak{R}_4^{\text{CF}} \\ \Downarrow \\ \text{finite four-ring carbon-framework classification} \\ \Downarrow \\ \text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \simeq_{\text{px}} G_{\text{chr}}^{\text{px}} \end{array} \quad (491)$$

The uniqueness conclusion is therefore separated from later claims about electronic decoupling, finite-field multiplication, spectral response, optimal transport, or physical boundary dynamics.

9.11 Characterization Hierarchy

The twelve-axiom foundation together with the characterization theorem may now be summarized as

$$\begin{array}{c} \mathfrak{R}_4 \\ \Downarrow \\ C_{2h}^{\text{std}}\text{-type canonical symmetry} \\ + \text{perimeter cipher} \\ + \text{directional inequivalence} \\ \Downarrow \\ \mathfrak{R}_4^{\text{CF}} \\ \Downarrow \\ G_{\text{chr}}^{\text{px}} \\ \text{(unique canonical carbon framework)} \\ \Downarrow \\ \mathbb{Z}^3\text{-fibrated tensor network} \\ \Downarrow \\ \mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3 \\ \Downarrow \\ \text{spectral noncommutative geometry} \\ \Downarrow \\ \text{multiple admissible physical/operator completions} \end{array} \quad (492)$$

Thus the revised formalism can be molecularly specific without collapsing its downstream operator, transport, response, and boundary theories into properties that hold by definition.

10 Logical Architecture and Chrysene Characterization

The twelve axioms are organized by mathematical prerequisite rather than by a claim of automatic physical causation.

Let

$$\mathcal{A} = \{A_1, \dots, A_{12}\}. \quad (493)$$

Partition the axiom system into five strata:

$$\mathcal{A}_G := \{A_1, A_2, A_3\}, \quad (494)$$

$$\mathcal{A}_T := \{A_4, A_5, A_6\}, \quad (495)$$

$$\mathcal{A}_Q := \{A_7, A_8\}, \quad (496)$$

$$\mathcal{A}_S := \{A_9\}, \quad (497)$$

$$\mathcal{A}_P := \{A_{10}, A_{11}, A_{12}\}. \quad (498)$$

The subscripts denote, respectively, molecular geometry, tensor-network construction, sequence/dual-cipher algebra, spectral noncommutative geometry, and physical response/admissibility structure.

Definition 80 (Prerequisite relation). For $A_i, A_j \in \mathcal{A}$, write

$$A_i \prec A_j \quad (499)$$

when the statement of A_j uses a mathematical object whose domain, codomain, or defining data are introduced by A_i . The relation records definitional prerequisite only and does not mean that the conclusions of A_j follow physically from properties stated in A_i .

A sufficient generating set is

$$A_1 \prec A_2, \quad A_1 \prec A_3, \quad A_2 \prec A_3, \quad (500)$$

$$A_3 \prec A_4, \quad A_4 \prec A_5, \quad A_5 \prec A_6, \quad (501)$$

$$A_6 \prec A_7, \quad A_1 \prec A_8, \quad A_6 \prec A_8, \quad A_7 \prec A_8, \quad (502)$$

$$A_7 \prec A_9, \quad A_4 \prec A_{10}, \quad A_9 \prec A_{10}, \quad (503)$$

$$A_9 \prec A_{12}. \quad (504)$$

Remark 45 (Expository order versus prerequisite). Axiom A_{11} is presented after Axiom A_{10} because polarization and response are physically downstream topics. Under Definition 80, however, the response realization is introduced independently and therefore does not require a generating prerequisite edge $A_7 \prec A_{11}$ or $A_{10} \prec A_{11}$. The sequential placement of A_{10} – A_{12} is therefore partly expository rather than a claim of definitional dependence.

Proposition 39 (Acyclicity of the prerequisite architecture). The prerequisite relation generated by Equations (500)–(504) contains no directed cycle.

Proof. Assign the numerical rank

$$r(A_i) = i.$$

Every generating prerequisite relation satisfies $i < j$. The rank therefore increases strictly along every directed path, which precludes a directed cycle. $\square$

11 Deferred Theorem Program

The twelve-axiom system defines the foundational Chrysene Formalism, and Theorem 36 identifies the canonical chrysene carbon framework, within the specified four-ring polyhex comparison class, as the unique canonical carbon-framework class realized by every full model.

The remaining principal claims of the Chrysene program are of a different logical type. They concern special arithmetic closure, finite-field multiplication, directional transfer, optical response, spectral metric reconstruction, higher algebraic structure, emergent chirality, and dynamical boundary behavior. None of these properties is required merely for an object to satisfy Axioms A_1 – A_{12} .

Accordingly, let

$$\mathfrak{R} \in \text{Ext}_{12}(\mathfrak{C}_{\text{chr}}) \quad (505)$$

be a specified full realization over the chrysene molecular reduct.

Let

$$\mathcal{H}_{\mathfrak{R}} = \{H_1(\mathfrak{R}), \dots, H_m(\mathfrak{R})\} \quad (506)$$

denote any additional collection of explicitly stated geometric, registry, algebraic, spectral, Hamiltonian, response, or dynamical hypotheses associated with that realization.

A deferred theorem has the logical form

$$\boxed{\mathcal{A} \cup \mathcal{H}_{\mathfrak{R}} \models P_{\mathfrak{R}}}, \quad (507)$$

where

$$\mathcal{A} = \{A_1, \dots, A_{12}\}$$

and where $P_{\mathfrak{R}}$ is not true merely by virtue of satisfying the twelve axioms.

11.1 Directional Registry and the $N + 2$ Closure Law

Under the axially aligned homogeneous specialization of Definition 21, exact directional closure gives

$$K_{\alpha} = N_{\alpha}^{\star} + \delta_{\alpha}, \quad \alpha \in \{x, y\}. \quad (508)$$

Subtracting the two relations yields

$$\boxed{N_y^{\star} - N_x^{\star} = (K_y - K_x) + (\delta_x - \delta_y)}. \quad (509)$$

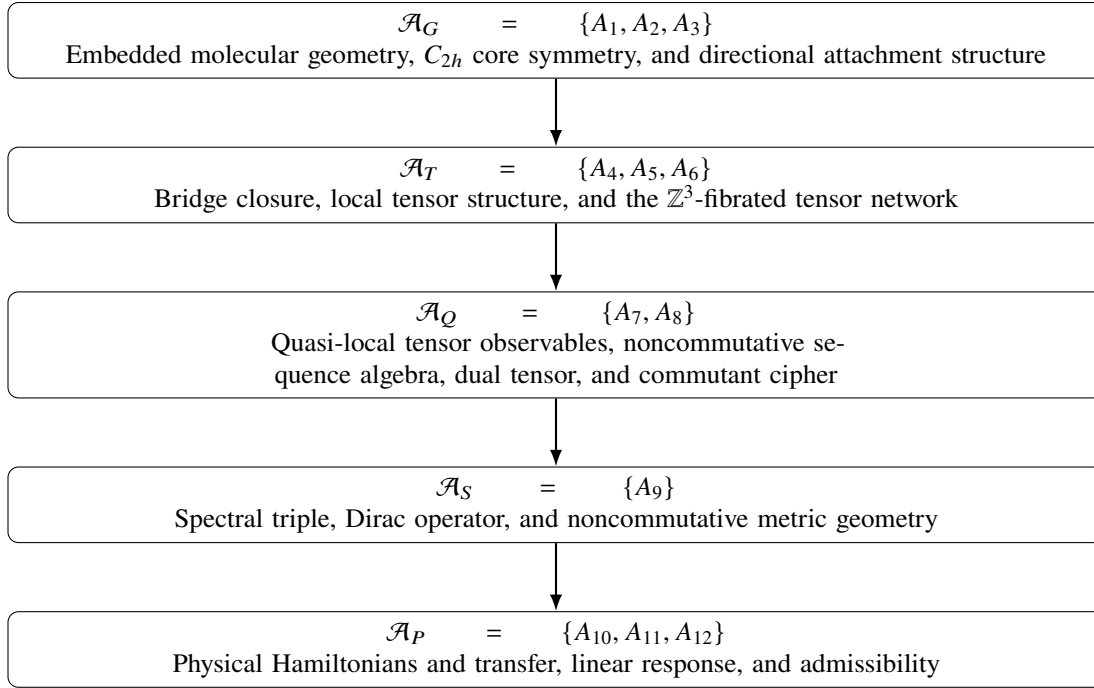


Figure 1: Stratification of the revised Chrysene Formalism. The arrows indicate availability of mathematical structure and definitional prerequisite. They do not assert automatic physical causation. In particular, molecular anisotropy does not by itself imply transfer or response anisotropy.

Definition 81 (Commensurate chrysene registry hypotheses). *A chrysene realization satisfies the commensurate registry hypotheses when*

$$\ell_x = \ell_y, \quad (510)$$

$$K_x = K_y, \quad (511)$$

$$\delta_x - \delta_y = 2. \quad (512)$$

Theorem 40 (Target: Chrysene $N + 2$ Closure Theorem). *If a full chrysene realization satisfies Definition 81, then*

$$N_y^* = N_x^* + 2. \quad (513)$$

Proof target. Under $K_x = K_y$, Equation (509) reduces to

$$N_y^* - N_x^* = \delta_x - \delta_y.$$

Thus $\delta_x - \delta_y = 2$ implies Equation (513).

The substantive theorem obligation is therefore the molecular and network derivation of

$$\delta_x - \delta_y = 2$$

for the relevant chrysene connector geometry. $\square$

Remark 46 (What must still be proved). *The algebraic subtraction is immediate. The nontrivial content of the future theorem is the demonstration that the chrysene attachment geometry and admissible connector family actually produce the required two-unit registry difference.*

11.2 Dual Cipher and the $GF(4)$ Field Law

Axiom A₈ already establishes the structural alphabet

$$\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z \quad (514)$$

with

$$|\Sigma_{\mathfrak{C}}| = 4 \quad (515)$$

and the canonical additive operation $\boxplus$.

Consequently,

$$(\Sigma_{\mathfrak{C}}, \boxplus) \cong \mathbb{Z}_2^2 \quad (516)$$

as an additive group.

A finite-field theorem requires additional multiplicative structure.

Definition 82 (Structurally induced multiplication). *A binary operation*

$$\boxtimes_{\mathfrak{R}} : \Sigma_{\mathfrak{C}} \times \Sigma_{\mathfrak{C}} \longrightarrow \Sigma_{\mathfrak{C}} \quad (517)$$

is called structurally induced when its value is determined by

an explicitly specified composition law of cipher and registry operations in the realization $\mathfrak{R}$, rather than transported arbitrarily from an external four-element field.

Theorem 41 (Target: Dual-Cipher $GF(4)$ Theorem). *Suppose a chrysene realization admits a structurally induced multiplication $\boxtimes_{\mathfrak{R}}$ for which*

$$(\Sigma_{\mathbb{C}}, \boxplus, \boxtimes_{\mathfrak{R}}) \quad (518)$$

*satisfies the field axioms.
Then*

$$(\Sigma_{\mathbb{C}}, \boxplus, \boxtimes_{\mathfrak{R}}) \cong GF(4) \cong \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1). \quad (519)$$

Remark 47 (Nontrivial content of the $GF(4)$ theorem). *Every four-element set can be equipped artificially with a field structure isomorphic to $GF(4)$. Such transport of structure is not the theorem sought here. The substantive question is whether the multiplication produced by the actual dual-cipher and registry composition laws satisfies the field axioms.*

11.3 Directional Electronic Transfer

For

$$|z| > \|H_{B,\alpha}^{(N)}\|,$$

Axiom A_{10} gives

$$\mathcal{T}_{\alpha}^{(N)}(z) = \mathcal{J}_{\alpha}^{(N)} + \sum_{k=0}^{\infty} \frac{M_{k,\alpha}^{(N)}}{z^{k+1}}, \quad (520)$$

with

$$M_{k,\alpha}^{(N)} = W_{L,\alpha}^{(N)\dagger} \left(H_{B,\alpha}^{(N)} \right)^k W_{R,\alpha}^{(N)}. \quad (521)$$

Assume unitary identifications of the x - and y -terminal Hilbert spaces have been specified so that the corresponding coefficients may be compared.

Theorem 42 (Target: Directional Transfer Anisotropy Theorem). *Suppose*

$$\mathcal{J}_x^{(N_x)} \neq \mathcal{J}_y^{(N_y)} \quad (522)$$

after the prescribed terminal identifications, or suppose that

$$M_{k,x}^{(N_x)} \neq M_{k,y}^{(N_y)} \quad (523)$$

for at least one $k \geq 0$.

Then the corresponding analytic transfer functions cannot

be identical:

$$\boxed{\mathcal{T}_x^{(N_x)}(\cdot) \neq \mathcal{T}_y^{(N_y)}(\cdot)}. \quad (524)$$

Proof target. After the specified unitary identifications, equality of the two analytic operator-valued functions on a common exterior domain would imply equality of their Laurent coefficients. Inequality of one coefficient therefore rules out identical transfer functions by uniqueness of analytic continuation. $\square$

Remark 48 (Geometry is not sufficient). *The condition*

$$\mathcal{D}_x \not\sim_{\Gamma_{\mathbb{C}}^{\text{core}}} \mathcal{D}_y$$

does not itself prove Equation (524). The bridge Hamiltonians and contact maps must carry the geometric inequivalence into the physical operator realization.

11.4 Polarization and Optical Anisotropy

For the finite-dimensional response model of Proposition 28, define the directional transition-dipole measures

$$dv_i(\Omega) := \sum_{n \neq 0} |\langle 0 | \mu_i | n \rangle|^2 \delta(\Omega - \omega_{n0}) d\Omega, \quad i \in \{x, y\}, \quad (525)$$

where

$$\omega_{n0} := \frac{E_n - E_0}{\hbar}. \quad (526)$$

Theorem 43 (Target: Polarization Anisotropy Theorem). *Under a common spectral and broadening prescription, suppose the directional transition spectra differ in at least one pole location or residue contributing to the response:*

$$\boxed{\nu_x \neq \nu_y}. \quad (527)$$

Under conditions sufficient for uniqueness of the corresponding spectral transforms,

$$\boxed{\alpha_{xx}(\cdot) \neq \alpha_{yy}(\cdot)}. \quad (528)$$

For an oriented finite realization, define absorption functions $A_x(\omega)$ and $A_y(\omega)$. Whenever

$$A_x(\omega) \neq A_y(\omega), \quad (529)$$

define the linear dichroism

$$\boxed{LD(\omega) := A_y(\omega) - A_x(\omega)}. \quad (530)$$

Remark 49 (Bulk optical response requires a further limit). *A molecular or finite-network anisotropy theorem does not establish a bulk dielectric anisotropy. Such a result additionally requires the existence and orientation dependence of the thermodynamic or homogenization limit introduced in Definition 62.*

11.5 Connes Distance and the Admissible Path Metric

Axiom A_{12} places two independently defined distances on the same structural state space:

$$d_D^{\text{str}} \quad \text{and} \quad d_C^*. \quad (531)$$

Definition 83 (Metric-calibrated Dirac realization). *A Dirac realization is called path calibrated on $Y \subseteq \Omega_{\text{adm}}$ when its Lipschitz seminorm*

$$L_D(a) = \|[D_{\mathfrak{C}}, \pi_{\omega}(a)]\|$$

reproduces the weighted graph-Lipschitz constraint associated with the admissible edge weights on Y .

Theorem 44 (Target: Spectral-Path Metric Compatibility Theorem). *For a path-calibrated Dirac realization satisfying the required separation and domain hypotheses, establish one or more of the relations*

$$d_D^{\text{str}}(x, y) = d_C^*(x, y), \quad (532)$$

$$c_- d_C^*(x, y) \leq d_D^{\text{str}}(x, y) \leq c_+ d_C^*(x, y), \quad (533)$$

$$d_D^{\text{str}}(x, y) = +\infty \iff d_C^*(x, y) = +\infty. \quad (534)$$

Remark 50 (The strongest relation need not always hold). *Exact equality is the strongest possible compatibility statement. Bi-Lipschitz equivalence or agreement only on disconnected components may be the appropriate theorem for a particular Dirac realization. The formalism does not require all three statements simultaneously.*

11.6 Dual Cipher and Real Spectral Structure

Axiom A_8 provides the conjugate-linear isometric involution

$$J_{\text{dc}} : \mathcal{H}_{\omega} \longrightarrow \mathcal{H}_{\omega} \quad (535)$$

satisfying

$$J_{\text{dc}}^2 = I \quad (536)$$

and

$$J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}} = \mathfrak{M}'_{\text{pri}}. \quad (537)$$

Equivalently, Axiom A_8 defines the representation of the opposite algebra by

$$\pi_{\text{dual}}(b^{\text{op}}) := J_{\text{dc}} \pi_{\omega}(b^*) J_{\text{dc}}^{-1}. \quad (538)$$

Consequently, the zero-order commutation relation

$$[\pi_{\omega}(a), \pi_{\text{dual}}(b^{\text{op}})] = 0 \quad (539)$$

holds for $a, b \in \mathfrak{A}_{\mathfrak{C}}$ at the von Neumann-algebraic level determined by the primary-dual commutant construction.

Axiom A_9 independently provides the spectral triple

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}). \quad (540)$$

The commutant relation of Axiom A_8 does not by itself establish compatibility of J_{dc} with the Dirac operator. That compatibility is therefore retained as a theorem-level problem.

Theorem 45 (Target: Real Spectral-Triple Theorem). *Assume the dual-cipher conjugation J_{dc} of Axiom A_8 , which already satisfies*

$$J_{\text{dc}}^2 = I.$$

Determine whether the following additional conditions hold. First, J_{dc} must preserve the Dirac domain:

$$J_{\text{dc}} \text{Dom}(D_{\mathfrak{C}}) = \text{Dom}(D_{\mathfrak{C}}). \quad (541)$$

Second, determine whether there exists a sign

$$\varepsilon' \in \{+1, -1\} \quad (542)$$

such that

$$J_{\text{dc}} D_{\mathfrak{C}} = \varepsilon' D_{\mathfrak{C}} J_{\text{dc}} \quad (543)$$

on $\text{Dom}(D_{\mathfrak{C}})$.

Third, determine whether the order-one condition

$$[[D_{\mathfrak{C}}, \pi_{\omega}(a)], \pi_{\text{dual}}(b^{\text{op}})] = 0 \quad (544)$$

holds for all

$$a, b \in \mathfrak{A}_{\mathfrak{C}}^{\circ}, \quad (545)$$

or on a specified dense involutive subalgebra thereof if a smaller common domain is required.

Equivalently, using Equation (538), the order-one condition is

$$[[D_{\mathfrak{C}}, \pi_{\omega}(a)], J_{\text{dc}} \pi_{\omega}(b^*) J_{\text{dc}}^{-1}] = 0. \quad (546)$$

If Equations (541), (543), and (544) hold, then J_{dc} provides a real structure compatible with the ungraded spectral triple

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}).$$

Remark 51 (Possible even grading). *No grading operator is imposed by Axioms A₁–A₉. If a later realization equips the spectral triple with a grading*

$$\Gamma = \Gamma^\dagger, \quad \Gamma^2 = I, \quad (547)$$

satisfying

$$\Gamma \pi_\omega(a) = \pi_\omega(a) \Gamma \quad (548)$$

and

$$\Gamma D_{\mathfrak{C}} = -D_{\mathfrak{C}} \Gamma, \quad (549)$$

then compatibility with the real structure requires an additional sign relation

$$J_{\text{dc}} \Gamma = \varepsilon'' \Gamma J_{\text{dc}}, \quad \varepsilon'' \in \{+1, -1\}. \quad (550)$$

Such an even grading is additional spectral structure and is not inferred merely from the three-dimensional lattice base.

Remark 52 (KO-dimension is derived). *No KO-dimension is assigned solely from the geometric relation*

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3. \quad (551)$$

The KO-dimension of a realized real spectral triple must instead be determined from the applicable real-structure sign data. In the present formulation,

$$J_{\text{dc}}^2 = I \quad (552)$$

is already fixed by Axiom A₈, whereas the sign ε' in

$$J_{\text{dc}} D_{\mathfrak{C}} = \varepsilon' D_{\mathfrak{C}} J_{\text{dc}} \quad (553)$$

remains to be derived. If an even grading is subsequently introduced, the additional sign ε'' in Equation (550) must also be determined.

Thus combinatorial lattice dimension, spectral dimension, and KO-dimension remain mathematically distinct invariants of the formalism.

11.7 Spectral Dimension of the $\mathbb{Z}^3$ -Fibrated Space

The discrete base satisfies

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3,$$

whereas Definition 52 defines the analytic dimension through eigenvalue growth of $D_{\mathfrak{C}}$.

Let

$$N_D(\Lambda) := \# \{ \lambda \in \sigma(|D_{\mathfrak{C}}|) : \lambda \leq \Lambda \}, \quad (554)$$

counting multiplicity.

Theorem 46 (Target: Spectral Dimension Theorem). *If a chosen Dirac realization satisfies a Weyl-type asymptotic law*

$$N_D(\Lambda) \sim C \Lambda^3 \quad (\Lambda \rightarrow \infty), \quad (555)$$

for some $C > 0$, then

$$d_{\text{sp}} = 3. \quad (556)$$

The substantive theorem problem is therefore whether the Dirac operator naturally generated by the chrysene tensor network possesses the spectral growth Equation (555).

11.8 Stacking Registry and Emergent Chirality

The isolated canonical molecular framework satisfies

$$\Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}}, \quad (557)$$

and is therefore achiral.

A stacked or extended tensor network can possess a different symmetry stabilizer.

Definition 84 (Assembly stabilizer). *Let $\mathfrak{S}$ be a specified stacked realization constructed from the $\mathbb{Z}^3$ -fibrated network and its registry sequence*

$$\{\eta_{\mathbf{k}}\}.$$

Let

$$\Gamma_{\mathfrak{S}} \quad (558)$$

denote the full Euclidean stabilizer of the assembled, labeled structure.

Theorem 47 (Target: Emergent Chirality Theorem). *A stacked realization $\mathfrak{S}$ is chiral precisely when*

$$\Gamma_{\mathfrak{S}} \cap \{g \in E(3) : \det R_g = -1\} = \emptyset. \quad (559)$$

The theorem program is to determine which admissible z -registry or dual-cipher sequences remove every improper symmetry of the complete assembly.

Remark 53 (Registry is not synonymous with chirality). *A signed or binary registry variable does not itself prove chirality. The full assembly stabilizer must be evaluated.*

11.9 Higher Algebraic Extensions and Braided Sequence Structure

The noncommutativity established in Axiom A_7 concerns multiplication:

$$AB \neq BA. \quad (560)$$

A distinct question concerns whether the Chrysene sequence algebra admits a coproduct and hence a Hopf-type or braided extension.

Definition 85 (Candidate higher-algebraic core). *Let*

$$\mathcal{H}_{\mathbb{C}} \subseteq \mathfrak{A}_{\mathbb{C}}^{\circ} \quad (561)$$

be a dense algebraic subalgebra on which a candidate coproduct

$$\Delta : \mathcal{H}_{\mathbb{C}} \longrightarrow \mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}} \quad (562)$$

has been defined from an explicit structural composition law.

Theorem 48 (Target: Noncocommutative Sequence Theorem). *Determine whether there exists a $a \in \mathcal{H}_{\mathbb{C}}$ such that*

$$\Delta^{\text{op}}(a) \neq \Delta(a), \quad (563)$$

where

$$\Delta^{\text{op}} := \tau \circ \Delta$$

and $\tau(x \otimes y) = y \otimes x$.

A stronger theorem would establish the existence of an invertible element

$$\mathcal{R} \in \mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}} \quad (564)$$

such that

$$\Delta^{\text{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad (565)$$

for all appropriate a .

If $\mathcal{R}$ also satisfies the required Yang–Baxter or quasitriangular identities, a braid-group representation may subsequently be derived.

Remark 54 (Higher algebra is not part of the foundation). *Neither a coproduct nor an R -matrix is required by Axioms A_1 – A_{12} . The noncommutative sequence algebra exists independently of whether a Hopf or braided extension can be constructed.*

11.10 Admissibility, Projection, and Dynamical Scission

Axiom A_{12} separates four different mathematical statements:

$$\begin{array}{c} \text{constraint violation} \\ \neq \\ \text{metric disconnection} \\ \neq \\ \text{Hilbert-space projection} \\ \neq \\ \text{physical removal.} \end{array} \quad (566)$$

Metric disconnection is already characterized by

$$d_{\mathbb{C}}^*(x, y) = +\infty \iff \mathcal{P}_{\text{adm}}(x, y) = \emptyset. \quad (567)$$

Definition 86 (Dynamical excision model). *A dynamical excision model is an explicitly specified evolution family*

$$\{\Phi_t\}_{t \geq 0} \quad (568)$$

on physical states such that the admissible and rejected sectors of Definition 67 have a stated dynamical role. The evolution may be unitary, dissipative, stochastic, measurement-induced, or otherwise explicitly defined.

Theorem 49 (Target: Dynamical Scission Theorem). *A physical scission or excision result requires sufficient dynamical hypotheses to prove that a rejected component is removed, absorbed, or dynamically isolated. A representative theorem target is*

$$\lim_{t \rightarrow \infty} \|Q_{\text{adm}}\Phi_t(\rho)Q_{\text{rej}}\| = 0, \quad (569)$$

together with a specified condition ensuring that rejected-state weight is irreversibly transferred out of the physical sector of interest.

Remark 55 (Idempotence is insufficient). *The algebraic identity*

$$Q_{\text{rej}}^2 = Q_{\text{rej}}$$

proves that Q_{rej} is a projection. It does not prove that a physical system executes that projection dynamically.

11.11 Dependency Structure of the Theorem Program

The principal theorem families are not independent of one another. Their natural logical organization is:

1. Chrysene Molecular Characterization

- Registry $\Rightarrow N_y = N_x + 2$
 - Dual cipher $\Rightarrow GF(4)$
 - Stacking registry $\Rightarrow$ Assembly chirality
2. Sequence Algebra
- Spectral triple $\Rightarrow$ Spectral dimension
 - $\Rightarrow$ Metric compatibility
 - Physical Hamiltonian $\Rightarrow$ Transfer anisotropy
 - $\Rightarrow$ Response anisotropy
 - Candidate coproduct $\Rightarrow$ Noncocommutativity
 - $\Rightarrow$ Braiding

3. Admissibility

- Metric disconnection $\Rightarrow$ Operator projection
- $\Rightarrow$ Dynamical scission

The arrows denote theorem dependencies or natural development paths, not automatic implications. In particular, the diagram records where additional information would have to enter a proof. The $N_y = N_x + 2$ relation requires a registry statement stronger than exact closure alone; a $GF(4)$ field law requires a multiplication generated by the structural operations rather than merely the existence of four states; and assembly chirality requires analysis of the stabilizer of the completed stacked realization rather than the symmetry of an isolated chrysene node. The molecular branch of Equation (570) therefore separates properties already encoded by the foundation from properties that depend upon additional realization-specific structure.

The same distinction governs the operator and admissibility branches. A spectral triple supplies the analytic setting in which spectral dimension and metric reconstruction may be investigated, but neither quantity is fixed by the existence of the triple alone. Likewise, a physical Hamiltonian makes directional transfer well defined without requiring the two directional transfer functions to differ, and a noncocommutative coproduct cannot be discussed until such a coproduct has first been constructed. In the admissibility branch, metric disconnection, Hilbert-space projection, and dynamical removal remain successive mathematical layers. This organization makes the deferred program modular: failure of a theorem at a later stage need not invalidate the foundational structure on which that theorem was posed.

11.12 Status of the Principal Chrysene Claims

The logical status of the principal results after the present restatement may be summarized as follows:

Statement	Status in the revised formalism
Canonical chrysene carbon framework	Established by the finite carbon-framework characterization theorem; no longer deferred.
$\mathbb{Z}^3$ -fibrated tensor network	Axiomatic structure in A_6 .
Noncommutative sequence algebra	Axiomatic construction in A_7 , with noncommutativity proved from translation/localization covariance.
2, 5, 8, 11 dual tensor and commutant	Axiomatic structure in A_8 .
Four-state structural alphabet	Axiomatic consequence of the dual-cipher and registry coordinates.
Spectral triple	Axiomatic structure in A_9 .
$N_y = N_x + 2$	Deferred theorem requiring a two-unit registry offset.
$GF(4)$ field multiplication	Deferred theorem requiring a structurally induced multiplication.
Directional transfer anisotropy	Deferred theorem requiring inequivalent Hamiltonian or transfer moments.
Optical/polarization anisotropy	Deferred theorem requiring inequivalent spectral-response data.
$d_D^{str} = d_C^*$	Deferred metric-compatibility theorem.
Real spectral structure / KO data	Deferred theorem concerning J_{dc} and $D_{\mathbb{C}}$.
Spectral dimension $d_{sp} = 3$	Deferred theorem requiring Dirac eigenvalue-growth estimates.
Supramolecular chirality	Deferred theorem requiring evaluation of the assembled stabilizer.
Noncocommutative coproduct and R -matrix	Optional higher-algebraic theorem program; not part of the twelve axioms.
Physical scission or excision	Deferred dynamical theorem; projection alone is insufficient.

The summary separates three logically different kinds of statement. First are results now established within the present article, most notably the canonical chrysene carbon-framework characterization and the non-vacuity of the complete twelve-axiom architecture. Second are structures deliberately placed at the axiomatic level, including the $\mathbb{Z}^3$ -indexed tensor network, sequence algebra, dual-cipher structure, and spectral triple. Their presence in the foundation does not mean that every stronger property associated with those structures has already been derived. The distinction is important because the axioms define the mathematical setting in which later propositions may be formulated, whereas the characterization and existence results are claims proved about that setting.

The remaining entries identify the principal boundaries of the present paper. They are not omitted consequences of the twelve axioms but explicit theorem targets whose proofs require additional hypotheses or realization data. Thus the four-state

alphabet is established without a distinguished field multiplication, directional Hamiltonians are defined without assuming transfer or optical anisotropy, and both spectral and admissible path geometries exist without being identified. Similarly, assembly chirality, higher-algebraic braiding, and physical scission require structures not supplied merely by the isolated molecular framework. Recording these distinctions in one place makes the scope of the revised formalism explicit and prevents deferred research objectives from being mistaken for assumptions of the foundation.

11.13 Role of the Deferred Program

The purpose of the theorem program is not to weaken the Chrysene Formalism. It is to separate statements of different logical types.

The foundation now fixes

chrysene molecular architecture
+
$\mathbb{Z}^3$ -fibrated tensor structure
+
sequence-dependent noncommutative algebra
+
dual-cipher commutant
+
spectral noncommutative geometry
+
physical operator and admissibility domains.

(571)

The specialized properties

$$N_y = N_x + 2, \quad GF(4), \quad \mathcal{T}_x \not\equiv_u \mathcal{T}_y, \quad \alpha_{xx} \not\equiv \alpha_{yy},$$

together with metric compatibility, real spectral structure, higher-algebraic braiding, emergent chirality, and dynamical scission, are therefore posed as explicit mathematical questions over that foundation.

This separation gives the subsequent Chrysene program the form

fixed foundation + explicit realization hypotheses
$\Rightarrow$ independently falsifiable theorems.

(572)

12 Relation to the Earlier Chrysene Formalism

The present formulation is a reconstruction of the earlier Chrysene Formalism rather than an independent abstract system bearing only a historical relation to chrysene. It restores several structures that were weakened or omitted in the preceding restatement while retaining the logical corrections that motivated that revision.

The governing principle is

restore primitive structure
without restoring unsupported implication

(573)

Thus the chrysene molecular architecture, tensor space, $\mathbb{Z}^3$ fibration, dual cipher, noncommutative sequence algebra, and spectral triple are returned to the foundational architecture. By contrast, quantum-channel isolation, $GF(4)$ multiplication, $N + 2$ closure, optical anisotropy, higher-algebraic braiding, and physical scission remain derived or conditional.

The revised concordance therefore contains both restorations and logical demotions. A concept is restored when it can be given a precise foundational definition without forcing an unsupported physical consequence. A stronger historical claim is retained only as a theorem target when its validity depends upon additional geometric, spectral, algebraic, or dynamical hypotheses.

13 Discussion

13.1 What the Reconstruction Restores

The principal change from the preceding restatement is that the Chrysene Formalism is once again explicitly a molecularly anchored, tensorial, three-dimensional, noncommutative-geometric construction.

The molecular graph is not arbitrary; it belongs to a restricted four-ring polyhex comparison class. The 3, 6, 9, 12 primary structure and 2, 5, 8, 11 dual cipher are again part of the foundational language. The planar bridge-closure equations are embedded within a full $\mathbb{Z}^3$ fibration, and the phrase *tensor space* denotes an actual tensor-product construction rather than a geometric analogy.

Most importantly, the noncommutative nature of the formalism now has an explicit mathematical origin. Local tensor observables can commute at distinct sites while the complete sequence algebra remains noncommutative because translations alter localization:

$$u_{\mathbf{m}} P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}. \quad (574)$$

This separates sequence noncommutativity from claims about geometrically perpendicular molecular axes.

The spectral triple of Axiom A_9 then turns this operator algebra into a noncommutative metric geometry. The Dirac operator is not identified with an electronic Hamiltonian; it supplies spectral geometry, while physical Hamiltonians are introduced separately in Axiom A_{10} .

13.2 What the Reconstruction Does Not Restore

The present reconstruction does not reinstate every strong conclusion appearing in earlier formulations.

In particular, C_{2h} symmetry is not described as molecular chirality. Because the group contains improper spatial symmetry, the isolated C_{2h} molecular core is achiral. Chirality may emerge only at the level of an assembled structure whose complete stabilizer contains no improper operation.

Likewise, geometric directional inequivalence does not establish quantum-channel isolation, electronic transfer anisotropy, or optical anisotropy. These conclusions concern different mathematical layers and require separate operator or response information.

The native four-state encoding is restored, but its cardinality does not automatically supply multiplication. Consequently,

$$\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^2$$

is foundational, while

$$\Sigma_{\mathfrak{C}} \cong GF(4)$$

as a field remains a theorem about a structurally induced multiplication.

Finally, a constraint boundary does not execute dynamics. The existence of an inadmissible state, an infinite path distance, an orthogonal rejection projection, and an actual physical scission process are four distinct statements.

13.3 Molecular Specificity and Model Plurality

The molecular characterization theorem changes the interpretation of $\text{Mod}(\mathcal{A})$.

All full models possess the same canonical carbon framework:

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \simeq_{\text{px}} G_{\text{chr}}^{\text{px}}. \quad (575)$$

This statement concerns molecular carbon-framework identity, not Euclidean congruence of every realization-specific embedding or uniqueness of every locant decoration.

The standard numbered chrysene realization remains distinguished by Definition 79 and supplies the coordinate frame used by the downstream Chrysene theorem program.

Nevertheless, the complete theory is not categorical. The same canonical chrysene framework may support different embeddings, bridge families, spectral triples, physical Hamiltonians, contact operators, environmental states, response models, and admissibility structures.

Hence

$$\begin{array}{c} \text{unique canonical molecular framework} \\ \Rightarrow \text{unique downstream physics.} \end{array} \quad (576)$$

13.4 Anisotropy as a Layered Property

Directional anisotropy occurs at several logically different levels.

At the canonical molecular level,

$$\widehat{\mathcal{D}}_x \not\sim_{\Gamma_{\mathfrak{C}}^{\text{core}}} \widehat{\mathcal{D}}_y. \quad (577)$$

At the closure level it may appear as unequal bridge counts, registries, or lattice pitches.

At the spectral-geometric level it may appear as

$$\| [D_{\mathfrak{C}}, U_{e_x}] \| \neq \| [D_{\mathfrak{C}}, U_{e_y}] \|. \quad (578)$$

At the physical transport level it may appear as inequivalent transfer functions,

$$\mathcal{T}_x \neq_u \mathcal{T}_y. \quad (579)$$

At the response level it may appear as

$$\alpha_{xx}(\omega) \neq \alpha_{yy}(\omega). \quad (580)$$

No adjacent implication in this sequence is automatic:

$$\begin{array}{c} \text{molecular anisotropy} \\ \Rightarrow \\ \text{closure anisotropy} \\ \Rightarrow \\ \text{spectral/transfer anisotropy} \\ \Rightarrow \\ \text{response anisotropy.} \end{array} \quad (581)$$

13.5 The Role of Falsifiability

The revised architecture localizes failure.

If the predicted two-unit registry difference is not obtained, the $N_y = N_x + 2$ theorem fails for that realization without invalidating the tensor network.

If the x - and y -transfer operators prove unitarily equivalent, the directional-transfer theorem fails without invalidating molecular anisotropy.

If no structurally natural multiplication on $\Sigma_{\mathfrak{C}}$ satisfies the field axioms, the $GF(4)$ theorem fails while the four-state cipher remains intact.

If a candidate Dirac operator does not reproduce the admissible path metric, exact spectral–path equivalence fails without removing the spectral triple itself.

This modularity is a mathematical advantage: increasingly specialized claims can be tested without requiring the entire foundational architecture to stand or fall with each one.

14 Conclusion

The Chrysene Formalism has been reconstructed as a twelve-axiom foundation in which its molecular, tensorial, noncommutative, spectral, and physical layers are mathematically distinguished but structurally connected.

The first three axioms define an embedded four-ring polyhex molecular node with a C_{2h} carbon-framework stabilizer, the distinguished 3, 6, 9, 12 primary attachment set, and intrinsically inequivalent directional data. Axioms A_4 – A_6 extend that node through exact bridge closure into a tensor network over

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3,$$

with

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \xrightarrow{\pi_{xy}} \mathbb{Z}^2.$$

Axiom A_7 constructs the noncommutative sequence algebra

$$\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3,$$

whose noncommutativity follows from the ordered action of translation on localized tensor observables. Axiom A_8 restores the 2, 5, 8, 11 dual tensor, the native four-state cipher, and the primary–dual commutant relation.

Axiom A_9 restores the explicitly noncommutative-geometric character of the formalism through the spectral triple

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}).$$

The final three axioms introduce physical Hamiltonians and effective transfer, causal polarization response, and admissible spectral/path-boundary geometry without identifying those physical constructions with the Dirac geometry by definition.

The molecular-characterization theorem separates canonical carbon-framework uniqueness from realization-level plurality. The finite classification of Appendix A establishes

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \simeq_{\text{px}} G_{\text{chr}}^{\text{px}}(\mathfrak{R} \in \text{Mod}(\mathcal{A})), \quad (582)$$

while the explicit construction of Appendix B establishes

$$\text{Mod}(\mathcal{A}) \neq \emptyset. \quad (583)$$

Thus both the finite-classification and analytic-extension obligations are discharged in the present work.

The standard 3, 6, 9, 12 $\leftrightarrow$ 2, 5, 8, 11 chrysene locant frame remains fixed on the reference molecular node and on the standard-locant realization class used for subsequent Chrysene-specific theorem development. The complete operator-theoretic and physical realization, however, need not be unique.

The reconstruction therefore restores the original molecular, tensor-space, $\mathbb{Z}^3$, dual-cipher, and noncommutative geometric identities of the Chrysene Formalism while retaining the principal logical corrections of the preceding restatement. Specifically,

unique chrysene molecular core + tensorial $\mathbb{Z}^3$ sequence architecture + noncommutative spectral geometry ≠ automatic determination of all downstream physics.

(584)

Relations such as

$$N_y = N_x + 2, \quad \Sigma_{\mathfrak{C}} \cong GF(4), \quad \mathcal{T}_x \neq_u \mathcal{T}_y, \quad \alpha_{xx} \neq \alpha_{yy},$$

together with spectral–path metric reconstruction, real spectral structure, higher-algebraic braiding, emergent assembly chirality, and dynamical scission, remain explicit theorem-level questions.

The resulting formalism is therefore neither a generic operator framework merely inspired by chrysene nor a system in which every historical chrysene claim is true by definition. It is a molecularly anchored tensorial noncommutative geometry whose specialized physical and algebraic consequences can be derived, tested, and, where necessary, falsified over a stable common foundation.

A Reference Chrysene Graph and Complete Four-Ring Classification

This appendix discharges the finite-classification condition used in the molecular-characterization argument. The construction is entirely finite and combinatorial. We first define a canonical coordinate representation for free four-cell polyhexes, enumerate the resulting seven equivalence classes, and compute their reference symmetry groups. We then identify the unique C_{2h} class with the reference chrysene molecular graph and verify the primary–dual cipher and directional conditions required by ChrSig.

No $GF(4)$ multiplication, quantum-channel isolation, optimal-transport condition, Hopf-algebra structure, response anisotropy, or scission mechanism enters the classification.

A.1 Canonical coordinates for the hexagonal tiling

Let

$$\mathbb{H}_{\text{cell}} := \mathbb{Z}^2 \quad (585)$$

denote the axial-coordinate set for the cells of the regular hexagonal tiling.

For

$$h = (q, r) \in \mathbb{Z}^2,$$

the six edge-neighboring cells are

$$\mathcal{N}(h) = \{(q+1, r), (q+1, r-1), (q, r-1), (q-1, r), (q-1, r+1), (q, r+1)\}. \quad (586)$$

A finite cell set

$$S \subset \mathbb{H}_{\text{cell}}$$

is called *cell connected* when the graph obtained by joining edge-neighboring cells is connected.

A.1.1 Dihedral action

Let

$$R(q, r) := (-r, q + r), \quad (587)$$

$$S(q, r) := (q, -q - r). \quad (588)$$

Then

$$R^6 = I, \quad S^2 = I, \quad SRS = R^{-1}, \quad (589)$$

so R and S generate the dihedral symmetry group D_6 of the regular hexagonal tiling.

Two finite cell sets $S, T \subset \mathbb{Z}^2$ are called *freely polyhex equivalent* when there exist $g \in D_6$ and $\mathbf{t} \in \mathbb{Z}^2$ such that

$$T = gS + \mathbf{t}. \quad (590)$$

Definition 87 (Translation normalization). *For a nonempty finite cell set $S \subset \mathbb{Z}^2$, let*

$$(q_0, r_0) := \min_{\text{lex}} S.$$

Define

$$\nu(S) := \{(q - q_0, r - r_0) : (q, r) \in S\}. \quad (591)$$

Definition 88 (Canonical free-polyhex representative). *Define*

$$\kappa(S) := \min_{\text{lex}} \{\nu(gS) : g \in D_6\}. \quad (592)$$

Thus

$$\kappa(S) = \kappa(T)$$

if and only if S and T represent the same free polyhex class.

A.2 Exhaustive enumeration of four-cell polyhexes

Let

$$\mathcal{P}_n$$

denote the set of canonical representatives of the free, cell-connected n -cell polyhexes.

Set

$$\mathcal{P}_1 := \{(0, 0)\}. \quad (593)$$

For a finite cell set S , define its cell frontier by

$$\partial_{\text{cell}} S := \left(\bigcup_{h \in S} \mathcal{N}(h) \right) \setminus S. \quad (594)$$

Recursively define

$$\mathcal{P}_{n+1} := \{\kappa(S \cup \{h\}) : S \in \mathcal{P}_n, h \in \partial_{\text{cell}} S\}. \quad (595)$$

Lemma 50 (Completeness of the recursive enumeration). *Equation (595) produces every free cell-connected $(n + 1)$ -cell polyhex exactly up to free-polyhex equivalence.*

Proof. Let T be any cell-connected $(n + 1)$ -cell polyhex. Its cell-adjacency graph is a finite connected graph. Every finite connected graph with at least two vertices possesses a vertex whose deletion leaves a connected graph. Choose a cell $h \in T$ corresponding to such a vertex.

Then

$$T \setminus \{h\}$$

is a cell-connected n -cell polyhex. By induction, its free-polyhex class has a representative

$$S \in \mathcal{P}_n.$$

Under the corresponding tiling isometry and translation, the deleted cell h becomes an edge-neighbor of S , hence belongs to

$$\partial_{\text{cell}} S.$$

Therefore the class of T appears among the sets generated in Equation (595). Finally, κ identifies exactly those generated sets differing by a translation or an element of D_6 . $\square$

Direct application of the recursion gives

$$|\mathcal{P}_1| = 1, \quad |\mathcal{P}_2| = 1, \quad |\mathcal{P}_3| = 3, \quad |\mathcal{P}_4| = 7. \quad (596)$$

A convenient set of canonical trihex representatives is

$$T_1 := \{(0, 0), (0, 1), (0, 2)\}, \quad (597)$$

$$T_2 := \{(0, 0), (0, 1), (1, -1)\}, \quad (598)$$

$$T_3 := \{(0, 0), (0, 1), (1, 0)\}. \quad (599)$$

The canonical children produced from their cell frontiers are

$$T_1 \longrightarrow \{S_1, S_2, S_3\}, \quad (600)$$

$$T_2 \longrightarrow \{S_2, S_3, S_4, S_5, S_6, S_7\}, \quad (601)$$

$$T_3 \longrightarrow \{S_3, S_5\}. \quad (602)$$

After canonicalization, the complete set $\mathcal{P}_4$ therefore consists of the following seven classes:

$$S_1 = \{(0, 0), (0, 1), (0, 2), (0, 3)\}, \quad (603)$$

$$S_2 = \{(0, 0), (0, 1), (0, 2), (1, -1)\}, \quad (604)$$

$$S_3 = \{(0, 0), (0, 1), (0, 2), (1, 0)\}, \quad (605)$$

$$S_4 = \{(0, 0), (0, 1), (1, -2), (1, -1)\}, \quad (606)$$

$$S_5 = \{(0, 0), (0, 1), (1, -1), (1, 0)\}, \quad (607)$$

$$S_6 = \{(0, 0), (0, 1), (1, -1), (1, 1)\}, \quad (608)$$

$$S_7 = \{(0, 0), (1, -2), (1, -1), (2, -1)\}. \quad (609)$$

Theorem 51 (Complete free-tetrihex enumeration). *Up to translation, rotation, and reflection of the regular hexagonal tiling, there exist exactly seven connected four-cell polyhexes:*

$$\mathcal{P}_4 = \{S_1, \dots, S_7\}. \quad (610)$$

Proof. Completeness follows from Lemma 50. Equations (600)–(602) give the complete canonicalized offspring of the three free trihex classes. Their union contains precisely the seven distinct canonical sets (603)–(609). $\square$

A.3 Exact carbon-skeleton construction

To avoid floating-point ambiguity, associate each cell

$$h = (q, r)$$

with the integer center coordinate

$$c_{\mathbb{Z}}(q, r) := (3q, 2r + q). \quad (611)$$

Let

$$C := \{(2, 0), (1, 1), (-1, 1), (-2, 0), (-1, -1), (1, -1)\}. \quad (612)$$

The six carbon vertices of the cell h are

$$V_h := c_{\mathbb{Z}}(q, r) + C. \quad (613)$$

The exact Euclidean realization is

$$\epsilon(X, Y) := \left(\frac{X}{2}, \frac{\sqrt{3}Y}{2}, 0 \right). \quad (614)$$

For a cell set S , define

$$V(G(S)) := \bigcup_{h \in S} V_h, \quad (615)$$

with edges joining cyclically consecutive corner coordinates of every hexagonal cell.

Coincident vertices and coincident edges are identified. This gives the carbon-skeleton graph

$$G(S).$$

Lemma 52 (Bipartiteness). *Each*

$$G(S_i), \quad i = 1, \dots, 7,$$

is bipartite.

Proof. Each graph is a finite subgraph of the bipartite hexagonal tiling graph. Restriction of the tiling bipartition gives a bipartition of the induced carbon skeleton. $\square$

A.4 Symmetry classification of the seven polyhexes

For a canonical cell set S_i , define its reference planar stabilizer by

$$\Gamma_i^{\text{px}} := \{g \in D_6 : v(gS_i) = S_i\}. \quad (616)$$

Because the reference carbon embedding is planar, reflection σ_h through the molecular plane is also a symmetry of the unsubstituted carbon framework. Define the corresponding three-dimensional reference point group by

$$\Pi_i^{\text{px}} := \langle \Gamma_i^{\text{px}}, \sigma_h \rangle. \quad (617)$$

Remark 56 (Meaning of the point-group labels). *The Schoenflies symbols in Table 1 refer to conjugacy classes of finite orthogonal subgroups of $O(3)$, not merely to abstract group-isomorphism types.*

Accordingly, the C_{2h} classification criterion used below is

$$\Pi^{\text{px}} \sim_{O(3)} C_{2h}^{\text{std}},$$

rather than abstract group isomorphism $\Pi^{\text{px}} \cong C_{2h}$.

A direct evaluation of the twelve elements of D_6 gives the classification in Table 1.

Remark 57 (Chemical names versus graph classes). *The symbols $S_1, \dots, S_7$ are retained as the primary identifiers because $\mathcal{B}_4$ is a combinatorial graph comparison class rather than a catalogue of neutral isolable hydrocarbons. Where an unambiguous conventional chemical identification is useful, the corresponding familiar skeleton may be noted parenthetically; no such chemical name enters the finite-classification proof.*

For reproducibility, the stabilizing elements of D_6 are

$$\Gamma_1^{\text{px}} = \{I, R^3, S, R^3S\}, \quad (618)$$

$$\Gamma_2^{\text{px}} = \{I\}, \quad (619)$$

$$\Gamma_3^{\text{px}} = \{I\}, \quad (620)$$

$$\Gamma_4^{\text{px}} = \{I, R^3\}, \quad (621)$$

$$\Gamma_5^{\text{px}} = \{I, R^3, RS, R^4S\}, \quad (622)$$

$$\Gamma_6^{\text{px}} = \{I, S\}, \quad (623)$$

$$\Gamma_7^{\text{px}} = \{I, R^2, R^4, RS, R^3S, R^5S\}. \quad (624)$$

Proposition 53 (Unique C_{2h} reference tetrihex). *Among all free four-cell polyhexes in $\mathcal{B}_4$, the class S_4 is the unique class whose three-dimensional reference carbon-framework*

Table 1: Complete symmetry classification of the seven free four-cell polyhexes. V_A, V_B denote the two carbon-skeleton bipartition classes. The final column records whether the reference three-dimensional carbon-framework group is C_{2h} .

Class	$ V , E $	(V_A , V_B)	Planar stabilizer	Reference point group	C_{2h}
S_1	18, 21	9, 9	D_2	D_{2h}	No
S_2	18, 21	9, 9	C_1	C_s	No
S_3	17, 20	9, 8	C_1	C_s	No
S_4	18, 21	9, 9	C_2	C_{2h}	Yes
S_5	16, 19	8, 8	D_2	D_{2h}	No
S_6	18, 21	9, 9	C_s	C_{2v}	No
S_7	18, 21	9, 9	D_3	D_{3h}	No

point group satisfies

$$\Pi_4^{\text{px}} \sim_{O(3)} C_{2h}^{\text{std}}. \quad (625)$$

Proof. By Theorem 51, every free four-cell polyhex is represented by exactly one of

$$S_1, \dots, S_7.$$

The explicit D_6 -stabilizer calculation gives

$$\Gamma_4^{\text{px}} = \{I, R^3\} \cong C_2.$$

After adjoining reflection through the molecular plane, the resulting orthogonal point group is conjugate in $O(3)$ to C_{2h}^{std} . The remaining six reference point groups are those listed in Table 1, and none is $O(3)$ -conjugate to C_{2h}^{std} . Hence S_4 is unique. $\square$

A.5 External identification of the surviving polyhex

Let

$$S_\star := S_4 = \{(0, 0), (0, 1), (1, -2), (1, -1)\} \quad (626)$$

and let

$$G_\star := G(S_\star) \quad (627)$$

be its carbon-skeleton graph.

At this stage $G_\star$ is defined purely by the finite polyhex enumeration; it has not yet been identified by chemical name.

Definition 89 (External chrysene connectivity graph). *Let G_{ext} be the carbon-connectivity graph encoded by the canonical SMILES for chrysene reported by PubChem:*

$$\text{C1=CC=C2C(=C1)C=CC3=C2C=CC4=CC=CC=C43.} \quad (628)$$

PubChem identifies this structure as chrysene, CID 9171, molecular formula $C_{18}H_{12}$ [Nat26a]. The same molecular

identity and standard connectivity descriptor are independently reported by the NIST Chemistry WebBook [Nat26b].

Proposition 54 (External chrysene identity certificate). *The polyhex carbon graph $G_\star$ is graph-isomorphic to the externally identified chrysene connectivity graph:*

$$G_\star \cong G_{\text{ext}}. \quad (629)$$

Proof. Number the eighteen carbon atoms appearing in Equation (628) in their SMILES parsing order by

$$c_1, \dots, c_{18}.$$

Define $\Phi : V(G_\star) \rightarrow V(G_{\text{ext}})$ by

$$\begin{aligned} & (v_2, v_3, v_4, w_3, w_1, v_1, v_{12}, v_{11}, w_2, \\ & \quad w_5, v_5, v_6, w_6, v_7, v_8, v_9, v_{10}, w_4) \\ & \quad \mapsto (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, \\ & \quad \quad c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}). \end{aligned} \quad (630)$$

Parsing the bond and ring-closure symbols in Equation (628) gives the carbon neighborhoods

$$N(c_1) = \{c_2, c_6\}, \quad N(c_2) = \{c_1, c_3\}, \quad (631)$$

$$N(c_3) = \{c_2, c_4\}, \quad N(c_4) = \{c_3, c_5, c_{10}\}, \quad (632)$$

$$N(c_5) = \{c_4, c_6, c_7\}, \quad N(c_6) = \{c_1, c_5\}, \quad (633)$$

$$N(c_7) = \{c_5, c_8\}, \quad N(c_8) = \{c_7, c_9\}, \quad (634)$$

$$N(c_9) = \{c_8, c_{10}, c_{18}\}, \quad N(c_{10}) = \{c_4, c_9, c_{11}\}, \quad (635)$$

$$N(c_{11}) = \{c_{10}, c_{12}\}, \quad N(c_{12}) = \{c_{11}, c_{13}\}, \quad (636)$$

$$N(c_{13}) = \{c_{12}, c_{14}, c_{18}\}, \quad N(c_{14}) = \{c_{13}, c_{15}\}, \quad (637)$$

$$N(c_{15}) = \{c_{14}, c_{16}\}, \quad N(c_{16}) = \{c_{15}, c_{17}\}, \quad (638)$$

$$N(c_{17}) = \{c_{16}, c_{18}\}, \quad N(c_{18}) = \{c_9, c_{13}, c_{17}\}. \quad (639)$$

Under Φ , these neighborhoods agree exactly with the adjacency certificate for $G_\star$ given below. Thus Φ preserves every carbon-carbon edge and is a graph isomorphism. $\square$

Corollary 7 (Identification of the reference molecular graph). *The unique C_{2h} four-cell polyhex survivor is the carbon-connectivity graph of chrysene. Accordingly, henceforth define*

$$S_{\text{chr}} := S_{\star}, \quad G_{\text{chr}} := G_{\star}. \quad (640)$$

The locant orientation used below is chosen to agree with the established fused-ring numbering convention for chrysene [Mos98].

Remark 58 (Chrysene locant convention). *The numerical locants used below follow the standard fused-ring numbering convention for chrysene [Mos98]. The coordinate table fixes the orientation of that numbering explicitly for the reference embedding. The atom order appearing in the SMILES string used for the external connectivity certificate is a graph-parsing order only and is not used as the chrysene locant order.*

A.5.1 Distinguished carbon coordinates

The twelve numerically labeled perimeter carbons are fixed by

j	(X_j, Y_j)	j	(X_j, Y_j)
1	$(-2, 2)$	7	$(5, -3)$
2	$(-1, 3)$	8	$(4, -4)$
3	$(1, 3)$	9	$(2, -4)$
4	$(2, 2)$	10	$(1, -3)$
5	$(4, 0)$	11	$(-1, -1)$
6	$(5, -1)$	12	$(-2, 0)$

(641)

and

$$\widehat{\mathbf{r}}_j := \widehat{\iota}_{\text{px}}(v_j) = \epsilon(X_j, Y_j). \quad (642)$$

The six remaining fusion vertices are denoted

$$w_1 := (-1, 1), \quad w_2 := (1, -1), \quad w_3 := (1, 1), \quad (643)$$

$$w_4 := (2, -2), \quad w_5 := (2, 0), \quad w_6 := (4, -2), \quad (644)$$

where the displayed pairs again denote integer coordinates prior to application of ϵ .

A.5.2 Explicit adjacency certificate

The complete reference adjacency is equivalently specified by

$$\begin{aligned} N(v_1) &= \{v_2, w_1\}, & N(v_7) &= \{w_6, v_8\}, \\ N(v_2) &= \{v_1, v_3\}, & N(v_8) &= \{v_7, v_9\}, \\ N(v_3) &= \{v_2, v_4\}, & N(v_9) &= \{v_8, v_{10}\}, \\ N(v_4) &= \{v_3, w_3\}, & N(v_{10}) &= \{v_9, w_4\}, \\ N(v_5) &= \{w_5, v_6\}, & N(v_{11}) &= \{w_2, v_{12}\}, \\ N(v_6) &= \{v_5, w_6\}, & N(v_{12}) &= \{v_{11}, w_1\}, \\ N(w_1) &= \{v_{12}, v_1, w_3\}, & N(w_4) &= \{v_{10}, w_2, w_6\}, \\ N(w_2) &= \{v_{11}, w_4, w_5\}, & N(w_5) &= \{v_5, w_2, w_3\}, \\ N(w_3) &= \{v_4, w_1, w_5\}, & N(w_6) &= \{v_6, v_7, w_4\}. \end{aligned} \quad (645)$$

This adjacency list fixes the reference graph used by Definition 74.

A.5.3 Reference perimeter cycle

The external perimeter cycle is

$$\partial G_{\text{chr}} = (v_1, v_2, v_3, v_4, w_3, w_5, v_5, v_6, w_6, v_7, v_8, v_9, v_{10}, w_4, w_2, v_{11}, v_{12}, w_1). \quad (646)$$

The orientation in Equation (646) is the reference perimeter orientation used for the dual-cipher construction.

A.6 Balanced bipartition of the reference graph

A bipartition of the reference chrysene carbon graph is

$$V_A = \{v_1, v_3, v_5, v_8, v_{10}, v_{12}, w_2, w_3, w_6\}, \quad (647)$$

$$V_B = \{v_2, v_4, v_6, v_7, v_9, v_{11}, w_1, w_4, w_5\}. \quad (648)$$

Hence

$$|V_A| = |V_B| = 9. \quad (649)$$

Thus the reference chrysene graph is balanced bipartite. As shown by Table 1, balanced bipartiteness is not asserted universally for the entire comparison class $\mathcal{B}_4$.

A.7 Primary–dual perimeter-cipher certificate

The primary and dual locant sets are

$$S_{\text{pri}} = \{3, 6, 9, 12\}, \quad (650)$$

$$S_{\text{dual}} = \{2, 5, 8, 11\}. \quad (651)$$

Inspection of Equation (646) gives

$$\begin{aligned} \{v_3, v_2\} &\in E(\partial G_{\text{chr}}), \\ \{v_6, v_5\} &\in E(\partial G_{\text{chr}}), \\ \{v_9, v_8\} &\in E(\partial G_{\text{chr}}), \\ \{v_{12}, v_{11}\} &\in E(\partial G_{\text{chr}}). \end{aligned} \quad (652)$$

Consequently, the map

$$\boxed{3 \mapsto 2, \quad 6 \mapsto 5, \quad 9 \mapsto 8, \quad 12 \mapsto 11} \quad (653)$$

is a uniform one-step displacement along the oriented external perimeter.

Thus the dual cipher is a graph-theoretically realized perimeter operation of G_{chr} , not an arbitrary reassignment of locant labels.

A.8 Canonical directional independence and orbit inequivalence

The directional quantities used in the finite molecular classification are those of the canonical regular-polyhex reference embedding, rather than those of an arbitrary realization-specific embedding.

Recall from Equation (642) that

$$\widehat{\mathbf{r}}_j = \widehat{\mathbf{t}}_{\text{px}}(v_j).$$

The two canonical primary directional vectors are therefore

$$\begin{aligned} \widehat{\mathbf{d}}_x &:= \widehat{\mathbf{r}}_9 - \widehat{\mathbf{r}}_3 \\ &= \left(\frac{1}{2}, -\frac{7\sqrt{3}}{2}, 0 \right), \end{aligned} \quad (654)$$

$$\begin{aligned} \widehat{\mathbf{d}}_y &:= \widehat{\mathbf{r}}_6 - \widehat{\mathbf{r}}_{12} \\ &= \left(\frac{7}{2}, -\frac{\sqrt{3}}{2}, 0 \right). \end{aligned} \quad (655)$$

Consequently,

$$\boxed{\widehat{\mathbf{d}}_x \times \widehat{\mathbf{d}}_y = 12\sqrt{3} \mathbf{e}_z \neq \mathbf{0}.} \quad (656)$$

Thus the two canonical primary directions are linearly independent, verifying condition (C4) of the chrysene characterization predicate.

For reference, their Euclidean invariants are

$$\|\widehat{\mathbf{d}}_x\|_2 = \sqrt{37}, \quad (657)$$

$$\|\widehat{\mathbf{d}}_y\|_2 = \sqrt{13}, \quad (658)$$

$$\widehat{\mathbf{d}}_x \cdot \widehat{\mathbf{d}}_y = 7. \quad (659)$$

In particular, the canonical reference geometry is not assumed to possess exact Euclidean orthogonality between the two primary directions.

A.8.1 Action of the canonical C_2 generator

The nontrivial planar symmetry of the surviving polyhex $\mathcal{S}_4$ is the half-turn

$$\boxed{\rho(X, Y) := (3 - X, -1 - Y).} \quad (660)$$

On the numerical locants this symmetry induces

$$\boxed{(1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12).} \quad (661)$$

Hence

$$\{v_3, v_9\} \quad (662)$$

and

$$\{v_{12}, v_6\} \quad (663)$$

form distinct orbits of the canonical planar C_2 action.

Define the canonical reference directional data by

$$\widehat{\mathcal{D}}_x := (v_3, v_9; \widehat{\mathbf{d}}_x), \quad (664)$$

$$\widehat{\mathcal{D}}_y := (v_{12}, v_6; \widehat{\mathbf{d}}_y). \quad (665)$$

No element of the canonical C_{2h} carbon-framework stabilizer $\Gamma_{\mathbb{C}_{\text{chr}}}^{\text{core}}$ sends simultaneously

$$v_3 \mapsto v_{12}, \quad v_9 \mapsto v_6. \quad (666)$$

Therefore the equivalence required to exchange the two reference directional data cannot occur:

$$\boxed{\widehat{\mathcal{D}}_x \not\sim_{\Gamma_{\mathbb{C}_{\text{chr}}}^{\text{core}}} \widehat{\mathcal{D}}_y.} \quad (667)$$

This verifies condition (C5) of the chrysene characterization predicate.

The conclusion is already forced by the action of the canonical carbon-framework stabilizer on the distinguished vertices and reference displacement vectors. No assumption concerning realization-specific attachment tangents is needed.

The quantities

$$\widehat{\mathbf{d}}_x, \quad \widehat{\mathbf{d}}_y, \quad \widehat{\mathcal{D}}_x, \quad \widehat{\mathcal{D}}_y$$

belong to the canonical molecular-classification layer. They are distinct from the unhatted realization-dependent quantities

$$\mathbf{d}_x, \quad \mathbf{d}_y, \quad \mathcal{D}_x^{\text{emb}}, \quad \mathcal{D}_y^{\text{emb}}$$

introduced in Axiom 3.

A.9 Named bay- and K-region perimeter geometry

The reference chrysene perimeter contains two chemically distinct types of named local perimeter geometry that may be useful in later registry, stacking, folding, and directional-closure constructions.

The terminology used here follows the standard distinction in polycyclic-aromatic-hydrocarbon geometry between inward-concave *bay regions* and the distinct *K-region* bonds.

Definition 90 (Chrysene bay regions). *The two inward-concave bay regions of the reference chrysene framework are the symmetry-related clefts associated with the 4–5 and 10–11 perimeter neighborhoods.*

Define

$$\mathcal{B}_{\text{bay}}(\mathfrak{C}_{\text{chr}}) := \left\{ B_{\text{bay}}^{(4,5)}, B_{\text{bay}}^{(10,11)} \right\}. \quad (668)$$

Under the canonical C_{2h} carbon-framework symmetry, these two bay regions belong to the same symmetry orbit.

Definition 91 (Chrysene K-region bonds). *The geometrically distinct, locally exposed perimeter bonds associated with the 5–6 and 11–12 neighborhoods are recorded as the reference chrysene K-region pair:*

$$\mathcal{K}_{\text{reg}}(\mathfrak{C}_{\text{chr}}) := \left\{ K^{(5,6)}, K^{(11,12)} \right\}. \quad (669)$$

These two K-region bonds likewise form a symmetry-related pair in the canonical carbon framework.

For later use, define the complete named perimeter-motif set by

$$\mathcal{M}_{\partial}(\mathfrak{C}_{\text{chr}}) := \mathcal{B}_{\text{bay}}(\mathfrak{C}_{\text{chr}}) \sqcup \mathcal{K}_{\text{reg}}(\mathfrak{C}_{\text{chr}}). \quad (670)$$

Remark 59 (Geometric status of the named perimeter motifs). *The sets*

$$\mathcal{B}_{\text{bay}}(\mathfrak{C}_{\text{chr}}) \quad \text{and} \quad \mathcal{K}_{\text{reg}}(\mathfrak{C}_{\text{chr}})$$

record named local geometry of the reference molecular reduct.

They do not imply a rotational operator

$$R(\theta),$$

a prescribed angular displacement, folding transformation, stacking offset, steric potential, electronic coupling, or energetic preference.

Any such consequence requires additional geometric, chemical, energetic, electronic, or dynamical hypotheses in the relevant realization.

A.10 Unconditional finite-classification theorem

Recall the characterization predicate

$$\text{ChrSig}(\mathfrak{C})$$

of Definition 76 and the characterization node class

$$\mathfrak{N}_4^{\text{CF}} = \{ \mathfrak{C} \in \mathfrak{N}_4 : \text{ChrSig}(\mathfrak{C}) \} \quad (671)$$

of Definition 77.

Theorem 55 (Finite Chrysene Carbon-Framework Classification Theorem). *For the canonical regular-polyhex embeddings defined in this appendix,*

$$\left\{ [\text{CanGraph}(\mathfrak{C})]_{\approx_{\text{px}}} : \mathfrak{C} \in \mathfrak{N}_4^{\text{CF}} \right\} = \left\{ [G_{\text{chr}}^{\text{px}}]_{\approx_{\text{px}}} \right\}. \quad (672)$$

Hence the finite-classification condition (FC), understood as canonical carbon-framework classification, holds unconditionally.

Proof. Let

$$\mathfrak{C} \in \mathfrak{N}_4^{\text{CF}}.$$

By condition (C1),

$$G_{\mathfrak{C}} \in \mathcal{B}_4.$$

By Theorem 51, its free polyhex class is represented by one of

$$\mathcal{S}_1, \dots, \mathcal{S}_7.$$

Condition (C2) requires

$$\Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}}.$$

By Proposition 53, only the class $\mathcal{S}_4$ has this canonical reference point group.

By Proposition 54, the carbon graph generated by $\mathcal{S}_4$ is graph-isomorphic to the externally specified chrysene carbon-connectivity graph. Therefore

$$\text{CanGraph}(\mathfrak{C}) \approx_{\text{px}} G_{\text{chr}}^{\text{px}}. \quad (673)$$

Conversely, the standard numbered reference chrysene node satisfies conditions (C1)–(C5) by the explicit perimeter-cipher and directional certificates of this appendix.

Thus the canonical carbon-framework quotient contains exactly the reference chrysene class. $\square$

Corollary 8 (Uniqueness of the canonical molecular framework). *Every full twelve-axiom model has canonical carbon-framework reduct polyhex-congruent to the reference chrysene framework:*

$$\text{CanGraphRed}_{\text{mol}}(\mathfrak{R}) \approx_{\text{px}} G_{\text{chr}}^{\text{px}} \quad (\mathfrak{R} \in \text{Mod}(\mathcal{A})). \quad (674)$$

A.11 Reproducibility of the finite enumeration

The classification above can be reproduced using only integer arithmetic.

Starting from

$$\mathcal{P}_1 = \{ \{(0,0)\} \},$$

repeat the following operation until $n = 4$:

1. for each $S \in \mathcal{P}_n$, construct the frontier $\partial_{\text{cell}} S$;
2. for every $h \in \partial_{\text{cell}} S$, form $S \cup \{h\}$;
3. apply all twelve elements of D_6 ;
4. translation-normalize each image using ν ;
5. retain the lexicographically minimal image $\kappa(S \cup \{h\})$;
6. delete duplicate canonical representatives.

The resulting class counts are

$$1, 1, 3, 7.$$

For every member of $\mathcal{P}_4$, the carbon graph is generated exactly by Equations (612)–(614), and the reference stabilizer is determined by testing the twelve elements

$$I, R, \dots, R^5, S, RS, \dots, R^5 S.$$

Thus both the exhaustion and the symmetry classification are finite and independently reproducible.

B Explicit Twelve-Axiom Chrysene Model

The finite classification of Appendix A identifies the reference chrysene molecular node $\mathfrak{C}_{\text{chr}}$ as the unique canonical chrysene carbon framework satisfying the molecular characterization conditions.

It remains to establish that this molecular reduct admits at least one full extension satisfying Axioms A_1 – A_{12} . The purpose of the present appendix is to construct such an extension explicitly.

The construction is an existence witness. It is not intended to represent a quantitatively realistic electronic, optical, or supramolecular model of a synthesized chrysene material. In particular, the physical Hamiltonians and response operators are chosen in the simplest form sufficient to satisfy the corresponding axioms.

We denote the resulting full model by

$$\mathfrak{R}_{\text{ex}}. \quad (675)$$

B.1 Molecular reduct: Axioms A_1 – A_3

Set

$$\mathfrak{C} := \mathfrak{C}_{\text{chr}}, \quad (676)$$

where $\mathfrak{C}_{\text{chr}}$ is the standard numbered reference chrysene node specified in Appendix A.

Thus

$$G_{\mathfrak{C}} = G_{\text{chr}} \in \mathcal{B}_4. \quad (677)$$

Use the canonical regular-polyhex embedding

$$\iota := \widehat{\iota}_{\text{px}}. \quad (678)$$

Define λ_{loc} by the standard chrysene locant convention fixed in Appendix A.

Explicit chemical and attachment labels. For the present existence witness, choose

$$\mathcal{L}_{\text{chem}} := \{\ell_C, \ell_{CC}\} \quad (679)$$

and a one-element non-null attachment-label set

$$\mathcal{L}_{\text{att}} := \{a\}, \quad (680)$$

with the augmented attachment set

$$\overline{\mathcal{L}}_{\text{att}} := \mathcal{L}_{\text{att}} \sqcup \{\emptyset\}. \quad (681)$$

Define the carbon-framework chemical labeling by

$$\lambda_{\text{chem}}(v) := \ell_C, \quad v \in V_{\mathfrak{C}}, \quad (682)$$

$$\lambda_{\text{chem}}(e) := \ell_{CC}, \quad e \in E_{\mathfrak{C}}. \quad (683)$$

Define the attachment labeling by

$$\lambda_{\text{att}}(v) := \begin{cases} a, & v \in A_{\mathfrak{C}}^{\text{pri}}, \\ \emptyset, & v \notin A_{\mathfrak{C}}^{\text{pri}}. \end{cases} \quad (684)$$

Let the orthogonal action on the attachment labels be trivial:

$$\rho_R(a) = a, \quad \rho_R(\emptyset) = \emptyset, \quad R \in O(3). \quad (685)$$

Thus every primary attachment site carries the same specified non-null attachment label, while all remaining carbon sites carry the null attachment label. These choices supply explicitly the chemical and attachment data required by Axiom A_1 .

By Appendix A, the canonical reference point group satisfies

$$\Pi_{\mathfrak{C}}^{\text{core}} \sim_{O(3)} C_{2h}^{\text{std}}. \quad (686)$$

The canonical directional data satisfy

$$\widehat{\mathbf{d}}_x \times \widehat{\mathbf{d}}_y \neq \mathbf{0} \quad (687)$$

and

$$\widehat{\mathcal{D}}_x \not\sim_{\Gamma_{\mathfrak{C}}^{\text{core}}} \widehat{\mathcal{D}}_y \quad (688)$$

by Appendix A.8.

Because the witness chooses

$$\iota = \widehat{\iota}_{\text{px}},$$

one has

$$\mathbf{r}_j = \widehat{\mathbf{r}}_j, \quad \mathbf{d}_\alpha = \widehat{\mathbf{d}}_\alpha \quad (\alpha \in \{x, y\}). \quad (689)$$

Thus the canonical and embedded displacement geometry coincide in this particular existence witness.

Let

$$\mathbf{c}_{\text{mol}} := \frac{1}{|V_{\mathbb{C}}|} \sum_{v \in V_{\mathbb{C}}} \iota(v) \quad (690)$$

be the molecular centroid. For the four primary attachment sites define

$$\mathbf{t}_j := \frac{\mathbf{r}_j - \mathbf{c}_{\text{mol}}}{\|\mathbf{r}_j - \mathbf{c}_{\text{mol}}\|_2}, \quad j \in \{3, 6, 9, 12\}. \quad (691)$$

Because the witness uses the canonical embedding, uniform carbon-framework chemical labels, and the common attachment label on the primary set, every fully labeled symmetry relevant to this witness is contained in the canonical reference stabilizer.

Therefore the canonical directional inequivalence above also excludes an embedded labeled symmetry exchanging the two ordered primary directions. Together with the covariant centroidal tangent construction, this verifies the embedded clause of Axiom A_3 .

Hence Axioms A_1 – A_3 are satisfied.

B.2 Bridge closure and the $\mathbb{Z}^3$ network: Axioms A_4 – A_6

Let

$$D_{\text{mol}} := \max_{v, w \in V_{\mathbb{C}}} \|\iota(v) - \iota(w)\|_2 \quad (692)$$

be the Euclidean diameter of the reference molecular embedding.

Choose

$$L > D_{\text{mol}} + 1 \quad (693)$$

and define

$$\mathbf{a}_x := L\mathbf{e}_x, \quad (694)$$

$$\mathbf{a}_y := L\mathbf{e}_y, \quad (695)$$

$$\mathbf{a}_z := L\mathbf{e}_z. \quad (696)$$

Then

$$\det(\mathbf{a}_x \quad \mathbf{a}_y \quad \mathbf{a}_z) = L^3 \neq 0. \quad (697)$$

For each $\alpha \in \{x, y\}$, set

$$\mathcal{N}_\alpha := \{1\}, \quad \mathbf{c}_\alpha := \mathbf{0}. \quad (698)$$

Explicit bridge-interface compatibility data. Choose the bridge-terminal label set

$$\mathcal{L}_{\mathcal{B}} := \{b^-, b^+\}. \quad (699)$$

For each $\alpha \in \{x, y\}$, define the compatibility relation of Definition 14 to be the singleton

$$\text{Comp}_\alpha := \{(b^-, b^+; a, a; \mathbf{t}_{v_\alpha^+}, \mathbf{t}_{v_\alpha^-})\}. \quad (700)$$

Thus, in this existence witness, compatibility requires the ordered bridge terminals to carry the labels (b^-, b^+) and the two molecular interface vertices to carry the common non-null attachment label a , together with the already specified ordered interface tangents.

Define

$$\mathbf{B}_\alpha(1) := \mathbf{a}_\alpha - \mathbf{d}_\alpha. \quad (701)$$

Let $B_\alpha(1)$ be the two-vertex one-edge graph

$$p_\alpha^- \text{ --- } p_\alpha^+$$

embedded by

$$\iota_{B_\alpha}(p_\alpha^-) = \mathbf{0}, \quad (702)$$

$$\iota_{B_\alpha}(p_\alpha^+) = \mathbf{B}_\alpha(1), \quad (703)$$

and equipped with the terminal-label map

$$\lambda_{B_\alpha}(p_\alpha^-) = b^-, \quad \lambda_{B_\alpha}(p_\alpha^+) = b^+. \quad (704)$$

By Equations (684) and (700), the tuple

$$(\lambda_{B_\alpha}(p_\alpha^-), \lambda_{B_\alpha}(p_\alpha^+); \lambda_{\text{att}}(v_\alpha^+), \lambda_{\text{att}}(v_\alpha^-); \mathbf{t}_{v_\alpha^+}, \mathbf{t}_{v_\alpha^-}) \quad (705)$$

belongs to Comp_α . Therefore

$$B_\alpha(1) \in \mathbb{B}_\alpha(\mathbb{C}). \quad (706)$$

Hence the bridge-generation map

$$B_\alpha : \mathcal{N}_\alpha \longrightarrow \mathbb{B}_\alpha(\mathbb{C}), \quad 1 \longmapsto B_\alpha(1), \quad (707)$$

is explicitly well typed.

Remark 60 (Chemical status of the witness bridge). *The single edge of $B_\alpha(1)$ is an abstract connector edge introduced solely for the explicit existence construction. It is not assigned the carbon–carbon chemical label ℓ_{CC} used for edges of the reference chrysene carbon skeleton, and its Euclidean terminal displacement*

$$\mathbf{B}_\alpha(1)$$

is not asserted to be a physical covalent-bond length. Accordingly, this one-edge bridge is not a claim that a

corresponding single-bond molecular linker is synthetically realizable. A chemical instantiation of the bridge requires an independently specified linker structure, bonding pattern, geometry, and energetic model.

The closure residual is then

$$\rho_\alpha(1) = \mathbf{a}_\alpha - \mathbf{d}_\alpha - \mathbf{c}_\alpha - \mathbf{B}_\alpha(1) \quad (708)$$

$$= \mathbf{0}. \quad (709)$$

Thus

$$\boxed{N_x^\star = N_y^\star = 1} \quad (710)$$

is an exact-closure choice.

The translation lattice is

$$\Lambda_{\mathfrak{C}} = L\mathbb{Z}^3. \quad (711)$$

The global placement

$$\iota_{\mathfrak{R}}(\mathbf{k}, v) = \iota(v) + L\mathbf{k} \quad (712)$$

is injective. Indeed, if two vertices in distinct lattice cells coincided, then a nonzero lattice displacement of norm at least L would equal the difference of two molecular vertices, whose norm is at most $D_{\text{mol}} < L$, a contradiction.

Assign the constant stacking registry

$$\boxed{\eta_{\mathbf{k}} := 0 \in \mathbb{Z}_2} \quad (713)$$

for every z -edge.

The translation action is

$$U_{\ell}(\mathbf{k}, v) := (\mathbf{k} + \ell, v). \quad (714)$$

Hence Axioms A_4 – A_6 are satisfied.

B.3 Explicit local tensor data

Choose

$$V_3 = V_9 = V_{12} = V_6 = V_{z^-} = V_{z^+} = \mathbb{C}. \quad (715)$$

For the internal structural factor take

$$\boxed{K_{\mathfrak{C}} := K_{\text{aux}} \otimes K_{\text{dc}} \otimes K_z := \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4.} \quad (716)$$

Therefore

$$\mathbb{T}_{\mathfrak{C}} \cong \mathbb{C}^4. \quad (717)$$

Choose

$$\boxed{\Psi_{\mathfrak{C}} := 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes |0\rangle_{\text{dc}} \otimes |0\rangle_z.} \quad (718)$$

The directional contractions are ordinary multiplication:

$$C_x(z \otimes w) := zw, \quad (719)$$

$$C_y(z \otimes w) := zw, \quad (720)$$

$$C_z^{(\eta)}(z \otimes w) := zw \quad (\eta \in \mathbb{Z}_2). \quad (721)$$

All three contraction maps are nonzero.

Thus the tensor-network structures required by Axiom A_6 are explicitly realized.

B.4 The quasi-local sequence algebra: Axiom A_7

For every finite $\Lambda \in \mathbb{Z}^3$, define

$$\mathcal{K}_{\Lambda} := \bigotimes_{\mathbf{k} \in \Lambda} \mathbb{C}^4 \quad (722)$$

and

$$\mathfrak{Q}_{\Lambda} := B(\mathcal{K}_{\Lambda}) \cong M_{4^{|\Lambda|}}(\mathbb{C}). \quad (723)$$

The canonical embeddings

$$A \mapsto A \otimes I$$

give the quasi-local UHF algebra

$$\boxed{\mathfrak{Q}_{\mathfrak{C}} := \overline{\lim_{\Lambda \in \mathbb{Z}^3} \mathfrak{Q}_{\Lambda}}.} \quad (724)$$

The lattice shift defines

$$\alpha : \mathbb{Z}^3 \longrightarrow \text{Aut}(\mathfrak{Q}_{\mathfrak{C}}) \quad (725)$$

by

$$\alpha_{\mathbf{m}}(\iota_{\mathbf{k}}(A)) := \iota_{\mathbf{k}+\mathbf{m}}(A). \quad (726)$$

Define the reduced crossed product

$$\boxed{\mathfrak{A}_{\mathfrak{C}} := \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha, r} \mathbb{Z}^3.} \quad (727)$$

Since $\mathbb{Z}^3$ is amenable, the reduced and full crossed products agree canonically, so Equation (727) realizes the crossed product of Axiom A_7 .

For finite Λ , let

$$\tau_{\Lambda}(A) := 4^{-|\Lambda|} \text{Tr}(A). \quad (728)$$

These traces are compatible with the inductive-limit embeddings and therefore determine a faithful tracial state

$$\tau : \mathfrak{Q}_{\mathfrak{C}} \longrightarrow \mathbb{C}. \quad (729)$$

Let

$$E : \mathfrak{A}_{\mathfrak{C}} \longrightarrow \mathfrak{Q}_{\mathfrak{C}} \quad (730)$$

be the canonical conditional expectation, defined on algebraic crossed-product elements by

$$E\left(\sum_{\mathbf{m}} A_{\mathbf{m}} u_{\mathbf{m}}\right) := A_0. \quad (731)$$

Define

$$\boxed{\omega := \tau \circ E.} \quad (732)$$

Lemma 56 (Faithfulness of the explicit state). *The state ω is faithful and tracial on $\mathfrak{A}_{\mathfrak{C}}$.*

Proof. The product trace τ is faithful on the quasi-local UHF algebra and invariant under every lattice shift $\alpha_{\mathbf{m}}$. The canonical expectation E on the reduced crossed product is faithful. Hence

$$\omega(A^*A) = \tau(E(A^*A)) = 0$$

implies

$$E(A^*A) = 0,$$

and therefore $A = 0$.

Shift invariance of τ implies that $\tau \circ E$ is tracial on the crossed product. $\square$

Thus the state required by Axiom A_7 is explicitly realized.

B.5 Dual tensor and commutant realization: Axiom A_8

Choose

$$V_2 = V_5 = V_8 = V_{11} = \mathbb{C}. \quad (733)$$

The unitary interface identifications

$$D_3, D_6, D_9, D_{12}$$

are chosen to be the canonical unitary maps between the corresponding one-dimensional copies of $\mathbb{C}$.

This defines the unitary cipher map

$$\mathcal{D}_{\text{dc}} : \mathbb{T}_{\mathbb{C}} \longrightarrow \mathbb{T}'_{\mathbb{C}} \quad (734)$$

and the nonzero dual tensor

$$\Psi'_{\mathbb{C}} := \mathcal{D}_{\text{dc}} \Psi_{\mathbb{C}}. \quad (735)$$

The perimeter realization

$$3 \mapsto 2, \quad 6 \mapsto 5, \quad 9 \mapsto 8, \quad 12 \mapsto 11 \quad (736)$$

is verified in Appendix A.7.

The native structural alphabet is

$$\Sigma_{\mathbb{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z. \quad (737)$$

On $K_{\text{dc}} \cong \mathbb{C}^2$, define

$$F_{\text{dc}}|0\rangle_{\text{dc}} = |1\rangle_{\text{dc}}, \quad F_{\text{dc}}|1\rangle_{\text{dc}} = |0\rangle_{\text{dc}}. \quad (738)$$

Then

$$F_{\text{dc}}^2 = I.$$

Set

$$\Delta_0 := I_{\text{aux}} \otimes F_{\text{dc}} \otimes I_z. \quad (739)$$

Its localized copy at $\mathbf{k}$ is

$$\Delta_{\mathbf{k}} := \iota_{\mathbf{k}}(\Delta_0). \quad (740)$$

B.5.1 Explicit commutant conjugation

Let

$$\mathcal{H}_{\omega} := L^2(\mathfrak{A}_{\mathbb{C}}, \omega) \quad (741)$$

be the GNS Hilbert space of the faithful trace, and let

$$\pi_{\omega}(A)[B] := [AB] \quad (742)$$

be the left regular representation.

On the dense algebraic GNS subspace define

$$J_{\text{dc}}[A] := [A^*]. \quad (743)$$

Because ω is tracial, J_{dc} extends to a conjugate-linear isometric involution on $\mathcal{H}_{\omega}$, and

$$J_{\text{dc}}^2 = I. \quad (744)$$

Moreover,

$$J_{\text{dc}}\pi_{\omega}(A)J_{\text{dc}} \quad (745)$$

acts by right multiplication by A^* .

Consequently,

$$J_{\text{dc}}\mathfrak{M}_{\text{pri}}J_{\text{dc}} = \mathfrak{M}'_{\text{pri}}, \quad (746)$$

where

$$\mathfrak{M}_{\text{pri}} = \pi_{\omega}(\mathfrak{A}_{\mathbb{C}})''. \quad (747)$$

Thus every structural requirement of Axiom A_8 is realized.

B.6 A compact-resolvent spectral realization: Axiom A_9

The spectral construction uses the same faithful GNS representation as the preceding subsection.

Define the algebraic crossed-product core

$$\mathfrak{A}_{\mathbb{C}}^{\text{alg}} := \left\{ \sum_{\mathbf{m} \in F} A_{\mathbf{m}} u_{\mathbf{m}} : \begin{array}{l} F \subseteq \mathbb{Z}^3, \\ A_{\mathbf{m}} \in \mathfrak{Q}_{\text{loc}} \end{array} \right\}. \quad (748)$$

This is a norm-dense unital involutive subalgebra of $\mathfrak{A}_{\mathbb{C}}$.

For $n \in \mathbb{Z}_{\geq 0}$, let

$$\Lambda_n := \{\mathbf{k} \in \mathbb{Z}^3 : \|\mathbf{k}\|_{\infty} \leq n\}. \quad (749)$$

Define

$$\mathcal{F}_n := \text{span} \{ [A u_{\mathbf{m}}] : A \in \mathfrak{Q}_{\Lambda_n}, \quad \|\mathbf{m}\|_{\infty} \leq n \} \subseteq \mathcal{H}_{\omega}. \quad (750)$$

Each $\mathcal{F}_n$ is finite dimensional, and

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \quad (751)$$

Furthermore,

$$\overline{\bigcup_{n=0}^{\infty} \mathcal{F}_n}^{\|\cdot\|_{\mathcal{H}_\omega}} = \mathcal{H}_\omega. \quad (752)$$

Let P_n be the orthogonal projection onto $\mathcal{F}_n$. Set

$$P_{-1} := 0 \quad (753)$$

and

$$Q_n := P_n - P_{n-1}. \quad (754)$$

Then

$$Q_n Q_m = 0 \quad (n \neq m)$$

and

$$\sum_{n=0}^{\infty} Q_n = I$$

in the strong operator topology.

Definition 92 (Explicit Chrysene Dirac operator). *Define*

$$D_{\mathfrak{C}} := \sum_{n=0}^{\infty} n Q_n \quad (755)$$

on the domain

$$\text{Dom}(D_{\mathfrak{C}}) := \left\{ \psi \in \mathcal{H}_\omega : \sum_{n=0}^{\infty} n^2 \|Q_n \psi\|^2 < \infty \right\}. \quad (756)$$

Proposition 57 (Self-adjointness and compact resolvent). *The operator $D_{\mathfrak{C}}$ is densely defined and self-adjoint, and*

$$(I + D_{\mathfrak{C}}^2)^{-1/2} \in \mathcal{K}(\mathcal{H}_\omega). \quad (757)$$

Proof. The finite-dimensional subspaces

$$Q_n \mathcal{H}_\omega$$

are pairwise orthogonal and their algebraic direct sum is dense by Equation (752).

Thus $D_{\mathfrak{C}}$ is the diagonal self-adjoint operator with eigenvalue n on $Q_n \mathcal{H}_\omega$.

Moreover,

$$(I + D_{\mathfrak{C}}^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{1+n^2}} Q_n. \quad (758)$$

Every Q_n has finite rank and

$$(1 + n^2)^{-1/2} \longrightarrow 0.$$

Hence the displayed operator is the norm limit of the finite-rank operators

$$\sum_{n=0}^N \frac{1}{\sqrt{1+n^2}} Q_n,$$

and is therefore compact. $\square$

B.6.1 Bounded commutators

Lemma 58 (Finite filtration propagation). *For every*

$$A \in \mathfrak{A}_{\mathfrak{C}}^{\text{alg}},$$

there exists an integer

$$c(A) \geq 0$$

such that

$$\pi_\omega(A) \mathcal{F}_n \subseteq \mathcal{F}_{n+c(A)}, \quad (759)$$

$$\pi_\omega(A)^* \mathcal{F}_n \subseteq \mathcal{F}_{n+c(A)} \quad (760)$$

for every $n \geq 0$.

Proof. Write

$$A = \sum_{\mathbf{m} \in F} A_{\mathbf{m}} u_{\mathbf{m}} \quad (761)$$

with F finite and each $A_{\mathbf{m}}$ of finite tensor support.

Choose $c(A)$ sufficiently large that:

1. every coefficient $A_{\mathbf{m}}$ and every coefficient occurring in A^* is supported in $\Lambda_{c(A)}$; and
2. every translation index occurring in A or A^* has ℓ^∞ -norm at most $c(A)$.

For $B \in \mathfrak{Q}_{\Lambda_n}$, the crossed-product multiplication law gives

$$(A_{\mathbf{m}} u_{\mathbf{m}})(B u_{\mathbf{p}}) = A_{\mathbf{m}} \alpha_{\mathbf{m}}(B) u_{\mathbf{m}+\mathbf{p}}. \quad (762)$$

The tensor support of $\alpha_{\mathbf{m}}(B)$ lies in $\Lambda_{n+c(A)}$, and

$$\|\mathbf{m} + \mathbf{p}\|_\infty \leq n + c(A).$$

Hence

$$\pi_\omega(A) \mathcal{F}_n \subseteq \mathcal{F}_{n+c(A)}.$$

The same argument applied to A^* proves the second inclusion. $\square$

Lemma 59 (Finite band width). *Let*

$$T := \pi_\omega(A), \quad A \in \mathfrak{A}_{\mathfrak{C}}^{\text{alg}}.$$

Then

$$Q_m T Q_n = 0 \quad \text{whenever } |m - n| > c(A). \quad (763)$$

Proof. If $m > n + c(A)$, Equation (759) shows that $T Q_n \mathcal{H}_\omega$ is contained in $\mathcal{F}_{n+c(A)}$ and therefore orthogonal to $Q_m \mathcal{H}_\omega$. If $n > m + c(A)$, apply the same argument to $T^* = \pi_\omega(A^*)$. $\square$

Proposition 60 (Bounded Dirac commutators). *For every*

$$A \in \mathfrak{A}_\mathbb{C}^{\text{alg}},$$

the commutator

$$[D_\mathbb{C}, \pi_\omega(A)] \quad (764)$$

extends to a bounded operator on $\mathcal{H}_\omega$.

Proof. Let

$$T = \pi_\omega(A)$$

and

$$c = c(A).$$

For $-c \leq k \leq c$, define the k -th operator diagonal

$$T_k := \sum_{n \geq 0} Q_{n+k} T Q_n, \quad (765)$$

where terms with negative indices are omitted.

By Lemma 59,

$$T = \sum_{k=-c}^c T_k. \quad (766)$$

Because the domain and range blocks in each fixed diagonal are orthogonal,

$$\|T_k\| \leq \|T\|. \quad (767)$$

On the algebraic direct sum of the spectral subspaces,

$$[D_\mathbb{C}, T_k] = k T_k. \quad (768)$$

Therefore

$$\|[D_\mathbb{C}, T]\| \leq \sum_{k=-c}^c |k| \|T_k\| \quad (769)$$

$$\leq c(c+1) \|T\|. \quad (770)$$

Thus the commutator extends boundedly to $\mathcal{H}_\omega$. $\square$

Corollary 9 (Spectral core). *The choice*

$$\mathfrak{A}_\mathbb{C}^\circ := \mathfrak{A}_\mathbb{C}^{\text{alg}} \quad (771)$$

is a norm-dense unital involutive spectral core satisfying

Axiom A9.

Moreover,

$$\pi_\omega(A) \text{Dom}(D_\mathbb{C}) \subseteq \text{Dom}(D_\mathbb{C}) \quad (772)$$

for every $A \in \mathfrak{A}_\mathbb{C}^\circ$.

Proof. Density and involutive closure of $\mathfrak{A}_\mathbb{C}^{\text{alg}}$ follow from the algebraic crossed-product construction.

Let

$$A \in \mathfrak{A}_\mathbb{C}^{\text{alg}}, \quad T := \pi_\omega(A),$$

and let

$$B_A \in B(\mathcal{H}_\omega)$$

denote the bounded extension of $[D_\mathbb{C}, T]$ provided by Proposition 60.

Take

$$\psi \in \text{Dom}(D_\mathbb{C})$$

and define the spectral truncations

$$\psi_N := P_N \psi. \quad (773)$$

Because

$$P_N = \sum_{n=0}^N Q_n,$$

the definition of $\text{Dom}(D_\mathbb{C})$ gives

$$\psi_N \longrightarrow \psi \quad (774)$$

and

$$D_\mathbb{C} \psi_N \longrightarrow D_\mathbb{C} \psi \quad (775)$$

in $\mathcal{H}_\omega$.

By Lemma 58, there exists $c(A)$ such that

$$T \psi_N \in \mathcal{F}_{N+c(A)} \subset \text{Dom}(D_\mathbb{C}).$$

On the finite-filtration core,

$$D_\mathbb{C} T \psi_N = T D_\mathbb{C} \psi_N + B_A \psi_N. \quad (776)$$

The right-hand side converges to

$$T D_\mathbb{C} \psi + B_A \psi.$$

At the same time,

$$T \psi_N \longrightarrow T \psi.$$

Since $D_\mathbb{C}$ is closed, it follows that

$$T \psi \in \text{Dom}(D_\mathbb{C}) \quad (777)$$

and

$$D_\mathbb{C} T \psi = T D_\mathbb{C} \psi + B_A \psi. \quad (778)$$

Therefore

$$\pi_\omega(A) \text{Dom}(D_{\mathfrak{C}}) \subseteq \text{Dom}(D_{\mathfrak{C}}),$$

and the bounded commutator agrees on the full Dirac domain with B_A .

Hence

$$\mathfrak{A}_{\mathfrak{C}}^{\circ} := \mathfrak{A}_{\mathfrak{C}}^{\text{alg}}$$

is a spectral core satisfying Axiom A_9 . $\square$

Remark 61 (The explicit Dirac operator is not translation invariant). *The compact-resolvent operator $D_{\mathfrak{C}}$ is filtration confining and is not required to commute with lattice translations.*

For every translation generator $u_{\mathbf{m}}$, Proposition 60 nevertheless gives

$$[D_{\mathfrak{C}}, \pi_\omega(u_{\mathbf{m}})] \in B(\mathcal{H}_\omega). \quad (779)$$

Thus the infinite $\mathbb{Z}^3$ base and compact-resolvent spectral geometry coexist without compactifying the molecular network.

Consequently,

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_\omega, D_{\mathfrak{C}}) \quad (780)$$

is a spectral triple satisfying Axiom A_9 .

B.7 Explicit physical motif Hamiltonians: Axiom A_{10}

For

$$\alpha \in \{x, y\}, \quad N = 1,$$

choose the motif observable algebra

$$\mathfrak{A}_{\alpha,1}^{\text{mot}} := \mathbb{C}I \subseteq \mathfrak{A}_{\mathfrak{C}}^{\circ}. \quad (781)$$

Let

$$\mathcal{H}_{L,\alpha} = \mathcal{H}_{B,\alpha}^{(1)} = \mathcal{H}_{R,\alpha} = \mathbb{C}. \quad (782)$$

Thus

$$\mathcal{H}_\alpha^{(1)} \cong \mathbb{C}^3. \quad (783)$$

Represent $\mathbb{C}I$ by scalar multiplication:

$$\varpi_{\alpha,1}(\lambda I) := \lambda I_3. \quad (784)$$

Choose

$$H_{L,\alpha} = 0, \quad (785)$$

$$H_{B,\alpha}^{(1)} = 0, \quad (786)$$

$$H_{R,\alpha} = 0, \quad (787)$$

$$W_{L,\alpha}^{(1)} = 1, \quad (788)$$

$$W_{R,\alpha}^{(1)} = 1, \quad (789)$$

$$\mathcal{J}_\alpha^{(1)} = 0. \quad (790)$$

Thus

$$H_\alpha^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (791)$$

which is bounded and self-adjoint.

The choice

$$\mathcal{J}_\alpha^{(1)} = 0$$

is the explicit strictly bridge-mediated assumption for this witness model.

For

$$z \neq 0,$$

the bridge resolvent is

$$R_{B,\alpha}^{(1)}(z) = z^{-1} \quad (792)$$

and the effective transfer operator is

$$\mathcal{T}_\alpha^{(1)}(z) = \frac{1}{z}. \quad (793)$$

Hence Axiom A_{10} is satisfied.

B.8 Explicit linear-response realization: Axiom A_{11}

Choose

$$\mathfrak{A}_{\text{resp}} := \mathbb{C}I, \quad \mathcal{H}_{\text{resp}} := \mathbb{C}, \quad (794)$$

with the scalar representation.

Let

$$H_{\text{resp}} := 0 \quad (795)$$

and choose

$$|0\rangle := 1. \quad (796)$$

Define

$$\mu_x = \mu_y = \mu_z := 0. \quad (797)$$

Then

$$\mu_i(t) = 0$$

for all t , and therefore

$$\alpha_{ij}(z) = 0 \quad (798)$$

for every

$$i, j \in \{x, y, z\}, \quad \text{Im } z > 0.$$

Axiom A_{11} requires the existence of the causal response construction, not a nonzero or anisotropic response.

Hence Axiom A_{11} is satisfied.

B.9 Explicit admissibility geometry: Axiom A_{12}

Let the structural state set be

$$\mathcal{X} := \{x_0\}. \quad (799)$$

Define the spectral realization by

$$J_{\text{str}}(x_0) := \omega. \quad (800)$$

This map is injective because its domain is a singleton.

Let

$$\mathcal{E}_0 := \emptyset. \quad (801)$$

The unique function

$$w : \emptyset \longrightarrow (0, \infty)$$

is a symmetric positive edge-weight function.

Choose one constraint functional

$$C_1(x_0) := 0 \quad (802)$$

and threshold

$$\tau_1 := 1. \quad (803)$$

Then

$$\Omega_{\text{adm}} = \{x_0\}. \quad (804)$$

The trivial path at x_0 gives

$$d_C^*(x_0, x_0) = 0. \quad (805)$$

The spectral distance likewise satisfies

$$d_D^{\text{str}}(x_0, x_0) = d_D(\omega, \omega) = 0. \quad (806)$$

No additional equality between the two metric constructions is required.

Hence Axiom A_{12} is satisfied.

B.10 Explicit extension theorem

Theorem 61 (Explicit Twelve-Axiom Chrysene Model). *The structure $\mathfrak{R}_{\text{ex}}$ constructed in Sections B.1–B.9 is a full model of Axioms A_1 – A_{12} with molecular reduct $\mathfrak{C}_{\text{chr}}$. Consequently,*

$$\mathfrak{R}_{\text{ex}} \in \text{Ext}_{12}(\mathfrak{C}_{\text{chr}}). \quad (807)$$

Proof. Axioms A_1 – A_3 hold by Section B.1 and the verified molecular classification of Appendix A. Axioms A_4 – A_6 hold by the bridge and lattice construction of Section B.2; in particular,

$$\rho_x(1) = \rho_y(1) = \mathbf{0}$$

and the three lattice translations are linearly independent.

Axiom A_7 is realized by the quasi-local UHF algebra $\mathfrak{Q}_{\mathfrak{C}}$, the shift action α , the crossed product $\mathfrak{A}_{\mathfrak{C}}$, and the faithful tracial state ω .

Axiom A_8 holds by the explicitly defined primary–dual tensor spaces, cipher flip, four-state structural alphabet, and conjugate-linear commutant involution J_{dc} .

Axiom A_9 holds by Proposition 57, Proposition 60, and Corollary 9.

Axiom A_{10} holds by the bounded self-adjoint motif Hamiltonians and resolvent transfer operators of Section B.7.

Axiom A_{11} holds by the causal zero-response model of Section B.8.

Finally, Axiom A_{12} holds by the singleton admissibility model of Section B.9.

Thus all twelve axioms are simultaneously satisfied. $\square$

Corollary 10 (Analytic extension condition). *The analytic extension condition is satisfied:*

$$\text{Ext}_{12}(\mathfrak{C}_{\text{chr}}) \neq \emptyset. \quad (808)$$

Hence condition (AE) need no longer be retained as a hypothesis.

Remark 62 (Logical role of the explicit model). *The model $\mathfrak{R}_{\text{ex}}$ is an existence witness, not a claim that the chosen bridge Hamiltonians, zero dipole operators, constant z-registry, or singleton admissibility space constitute a chemically realistic description of a physical chrysene material. Its role is solely to prove*

$$\text{Ext}_{12}(\mathfrak{C}_{\text{chr}}) \neq \emptyset.$$

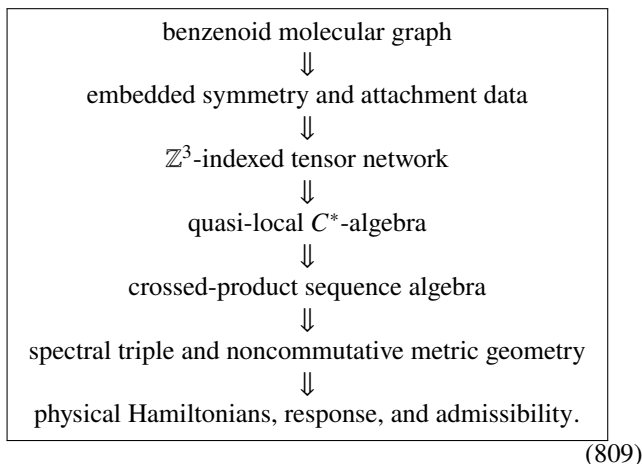
More specialized chrysene realizations may replace any of these minimal downstream choices while retaining the same molecular reduct and twelve-axiom architecture.

C Mathematical Literature and Relation to Established Theory

The revised Chrysene Formalism combines mathematical structures drawn from graph theory, benzenoid graph theory, multilinear algebra, operator algebras, crossed-product C^* -algebras, noncommutative geometry, spectral theory, finite-field theory, linear-response theory, and metric geometry. None of these mathematical subjects is claimed to be new in isolation. The purpose of this appendix is to identify the established mathematical literature supporting the individual constructions and to distinguish that established machinery from the specific organization proposed in the Chrysene Formalism.

The mathematical architecture developed in the main text may

be summarized schematically as



C.1 Molecular Graphs, Polyhexes, and Symmetry

The molecular foundation of Axioms A_1 – A_3 belongs first to the mathematical theory of finite graphs. Standard graph-theoretic notions used in the present work include graph isomorphism, automorphism groups, connectedness, cycles, paths, induced subgraphs, and graph embeddings; see, for example, Diestel [Die17].

For benzenoid hydrocarbons, the carbon skeleton can be represented combinatorially through fused hexagonal systems. The mathematical enumeration and structural analysis of benzenoid systems has an established literature; a systematic treatment is given by Gutman and Cyvin [GC89]. This literature motivates the use of a finite four-ring polyhex comparison class $\mathcal{B}_4$ in the present formalism.

The introduction of $\mathcal{B}_4$ does not imply that the standard theory of benzenoid hydrocarbons contains the Chrysene Characterization Theorem proved or proposed here. Rather, established benzenoid graph theory supplies the finite combinatorial domain in which the required classification can be performed.

The symmetry stabilizers

$$\Gamma_{\mathfrak{C}} \quad \text{and} \quad \Gamma_{\mathfrak{C}}^{\text{core}}$$

are group actions by graph-compatible Euclidean transformations. Finite group representations and their invariant structures belong to classical representation theory; see Serre [Ser77]. In the present work, however, the particular use of a C_{2h} carbon-framework stabilizer together with the distinguished 3, 6, 9, 12 and 2, 5, 8, 11 locant data is specific to the Chrysene Formalism.

It is also important that symmetry and chirality remain distinct. The group C_{2h} contains improper spatial symmetry and therefore corresponds to an achiral isolated molecular realization under the definition used in this article. Directional inequivalence of attachment data is a separate orbit property and need not imply chirality.

C.2 Tensor Products and the $\mathbb{Z}^3$ Tensor Network

The tensorial layer of the formalism uses ordinary finite-dimensional complex tensor products. For a local molecular node the structural space has the form

$$\mathbb{T}_{\mathfrak{C}} = V_3 \otimes V_9 \otimes V_{12} \otimes V_6 \otimes V_{z^-} \otimes V_{z^+} \otimes K_{\mathfrak{C}}. \quad (810)$$

For a finite subset $\Lambda \in \mathbb{Z}^3$, the corresponding regional Hilbert space is

$$\mathcal{K}_{\Lambda} = \bigotimes_{\mathbf{k} \in \Lambda} \mathcal{K}_{\mathbf{k}}. \quad (811)$$

Such tensor-product constructions and the associated local operator algebras are standard in the operator-algebraic formulation of extended quantum systems; see Bratteli and Robinson [BR87].

The discrete base

$$X_{\mathfrak{C}} \cong \mathbb{Z}^3 \quad (812)$$

is separated in the present formalism from the tensor fiber carried over each lattice coordinate. Thus the complete Chrysene Tensor Space is not identified literally with $\mathbb{Z}^3$. Instead, $\mathbb{Z}^3$ indexes translated copies of the local structural tensor.

The decomposition

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \xrightarrow{\pi_{xy}} \mathbb{Z}^2 \quad (813)$$

is an elementary short exact sequence of abelian groups at the level used here, with

$$\ker \pi_{xy} \cong \mathbb{Z}.$$

The terminology *fibration* refers to this discrete base–fiber organization and does not assert the existence of a smooth fiber bundle.

C.3 Quasi-Local Operator Algebras and Inductive Limits

For finite $\Lambda \in \mathbb{Z}^3$, the local observable algebra is

$$\mathfrak{Q}_{\Lambda} = B(\mathcal{K}_{\Lambda}). \quad (814)$$

If $\Lambda \subseteq \Lambda'$, the canonical embedding

$$A \mapsto A \otimes I_{\Lambda' \setminus \Lambda} \quad (815)$$

defines a directed system of finite-dimensional C^* -algebras.

The norm-completed inductive limit

$$\mathfrak{Q}_{\mathfrak{C}} = \varinjlim_{\Lambda \in \mathbb{Z}^3} \mathfrak{Q}_{\Lambda} \quad (816)$$

is a standard quasi-local construction of the type used in operator-algebraic statistical mechanics [BR87].

This established theory supports the mathematical existence of the quasi-local algebra. The particular interpretation of its local factors as structural degrees of freedom associated with substituted chrysene is additional model structure introduced in the present formalism.

C.4 Crossed Products and Sequence-Dependent Noncommutativity

Let a discrete group act on a C^* -algebra through automorphisms. The corresponding crossed-product construction is a standard part of C^* -algebra theory; a comprehensive treatment is given by Williams [Wil07].

In the present notation the translation action is

$$\alpha : \mathbb{Z}^3 \longrightarrow \text{Aut}(\mathfrak{Q}_{\mathfrak{C}}), \quad (817)$$

and the resulting sequence algebra is

$$\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3. \quad (818)$$

The covariance relation

$$u_{\mathbf{m}} A u_{\mathbf{m}}^* = \alpha_{\mathbf{m}}(A) \quad (819)$$

is the standard crossed-product relation. For a localized observable,

$$\alpha_{\mathbf{m}}(P_{\mathbf{k}}) = P_{\mathbf{k}+\mathbf{m}}, \quad (820)$$

and hence

$$u_{\mathbf{m}} P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}. \quad (821)$$

When $P_{\mathbf{k}+\mathbf{m}} \neq P_{\mathbf{k}}$, this relation gives

$$[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0. \quad (822)$$

The general crossed-product machinery is established mathematics. The specific interpretation of this noncommutativity as *sequence dependence* of the Chrysene Tensor Space is the organizational use made of that mathematics in this formalism.

C.5 C^* -Algebras, GNS Representations, and Commutants

States, representations, commutants, bicommutants, and von Neumann closures belong to the standard theory of operator algebras [BR87, Tak02].

Given a state

$$\omega : \mathfrak{A}_{\mathfrak{C}} \longrightarrow \mathbb{C}, \quad (823)$$

the Gelfand–Naimark–Segal construction provides a cyclic representation

$$(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega}) \quad (824)$$

such that

$$\omega(A) = \langle \Omega_{\omega}, \pi_{\omega}(A) \Omega_{\omega} \rangle. \quad (825)$$

The represented von Neumann algebra is

$$\mathfrak{M}_{\omega} = \pi_{\omega}(\mathfrak{A}_{\mathfrak{C}})''. \quad (826)$$

The commutant

$$\mathfrak{M}'_{\omega} = \{B \in B(\mathcal{H}_{\omega}) : [A, B] = 0 \text{ for all } A \in \mathfrak{M}_{\omega}\} \quad (827)$$

and the bicommutant relation are classical operator-algebraic constructions [Tak02].

Tomita–Takesaki theory supplies a particularly important precedent for conjugate-linear operators implementing commutant relations in suitable standard von Neumann-algebraic representations [Tak02]. The present axiom

$$J_{\text{dc}} \mathfrak{M}_{\text{pri}} J_{\text{dc}} = \mathfrak{M}'_{\text{pri}} \quad (828)$$

should therefore be read as an imposed dual-cipher commutant structure compatible with this operator-algebraic language. Standard operator-algebra theory does not itself identify a molecular 2, 5, 8, 11 substitution pattern with a commutant; that identification is specific to Axiom A_8 .

C.6 Connes' Noncommutative Geometry and Spectral Triples

The principal mathematical framework for interpreting a noncommutative algebra as a geometry is the spectral formulation of noncommutative geometry developed by Connes [Con94].

In the form used in this article, a spectral triple consists of a dense involutive algebra $\mathfrak{A}_{\mathfrak{C}}^{\circ}$, a faithful representation on a Hilbert space $\mathcal{H}_{\omega}$, and a densely defined self-adjoint operator $D_{\mathfrak{C}}$:

$$(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}). \quad (829)$$

The principal analytic conditions adopted here are

$$[D_{\mathfrak{C}}, \pi_{\omega}(a)] \in B(\mathcal{H}_{\omega}) \quad (a \in \mathfrak{A}_{\mathfrak{C}}^{\circ}) \quad (830)$$

and compact resolvent, expressed equivalently in the present formulation as

$$(I + D_{\mathfrak{C}}^2)^{-1/2} \in \mathcal{K}(\mathcal{H}_{\omega}). \quad (831)$$

These are standard spectral-triple conditions in noncommutative geometry [Con94].

The Dirac commutator defines the seminorm

$$L_D(a) := \|[D_{\mathfrak{C}}, \pi_{\omega}(a)]\|, \quad (832)$$

and the associated spectral distance between states is

$$d_D(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a = a^*, L_D(a) \leq 1 \}. \quad (833)$$

The interpretation of Lipschitz seminorms as metrics on state spaces, including in noncommutative settings, was developed further by Rieffel [Rie99]. That literature provides the broader metric context for Equation (833).

The present formalism does not assume that the Connes distance is automatically equal to the independently defined admissible path metric d_C^* . Such equality is a separate metric-reconstruction problem requiring a specifically calibrated Dirac operator.

C.7 Real Spectral Structure and the Dual Cipher

Connes subsequently introduced the notion of a real structure on spectral geometry, represented by an antilinear isometry J , with sign relations involving the Dirac operator and, in the even case, the grading operator [Con95].

The right action of the opposite algebra is represented in the standard real-spectral formalism through

$$b^{\text{op}} \mapsto J\pi(b^*)J^{-1}, \quad (834)$$

together with the zero-order commutation condition and the order-one relation

$$[[D, \pi(a)], J\pi(b^*)J^{-1}] = 0. \quad (835)$$

These structures and their relation to real spectral geometry are treated explicitly by Connes [Con95].

Accordingly, the deferred theorem concerning J_{dc} asks whether the already-defined dual-cipher commutant involution also satisfies the additional compatibility conditions required of a real spectral structure.

This distinction is important. Axiom A_8 already imposes

$$J_{\text{dc}}^2 = I \quad (836)$$

and

$$J_{\text{dc}}\mathfrak{M}_{\text{pri}}J_{\text{dc}} = \mathfrak{M}'_{\text{pri}}. \quad (837)$$

It does not automatically establish

$$J_{\text{dc}}D_{\mathfrak{C}} = \pm D_{\mathfrak{C}}J_{\text{dc}} \quad (838)$$

or the order-one condition.

The use of represented one-forms and bounded inner fluctuations

$$D_{\mathfrak{C}} \mapsto D_{\mathfrak{C}} + A \quad (839)$$

also belongs to the established noncommutative-geometric framework; see Connes [Con94, Con96]. The present article uses this construction only as spectral-geometric structure and does not identify every algebraic one-form with a specific molecular perturbation.

C.8 Self-Adjoint Operators, Resolvents, and Effective Transfer

The physical Hamiltonians of Axiom A_{10} belong to standard functional and spectral analysis. Self-adjoint operators, spectra, resolvent sets, and resolvent identities are treated systematically in Reed and Simon [RS80].

For a bridge Hamiltonian $H_{B,\alpha}^{(N)}$, the resolvent is

$$R_{B,\alpha}^{(N)}(z) = \left(zI - H_{B,\alpha}^{(N)} \right)^{-1}. \quad (840)$$

The elimination of an internal block of a block operator or matrix is naturally expressed through the Schur complement.

The mathematical theory and broad applications of Schur complements are reviewed by Zhang [Zha05].

Accordingly, the effective transfer operator

$$\mathcal{T}_{\alpha}^{(N)}(z) = \mathcal{J}_{\alpha}^{(N)} + W_{L,\alpha}^{(N)\dagger} R_{B,\alpha}^{(N)}(z) W_{R,\alpha}^{(N)} \quad (841)$$

is an instance of standard resolvent–Schur-complement reduction.

What is specific to the Chrysene Formalism is the assignment of such operator data to the directional node–bridge motifs generated from the molecular attachment structure. Standard Schur-complement theory does not imply that the x - and y -directional transfer operators are inequivalent.

C.9 Linear Response and Polarizability

The causal response tensor introduced in Axiom A_{11} belongs to standard linear-response theory. Kubo’s formulation expresses susceptibilities and response functions through time-dependent correlation and commutator functions [Kub57].

In the notation of the present article,

$$\alpha_{ij}(z) = \frac{i}{\hbar} \int_0^{\infty} e^{izt} \langle 0 | [\mu_i(t), \mu_j] | 0 \rangle dt, \quad \text{Im } z > 0, \quad (842)$$

is a standard Kubo-type causal response expression.

Insertion of a spectral resolution gives the associated sum-over-states representation in finite-dimensional realizations. The established linear-response theory therefore provides the mathematical basis for Axiom A_{11} .

It does not imply

$$\alpha_{xx}(\omega) \neq \alpha_{yy}(\omega) \quad (843)$$

merely because the molecular geometry is directionally anisotropic. Such anisotropy depends on the actual excitation spectrum and transition-dipole matrix elements.

C.10 Finite Fields and the Four-State Alphabet

Finite-field theory is classical; a standard comprehensive reference is Lidl and Niederreiter [LN97].

Up to isomorphism, the field with four elements can be represented as

$$GF(4) \cong \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1). \quad (844)$$

Its additive group is isomorphic to

$$\mathbb{Z}_2 \times \mathbb{Z}_2. \quad (845)$$

Axiom A_8 independently produces the structural alphabet

$$\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z. \quad (846)$$

Consequently,

$$(\Sigma_{\mathfrak{C}}, \boxplus) \cong (GF(4), +) \quad (847)$$

as additive groups.

The deferred theorem program will exploit the fact that the four states of $\Sigma_{\mathfrak{C}}$ are not introduced as four unrelated symbols. They arise from two distinguished binary structural coordinates: the primary–dual coordinate supplied by the dual cipher and the independent z -registry coordinate. Thus a state

$$s_{\epsilon, \eta}, \quad (\epsilon, \eta) \in \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z,$$

has an intrinsic two-coordinate origin. This provides a natural four-state, or quaternary, structural encoding without first imposing an external base-four labeling. The role of the dual cipher in the subsequent theorem program is therefore stronger than a mere counting argument: it supplies one of the two binary operations from which the native four-state alphabet is assembled.

The remaining question is whether composition of the cipher and registry operations also determines a distinguished multiplication on these four states. A future construction will seek such an operation directly from the structural sequence rules and then test the field axioms, rather than transporting the multiplication table of $GF(4)$ through an arbitrary bijection. In particular, one may ask whether the nontrivial structural states admit a naturally selected generator whose induced composition satisfies the analogue of $\omega^2 + \omega + 1 = 0$ over $\mathbb{F}_2$. If such a law is derived from the dual-cipher and registry architecture and is shown to satisfy distributivity and the required inverse properties, then the resulting four-state system would realize $GF(4)$ for structural reasons. Establishing that derivation, rather than the abstract existence of a four-element field, is the substantive content of the deferred $GF(4)$ theorem.

C.11 Path Metrics and Admissible Connectivity

The admissible distance of Axiom A_{12} is a weighted path-length construction. Length metrics and path-metric methods belong to standard metric geometry; see Burago, Burago, and Ivanov [BBI01]. The corresponding finite-path language on graphs is standard graph theory [Die17].

For an admissible path

$$\gamma = (x_0, \dots, x_r),$$

the length is

$$L_w(\gamma) = \sum_{k=1}^r w(x_{k-1}, x_k), \quad (848)$$

and the distance

$$d_C^*(x, y) = \inf_{\gamma \in \mathcal{P}_{\text{adm}}(x, y)} L_w(\gamma) \quad (849)$$

is assigned the extended value $+\infty$ when no admissible path exists.

This construction distinguishes a local violation of a constraint from global disconnection of the surviving state graph. It also explains why an algebraic projection onto a rejected Hilbert-space sector is mathematically distinct from a dynamical physical excision mechanism.

C.12 Role of the Established Mathematics

The revised formalism relies on the following established mathematical correspondences:

finite and benzenoid graph theory → molecular comparison class	(850)
group actions → symmetry stabilizers	
tensor products and inductive limits → local and quasi-local tensor structure	
crossed-product C^* -algebras → sequence algebra	
GNS and von Neumann theory → represented observable and commutant structure	
Connes' spectral geometry → noncommutative metric structure	
resolvent and Schur-complement theory → effective transfer	
Kubo linear response → polarization response	
finite-field theory → conditional $GF(4)$ realization	
metric and graph geometry → admissible connectivity	

The novelty asserted for the revised Chrysene Formalism therefore does not lie in the invention of graph automorphisms, tensor products, crossed products, C^* -algebras, spectral triples, Schur complements, Kubo response, finite fields, or path metrics.

Rather, the proposed contribution is their specific axiomatic organization around a molecularly characterized chrysene node:

$ \begin{array}{c} \mathfrak{C}_{\text{chr}} \\ \Downarrow \\ \Psi_{\mathfrak{C}} \\ \Downarrow \\ \mathbb{Z}^3\text{-fibrated tensor network} \\ \Downarrow \\ \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3 \\ \Downarrow \\ (\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}}) \\ \Downarrow \\ \text{physical transfer, response, and admissibility.} \end{array} $	(851)
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Likewise, the standard references cited in this appendix do not prove the chrysene-specific results deferred in the main text, including

$$N_y = N_x + 2, \quad \Sigma_{\mathfrak{C}} \cong GF(4) \text{ as a field,}$$

directional transfer anisotropy, optical anisotropy, spectral–path metric equality, emergent assembly chirality, higher-algebraic braiding, or dynamical scission.

Those statements remain separate theorem-level questions over the established mathematical framework reviewed above.

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Table 2: Concordance between major elements of the earlier Chrysene Formalism and the present twelve-axiom reconstruction.

Earlier element	Present status	Revised formulation
Chrysene as distinguished molecular architecture	Restored as molecular characterization	A1–A3 restrict the molecular reduct to a four-ring polyhex class with C_{2h} core symmetry and directional inequivalence; Theorem 36 is proved.
Geometric basis tensor Ψ_C	Restored and made literal	Ψ_C is an element of an explicitly defined local tensor product rather than a synonym for the molecular graph.
Chrysene Tensor Space I_C	Restored and refined	The tensorial geometric skeleton is a tensor-space fibration over $X_C \cong \mathbb{Z}^3$, with local tensor fibers and network contractions.
$I_C \cong \mathbb{Z}^3$	Restored in precise base-space form	$\mathbb{Z}^3$ is the discrete lattice base, with $\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$; the complete tensor space is not identified literally with its base.
3, 6, 9, 12 primary cipher	Retained	Primary locants and directional attachment data occur in A1–A3.
2, 5, 8, 11 dual cipher	Restored	A8 defines the one-step dual-cipher map, dual tensor Ψ'_C , four-state alphabet, and commutant representation.
Noncommutative observable space	Restored with explicit source	A7 constructs $\mathfrak{A}_C = \mathfrak{Q}_C \rtimes_{\alpha} \mathbb{Z}^3$; noncommutativity follows from ordered translation/localization.
Connes spectral triple and Dirac geometry	Restored	A9 introduces $(\mathfrak{A}_C^{\circ}, \mathcal{H}_{\omega}, D_C)$, bounded commutators, compact resolvent, and spectral distance.
C_{2h} molecular symmetry	Retained with chirality correction	The carbon framework is C_{2h} ; because C_{2h} contains improper symmetry, the isolated core is achiral.
Geometric orthogonality implies quantum isolation	Not retained as an axiom	Euclidean directions, Hilbert-space orthogonality, commuting observables, and dynamical reducing subspaces remain distinct.
Four-state $GF(4)$ sequence	Partly restored, field law deferred	A8 structurally produces four states $\mathbb{Z}_2 \times \mathbb{Z}_2$; a natural multiplication giving $GF(4)$ is a theorem target.
$N_y = N_x + 2$	Deferred	Exact closure is axiomatic; the two-unit registry offset must be independently proved.
Directional electronic anisotropy	Deferred	A10 defines the transfer operators; inequivalence requires distinct Hamiltonian/contact data.
Optical anisotropy	Deferred	A11 defines causal response; anisotropy requires unequal spectral response data.
Spectral distance equals path distance	Deferred	A9 and A12 define the two metrics independently; exact or bi-Lipschitz compatibility is a theorem target.
Noncocommutative coproduct / R -matrix	Optional higher theorem program	Global operator noncommutativity does not itself supply a coproduct, Hopf algebra, or braid structure.

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Earlier element	Present status	Revised formulation
Automatic scission after threshold violation	Not retained	A12 distinguishes constraint violation, metric disconnection, projection, and dynamical removal.
Balanced bipartite molecular structure	Retained as molecular/combinatorial fact	The four-ring polyhex class is bipartite by Lemma 1; the reference chrysene carbon skeleton has a balanced bipartition. No separate bipartiteness axiom is required.
Bay- and K-region perimeter geometry	Restored with corrected PAH nomenclature	Appendix A.9 distinguishes the symmetry-equivalent 4–5 and 10–11 bay clefts from the 5–6 and 11–12 K-region bonds. No rotational operator, folding law, or stacking offset is inferred from either motif.
Closed conjugated perimeter / perimeter conductivity	Retained as realization hypothesis	The Conjugated π -Framework Hypothesis $H_{\partial\pi}(\mathfrak{R})$ records delocalized perimeter π conjugation without making conductivity an axiom or deriving it from C_{2h} symmetry.