

Certified Mathematical Growth of the Chrysene Formalism: Concordance Frontiers and Minimal Scission

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Abstract

We give a compact theory of certified extension for a formal presentation whose identity is constrained by an external reference structure. A protected fragment of an initial presentation is held fixed under ordinary extension, while mathematics outside that fragment may grow strictly. Candidate extensions carry proof provenance and are admitted only after passage of logically distinct mathematical, concordance, realization, and domain conditions. For a theorem with several proof routes, the inclusion-minimal sets of uncertified admission-requiring premises form a certification frontier. If $\emptyset \notin \text{Front}_n(\theta)$, then no recorded route supports assertoric certification; the corresponding frontier deficits instead yield conditional theorems. Provenance-labeled proof support prevents uncertified premises from acquiring certified status through intermediate derivations. Finally, a directed proof-support hypergraph and a least-fixed-point construction define theoretical scission: when previously certified support is invalidated, exactly the unsupported dependency region is removed while every theorem possessing a surviving grounded proof is retained. Applied to the Chrysene Formalism, the construction permits strict mathematical growth while preserving its Canonical Molecular Anchor.

Keywords. certified extension; conservative extension; proof provenance; certification frontier; theoretical scission; formal presentation; Chrysene Formalism.

1 Introduction

Let a mathematical presentation be constrained by an externally fixed reference object. The principal problem is to permit strict mathematical growth without allowing a later extension to alter the criterion by which it is recognized as an extension of the original theory.

The Chrysene Formalism provides the motivating instance. Its detailed molecular characterization and twelve-axiom construction are taken as previously certified input; the present paper concerns only the mathematics of subsequent extension. We therefore distinguish

candidate mathematics from certified mathematics.

Our construction has three principal ingredients.

First, concordance is imposed only on a designated protected identity fragment, rather than on the complete old language. Hence new theorems concerning old objects remain possible.

Second, proof certification is route-sensitive. If $\mathbf{r}$ is a proof route for θ , the uncertified admission-requiring leaves of $\mathbf{r}$ form a deficit $\Gamma(\mathbf{r}; \theta)$. The inclusion-minimal deficits over all recorded

routes form an antichain

$\text{Front}(\theta)$.

Third, proof provenance is retained after admission. If certified support is later withdrawn, a fixed-point construction recomputes the maximal surviving grounded theorem system. This operation will be called *theoretical scission*.

The theory is relative: soundness of the underlying proof calculus, correctness of the protected reference data, and validity of accepted empirical certificates remain external hypotheses.

2 Anchored Presentations

We work in a fixed classical logical framework equipped with a many-sorted proof system that is both sound and finitary. Thus, for each signature Σ , formal derivability is written $\vdash_\Sigma$, and $\text{Sent}(\Sigma)$ denotes the set of closed Σ -formulas. Soundness means that every formally derivable sentence is valid in every model of its assumptions. Finitarity means that whenever

$\Gamma \vdash_\Sigma \varphi$,

there exists a finite subset

$$\Gamma_0 \subseteq \Gamma$$

such that

$$\Gamma_0 \vdash_{\Sigma} \varphi.$$

Definition 1. A presentation is a pair

$$\mathbf{P} = (\Sigma, A), \quad A \subseteq \text{Sent}(\Sigma).$$

Its consequence theory and model class are

$$\text{Th}(\mathbf{P}) := \{\varphi : A \vdash_{\Sigma} \varphi\}, \quad \text{Mod}(\mathbf{P}) := \{\mathfrak{M} : \mathfrak{M} \models A\}.$$

Presentations, rather than only deductively closed theories, are used because later repair acts on primitive support and then recomputes consequences.

Let

$$\mathbf{P}_0 = (\Sigma_0, A_0)$$

be the initial certified Chrysene presentation.

Fix subsignatures

$$\Sigma_{\text{anc}} \subseteq \Sigma_{\text{prot}} \subseteq \Sigma_0$$

and a protected primitive set

$$A_{\text{prot}} \subseteq A_0.$$

The molecular anchor is represented by an equivalence class

$$\mathfrak{C}_{\text{ref}}$$

of Σ_{anc} -structures. In the Chrysene instance this class contains the previously certified canonical molecular reference data; its internal reconstruction is not required here.

Definition 2. Let $\mathbf{P} = (\Sigma, A)$ with $\Sigma_{\text{anc}} \subseteq \Sigma$. A model $\mathfrak{M} \models \mathbf{P}$ is anchor-compatible if

$$\text{Red}_{\Sigma_{\text{anc}}}(\mathfrak{M}) \in \mathfrak{C}_{\text{ref}}.$$

Write

$$\text{Mod}_{\text{anc}}(\mathbf{P})$$

for the class of anchor-compatible models. The presentation is anchored when

$$\text{Mod}_{\text{anc}}(\mathbf{P}) \neq \emptyset.$$

We impose the following convention.

Convention 1 (Anchor fixity). The object $\mathfrak{C}_{\text{ref}}$ is fixed throughout every ordinary extension sequence. An operation

that changes $\mathfrak{C}_{\text{ref}}$ is a foundation revision, not an ordinary extension.

3 Protected Concordance

Choose a collection

$$\mathcal{B}_{\text{conc}} \subseteq \text{Sent}(\Sigma_{\text{prot}})$$

of identity-defining protected sentences, and let

$$\mathcal{L}_{\text{id}} := \text{Bool}(\mathcal{B}_{\text{conc}})$$

be its finite Boolean closure.

A stage presentation

$$\mathbf{P}_n = (\Sigma_n, A_n)$$

is equipped with an injective protected embedding

$$\iota_n : \Sigma_{\text{prot}} \hookrightarrow \Sigma_n.$$

Define

$$\text{IdTh}(\mathbf{P}_n) := \{\varphi \in \mathcal{L}_{\text{id}} : A_n \vdash_{\Sigma_n} (\iota_n)_*(\varphi)\}. \quad (1)$$

The purpose of $\mathcal{L}_{\text{id}}$ is not to freeze every sentence about protected objects.

Definition 3. Let

$$j : \Sigma_n \hookrightarrow \Sigma'$$

be an injective signature map satisfying

$$\iota' = j \circ \iota_n.$$

A presentation

$$\mathbf{P}' = (\Sigma', A')$$

is an ordinary concordant extension of $\mathbf{P}_n$ when:

1.

$$j_*(A_n) \subseteq A';$$

2.

$$\text{Mod}_{\text{anc}}(\mathbf{P}') \neq \emptyset;$$

3.

$$\text{IdTh}(\mathbf{P}') = \text{IdTh}(\mathbf{P}_n).$$

Thus concordance is restricted conservativity on $\mathcal{L}_{\text{id}}$, together with anchor non-vacuity.

Remark 1. Full conservativity is not required. It is permit-

ted that

$$\mathbf{P}_n \not\vdash \varphi, \quad \mathbf{P}' \vdash j_*(\varphi)$$

for

$$\varphi \in \text{Sent}(\Sigma_n) \setminus (\iota_n)_*(\mathcal{L}_{\text{id}}).$$

Hence the formalism may prove genuinely new theorems concerning previously existing objects.

4 Certified Extensions

Let $\mathcal{G}_n$ be a class of syntactically admissible candidate extensions over $\mathbf{P}_n$. Membership in $\mathcal{G}_n$ has no admission consequence.

A candidate $E \in \mathcal{G}_n$ consists, minimally, of:

$$E = (\Sigma_E, j_E, B_E, \Theta_E, \lambda_E),$$

where $j_E : \Sigma_n \hookrightarrow \Sigma_E$, B_E is a finite set of proposed primitive additions, Θ_E is a finite set of proposed derived claims, and λ_E records their logical status.

For each candidate item

$$x \in B_E \cup \Theta_E$$

and each gate

$$g \in \{M, C, P, R, D\},$$

write

$$g_n(x) \in \{0, 1\}$$

for the corresponding certification predicate whenever that gate is applicable to the logical status and scope of x .

Let

$$\mu_x : \{M, C, P, R, D\} \longrightarrow \{0, 1\}$$

record gate applicability, with

$$\mu_x(g) = 1$$

exactly when gate g is required for certification of x .

Definition 4 (Certified candidate item). *A candidate item x is certified at stage n when*

$$\text{Cert}_n(x) = 1 \iff \bigwedge_{\mu_x(g)=1} g_n(x) = 1.$$

Let

$$\text{Adm}(E) \subseteq B_E \cup \Theta_E$$

denote the items proposed by E for admission to the certified state.

Definition 5 (Certified candidate extension). *The candidate*

E is certified when

$$\text{Cert}_n(E) = 1 \iff \text{Cert}_n(x) = 1 \text{ for every } x \in \text{Adm}(E).$$

A nonapplicable gate is omitted rather than counted as a successful test. In particular, failure of one required gate cannot be compensated by success of another.

No numerical averaging is permitted: failure of one required gate cannot be compensated by success of another.

4.1 Contextual realization

Let ρ denote a realization context. A realization overlay is a presentation

$$\mathbf{R}_{n,\rho} = (\Sigma_{n,\rho}^{\text{real}}, A_{n,\rho}^{\text{real}}),$$

and

$$\mathbf{P}_n \langle \rho \rangle = (\Sigma_n \cup \Sigma_{n,\rho}^{\text{real}}, A_n \cup A_{n,\rho}^{\text{real}}).$$

A realization assertion

$$R \in A_{n,\rho}^{\text{real}}$$

is a contextual primitive premise and is not thereby a protected axiom.

Let $\mathfrak{E}_{\text{vid}}$ be a class of external evidence objects and write

$$e \Vdash_{\rho} R$$

when the designated external evidentiary procedure accepts e as support for R in context ρ .

If R is required to prove a target theorem Q , its evidentiary support must not be obtained from Q itself. In particular,

$$R \in \text{Supp}(Q) \not\Rightarrow R \in A_{n,\rho}^{\text{real}}. \quad (2)$$

This is the only empirical principle needed in the compact theory: a theorem may identify a premise worth testing, but cannot certify that premise by requiring it.

4.2 Certified state

Write

$$\mathfrak{X}_n = (\mathfrak{E}_{\text{ref}}, \mathbf{P}_n, \mathcal{R}_n, \mathcal{K}_n, \mathcal{Q}_n),$$

where $\mathcal{R}_n$ is the family of realization overlays, $\mathcal{K}_n$ the certified theorem registry, and $\mathcal{Q}_n$ the quarantine registry.

If

$$\text{Cert}_n(E) = 1,$$

the ordinary update

$$\mathcal{U}_n(\mathfrak{X}_n, E) = \mathfrak{X}_{n+1}$$

adds only those primitive objects eligible for their certified status, adds certified realization assertions only to their scoped overlays, and adds certified derived results to $\mathcal{K}_{n+1}$.

If $\text{Cert}_n(E) = 0$,
 define $\mathcal{Q}_n(\mathfrak{X}_n, E) = \widehat{\mathfrak{X}}_n$
 by $\widehat{\mathbf{P}}_n = \mathbf{P}_n, \quad \widehat{\mathcal{R}}_n = \mathcal{R}_n, \quad \widehat{\mathcal{K}}_n = \mathcal{K}_n,$
 while $\mathcal{Q}_n$ may be enlarged by the failed candidate and its diagnostic record.

5 Admission and Preservation

Theorem 1 (Non-Admittance). *If*

$$\text{Cert}_n(E) = 0,$$

then quarantine leaves the certified content unchanged:

$$\mathbf{P}_n, \quad \mathcal{R}_n, \quad \mathcal{K}_n, \quad \mathfrak{C}_{\text{ref}}$$

are identical before and after the disposition of E . Consequently,

$$\text{Th}(\widehat{\mathbf{P}}_n) = \text{Th}(\mathbf{P}_n).$$

Proof. This is immediate from the definition of the quarantine operator. $\square$

To state the preservation theorem, suppose the candidate presentation used for concordance is

$$\mathbf{P}_n[E] = (\Sigma_E, A_E),$$

whereas the actually admitted presentation satisfies

$$j_E(A_n) \subseteq A_{n+1} \subseteq A_E.$$

Lemma 2 (Concordance descent). *If*

$$\text{IdTh}(\mathbf{P}_n[E]) = \text{IdTh}(\mathbf{P}_n),$$

then

$$\text{IdTh}(\mathbf{P}_{n+1}) = \text{IdTh}(\mathbf{P}_n).$$

Proof. Since $j_E(A_n) \subseteq A_{n+1}$,

$$\text{IdTh}(\mathbf{P}_n) \subseteq \text{IdTh}(\mathbf{P}_{n+1}).$$

Since $A_{n+1} \subseteq A_E$,

$$\text{IdTh}(\mathbf{P}_{n+1}) \subseteq \text{IdTh}(\mathbf{P}_n[E]).$$

The outer terms are equal by concordance. $\square$

Lemma 3 (Anchor non-vacuity descent). *Let*

$$\Sigma_{\text{anc}} \subseteq \Sigma' \subseteq \Sigma$$

and let

$$A' \subseteq A \cap \text{Sent}(\Sigma').$$

If

$$\text{Mod}_{\text{anc}}(\Sigma, A) \neq \emptyset,$$

then

$$\text{Mod}_{\text{anc}}(\Sigma', A') \neq \emptyset.$$

Proof. Choose

$$\mathfrak{M} \models A$$

with anchor reduct in $\mathfrak{C}_{\text{ref}}$. Its Σ' -reduct satisfies every sentence of A' , since $A' \subseteq A$, and its Σ_{anc} -reduct is unchanged. Hence it is an anchor-compatible model of (Σ', A') . $\square$

Proof. Restrict any anchor-compatible model of the stronger presentation to the retained signature. Deletion of axioms preserves satisfaction, and the anchor reduct is unchanged. $\square$

Theorem 4 (Protected-Foundation Preservation). *Let E be a certified ordinary extension and let*

$$\mathfrak{X}_{n+1} = \mathcal{U}_n(\mathfrak{X}_n, E).$$

Then:

1.

$$\mathfrak{C}_{\text{ref}}^{n+1} = \mathfrak{C}_{\text{ref}}^n;$$

2. *there is an injective stage embedding*

$$j_{n,n+1} : \Sigma_n \hookrightarrow \Sigma_{n+1}$$

transporting the protected signature and protected axioms;

3.

$$\text{Mod}_{\text{anc}}(\mathbf{P}_{n+1}) \neq \emptyset;$$

4.

$$\text{IdTh}(\mathbf{P}_{n+1}) = \text{IdTh}(\mathbf{P}_n). \quad (3)$$

Proof. Anchor and protected-signature preservation are part of ordinary concordance. The admitted presentation is a weakening, with respect to nonadmitted candidate primitives, of the presentation on which concordance was audited. Anchor non-vacuity therefore descends by the preceding lemma, and (3) follows from concordance descent. $\square$

Corollary 1 (Finite stability). *For a finite sequence of certified ordinary updates*

$$\mathbf{P}_0 \longrightarrow \mathbf{P}_1 \longrightarrow \cdots \longrightarrow \mathbf{P}_N,$$

one has, for every $0 \leq n \leq N$,

$$\mathfrak{C}_{\text{ref}}^n = \mathfrak{C}_{\text{ref}}^0, \quad \text{IdTh}(\mathbf{P}_n) = \text{IdTh}(\mathbf{P}_0),$$

and

$$\text{Mod}_{\text{anc}}(\mathbf{P}_n) \neq \emptyset.$$

Proof. Induction using Theorem 4. $\square$

Remark 2 (Strict growth). *Let*

$$j_{m,n} := j_{n-1,n} \circ \cdots \circ j_{m,m+1}, \quad m < n,$$

denote the composite stage embedding. The preceding corollary is compatible with strict mathematical growth in the sense that, for some $m < n$,

$$(j_{m,n})_* \text{Th}(\mathbf{P}_m) \subseteq \text{Th}(\mathbf{P}_n).$$

Only the protected identity fragment is fixed. Preservation of the foundation therefore does not imply mathematical stasis.

6 Certification Frontiers

Let θ be a candidate-derived claim over a certified stage $\mathbf{P}_n$. We assume throughout this section that every proof under consideration is finite and formally valid.

Let

$$\mathfrak{P}_n(\theta)$$

denote the family of recorded finite proof routes for θ .

A route

$$\mathbf{r} \in \mathfrak{P}_n(\theta)$$

is a finite rooted proof tree with root θ . Whenever a leaf is itself a previously certified derived statement, recursively expand one recorded certified proof of that statement. Continue until every leaf is primitive.

Denote the resulting finite primitive support by

$$\text{PrimLeaf}(\mathbf{r}).$$

All certification deficits below are computed from this recursively expanded primitive support.

6.1 Admission-Requiring Support

Not every hypothesis occurring in a valid proof must be asserted as true by the certified theory. A theorem may legitimately retain an explicit hypothesis.

Definition 6. *For a route $\mathbf{r}$ to θ , define*

$$\text{AdmReq}_n(\mathbf{r}; \theta) \subseteq \text{PrimLeaf}(\mathbf{r}).$$

to be the set of primitive leaves whose certified truth is required by the scope in which θ is asserted.

Thus, for example, a realization premise used in an assertoric molecular claim belongs to AdmReq_n , whereas a hypothesis retained explicitly in a conditional theorem does not.

For

$$p \in \text{AdmReq}_n(\mathbf{r}; \theta),$$

define

$$\text{PremCert}_n(p; \theta) \in \{0, 1\}$$

according to the certification rule appropriate to the logical status of p .

Definition 7 (Route deficit). *The certification deficit of $\mathbf{r}$ for θ is*

$$\Gamma_n(\mathbf{r}; \theta) := \{p \in \text{AdmReq}_n(\mathbf{r}; \theta) : \text{PremCert}_n(p; \theta) = 0\}. \quad (4)$$

The deficit is finite because the proof route has finite support.

6.2 The Frontier Antichain

Define the family of route deficits

$$\mathfrak{D}_n(\theta) := \{\Gamma_n(\mathbf{r}; \theta) : \mathbf{r} \in \mathfrak{P}_n(\theta)\}.$$

For a family $\mathcal{A}$ of sets, write

$$\text{Min}_{\subseteq} \mathcal{A}$$

for its inclusion-minimal members.

Definition 8 (Certification frontier). *If $\mathfrak{P}_n(\theta) \neq \emptyset$, define*

$$\text{Front}_n(\theta) := \text{Min}_{\subseteq} \mathfrak{D}_n(\theta). \quad (5)$$

The frontier is an antichain. It need not possess a unique member.

Lemma 5 (Existence). *If*

$$\mathfrak{P}_n(\theta) \neq \emptyset,$$

then

$$\text{Front}_n(\theta) \neq \emptyset.$$

Proof. Choose $\Gamma_0 \in \mathfrak{D}_n(\theta)$. If it is not minimal, choose $\Gamma_1 \subsetneq \Gamma_0$ in the same family and continue. Since Γ_0 is finite, a strictly descending chain of its subsets terminates after finitely many steps. The terminal member is inclusion-minimal. $\square$

Proposition 6 (Frontier domination). *For every $\mathbf{r} \in \mathfrak{P}_n(\theta)$ there exists*

$$\Gamma \in \text{Front}_n(\theta)$$

such that

$$\Gamma \subseteq \Gamma_n(\mathbf{r}; \theta).$$

Proof. Apply the descending-subset argument of Lemma 5 starting from $\Gamma_n(\mathbf{r}; \theta)$. $\square$

6.3 Route-External Conditions

A deficit records missing primitive certification. It does not record every possible defect. A proof route may have empty deficit while the candidate still fails concordance or provenance.

Definition 9. *Write*

$$\text{ExtReady}_n(\theta) = 1$$

when every certification requirement applicable to θ that is not encoded by PremCert_n on the route leaves has passed.

This includes, as applicable, mathematical well-formedness, concordance, complete provenance, valid interpretation maps, and correct scope.

A route is *certificate-complete* when

$$\Gamma_n(\mathbf{r}; \theta) = \emptyset$$

and

$$\text{ExtReady}_n(\theta) = 1.$$

Theorem 7 (Certification-Frontier Theorem). *Assume*

$$\mathfrak{P}_n(\theta) \neq \emptyset \quad \text{and} \quad \text{ExtReady}_n(\theta) = 1.$$

Then:

1. *Every frontier member is the deficit of a recorded valid proof route.*
2. *Every recorded route has a deficit containing at least one frontier member.*
3. *A certificate-complete recorded route exists if and only if*

$$\emptyset \in \text{Front}_n(\theta). \quad (6)$$

4. *If*

$$\emptyset \notin \text{Front}_n(\theta),$$

then no recorded route supports assertoric certification of θ .

Proof. The first assertion is immediate from (5). The second is Proposition 6.

If some recorded route has empty deficit, then the empty set is inclusion-minimal and hence belongs to the frontier. Conversely, if $\emptyset$ belongs to the frontier, it is by definition the deficit of a recorded route. Under route-external readiness that route is certificate-complete. The fourth assertion is the contrapositive. $\square$

6.4 Conditionalization of a Frontier Route

For a finite set

$$\Gamma = \{p_1, \dots, p_k\},$$

write

$$\bigwedge \Gamma = p_1 \wedge \dots \wedge p_k, \quad \bigwedge \emptyset := \top.$$

We use the ordinary deduction theorem for finite hypotheses in the fixed classical proof calculus.

For a route $\mathbf{r}$, let

$$\mathbf{B}_{n,\mathbf{r}}$$

denote its *certified scope presentation*: the presentation obtained from $\mathbf{P}_n$ by adjoining every certified primitive premise in

$$\text{PrimLeaf}(\mathbf{r}) \setminus \Gamma_n(\mathbf{r}; \theta)$$

that is required by the declared scope of θ , including any applicable certified realization or formal-domain premises. All signatures are assumed compatible on their common subsignatures.

Proposition 8 (Frontier conditionalization). *Let*

$$\Gamma = \Gamma_n(\mathbf{r}; \theta)$$

for a valid recorded proof route $\mathbf{r}$. Then

$$\mathbf{B}_{n,\mathbf{r}} \vdash \left(\bigwedge \Gamma \right) \rightarrow \theta.$$

Proof. Recursively expand every derived statement occurring in $\mathbf{r}$ until primitive support is reached. By definition, every primitive support item outside Γ is contained in the certified scope presentation $\mathbf{B}_{n,\mathbf{r}}$. The only remaining assumptions are therefore the members of Γ . Repeated application of the deduction theorem gives

$$\mathbf{B}_{n,\mathbf{r}} \vdash \left(\bigwedge \Gamma \right) \rightarrow \theta.$$

$\square$

Thus

$$\emptyset \notin \text{Front}_n(\theta)$$

does not imply mathematical failure. It means that every recorded route retains at least one uncertified admission-requiring premise, so the corresponding hypotheses must remain explicit.

6.5 Frontier Closure

Let S be a set of previously uncertified premises that subsequently become independently certified. Assume the proof routes and all other certification values remain unchanged.

Then

$$\Gamma_{n'}(\mathbf{r}; \theta) = \Gamma_n(\mathbf{r}; \theta) \setminus S. \quad (7)$$

Consequently,

$$\text{Front}_{n'}(\theta) = \text{Min}_{\subseteq} \{ \Gamma_n(\mathbf{r}; \theta) \setminus S : \mathbf{r} \in \mathfrak{P}_n(\theta) \}. \quad (8)$$

Corollary 2 (Independent realization closure). *Suppose*

$$\Gamma = \{R_1, \dots, R_k\} \in \text{Front}_n(\theta)$$

consists of contextual realization premises. If each R_i later receives the independently required realization certificate and no other certification status changes, then the corresponding route has empty deficit after re-evaluation.

The conclusion does not reverse the direction of evidentiary justification. In particular,

$$R \in \Gamma_n(\mathbf{r}; \theta) \not\Rightarrow \text{PremCert}_n(R; \theta) = 1. \quad (9)$$

The fact that R would close the proof is not evidence for R .

7 Provenance and Non-Propagation

The same formula may enter reasoning under distinct logical provenances. Certification must therefore act on formulas together with their sources.

Definition 10 (Support token). *A support token is a pair*

$$\mathbf{p} = [p; s],$$

where p is a sentence and s records the provenance under which p is invoked.

For the present paper it is sufficient to distinguish the source classes

Base, Real, Def, Hyp, Thm, Quar.

A token may therefore have forms such as

$$[p; \text{Real}(\rho)], \quad [p; \text{Hyp}(\theta)], \quad [p; \text{Quar}(E)].$$

Tokens with the same underlying formula but different provenance are distinct:

$$[p; s_1] \neq [p; s_2] \quad (s_1 \neq s_2). \quad (10)$$

Definition 11 (Certified primitive token). *Let*

$$\text{CertPrim}_n$$

be the set of primitive tokens whose stated provenance possesses the certification required at stage n .

In particular,

$$[p; \text{Quar}(E)] \notin \text{CertPrim}_n$$

unless a later independent certification event creates a *new* token for the same formula under certified provenance.

7.1 Ancestral Support

Let $\mathbf{r}$ be a proof route for θ whose leaves are provenance-labeled. Recursively expand every theorem token appearing internally until primitive support is reached.

Definition 12. *The resulting set of primitive provenance tokens is the ancestral support,*

$$\text{AncSupp}(\mathbf{r}).$$

An assertoric proof route has *certified ancestry* when

$$\text{AncSupp}(\mathbf{r}) \subseteq \text{CertPrim}_n.$$

An explicit conditional hypothesis is permitted only if it remains visible as a hypothesis of the conclusion; it is not thereby converted into certified primitive support.

Lemma 9 (Ancestral exposure). *If a primitive token occurs anywhere in the recursively expanded dependency tree of $\mathbf{r}$, then it belongs to $\text{AncSupp}(\mathbf{r})$.*

Proof. The ancestral support is defined as the union of all primitive leaves obtained by recursive expansion. Hence every such token occurs in that union. $\square$

Theorem 10 (Non-Propagation of Uncertified Provenance). *Let $\mathbf{q}$ be a primitive provenance token satisfying*

$$\mathbf{q} \notin \text{CertPrim}_n.$$

Then no assertoric certified theorem θ can possess a certified proof route $\mathbf{r}$ for which

$$\mathbf{q} \in \text{AncSupp}(\mathbf{r}).$$

Proof. An assertoric certified route has certified ancestry, hence

$$\text{AncSupp}(\mathbf{r}) \subseteq \text{CertPrim}_n.$$

If

$$\mathbf{q} \in \text{AncSupp}(\mathbf{r})$$

while

$$\mathbf{q} \notin \text{CertPrim}_n,$$

this inclusion is contradicted. $\square$

Corollary 3 (No laundering through intermediate theorems). *If an uncertified token is an ancestor of a derived statement and no alternative proof removes that dependency, repeated introduction of intermediate lemmas cannot convert the descendant into an assertoric certified theorem.*

Remark 3 (Independent re-derivation). *The theorem concerns provenance, not formula identity. If the same formula p is later proved independently from certified support, then*

$$[p; \text{Thm}]$$

is a new certified token distinct from

$$[p; \text{Quar}(E)].$$

Thus prior quarantine does not permanently contaminate a formula.

8 Proof-Support Hypergraphs and Scission

The frontier controls missing support before admission. Scission treats support that was admitted and later invalidated.

At a certified stage n , let

$$V_n^{\text{prim}}$$

be the set of certified primitive support tokens and let

$$V_n^{\text{thm}} = \{\mathbf{t}_\theta : \theta \in \mathcal{K}_n\}$$

be the set of certified theorem tokens.

Set

$$V_n = V_n^{\text{prim}} \sqcup V_n^{\text{thm}}.$$

Definition 13 (Proof hyperedge). *A certified proof hyperedge is a triple*

$$e = (S_e, t_e, \pi_e),$$

where

$$S_e \subseteq V_n$$

is finite, $t_e \in V_n^{\text{thm}}$, and π_e is a valid proof of the formula underlying t_e from the formulas underlying S_e .

Let

$$\mathcal{E}_n$$

be the set of retained certified proof hyperedges and define

$$\mathcal{H}_n = (V_n, \mathcal{E}_n).$$

Several edges may have the same target. They represent alternative proofs.

The global hypergraph need not be acyclic: a later theorem may provide a new proof of an earlier one. What is required after invalidation is not global acyclicity but grounded derivability from surviving primitive support.

8.1 Grounded Proof Trees

Definition 14 (Complete proof tree). *A complete proof tree for $t \in V_n^{\text{thm}}$ is a finite rooted hyper-tree with root t such that each internal theorem vertex is justified by one incoming proof hyperedge and every leaf belongs to V_n^{prim} .*

Let

$$\mathfrak{T}_n(t)$$

be the family of complete certified proof trees for t .

A theorem in the certified registry is required to satisfy

$$\mathfrak{T}_n(t) \neq \emptyset.$$

8.2 Invalidation

A later failure may concern either a support assertion or a particular proof.

Definition 15 (Scission datum). *A scission datum is a pair*

$$\Delta = (Z_V, Z_E),$$

where

$$Z_V \subseteq V_n$$

is the set of support tokens whose certification is withdrawn and

$$Z_E \subseteq \mathcal{E}_n$$

is the set of proof hyperedges whose proof certificates are withdrawn.

Define the surviving primitive support

$$P^\Delta := V_n^{\text{prim}} \setminus Z_V$$

and the surviving proof-edge family

$$\mathcal{E}^\Delta := \mathcal{E}_n \setminus Z_E.$$

Edges having a target in Z_V are not permitted to restore that target within the same scission operation.

8.3 The Retention Fixed Point

Define

$$\Phi_\Delta : \mathcal{P}(V_n) \longrightarrow \mathcal{P}(V_n)$$

by

$$\Phi_\Delta(X) := P^\Delta \cup \{t \in V_n^{\text{thm}} \setminus Z_V : \exists e \in \mathcal{E}^\Delta, t_e = t, S_e \subseteq X\}. \quad (11)$$

Lemma 11. *The operator Φ_Δ is monotone.*

Proof. If $X \subseteq Y$, every finite support set contained in X is contained in Y . The primitive term P^Δ is fixed. $\square$

Define

$$R_0^\Delta := P^\Delta$$

and recursively

$$R_{k+1}^\Delta := \Phi_\Delta(R_k^\Delta), \quad k < \omega.$$

Lemma 12 (Finitary fixed-point closure). *Let*

$$R_\omega^\Delta := \bigcup_{k < \omega} R_k^\Delta.$$

Then

$$\Phi_\Delta(R_\omega^\Delta) = R_\omega^\Delta.$$

Consequently,

$$R_\omega^\Delta = \text{lfp}(\Phi_\Delta).$$

Proof. Since

$$R_0^\Delta = P^\Delta \subseteq \Phi_\Delta(R_0^\Delta) = R_1^\Delta,$$

monotonicity of Φ_Δ implies inductively that

$$R_k^\Delta \subseteq R_{k+1}^\Delta \quad (k < \omega).$$

Hence

$$R_\omega^\Delta \subseteq \Phi_\Delta(R_\omega^\Delta).$$

Conversely, let

$$t \in \Phi_\Delta(R_\omega^\Delta).$$

If $t \in P^\Delta$, then $t \in R_0^\Delta$. Otherwise there exists a surviving proof hyperedge $e \in \mathcal{E}^\Delta$ such that

$$t_e = t, \quad S_e \subseteq R_\omega^\Delta.$$

Since S_e is finite, there exists $k < \omega$ with

$$S_e \subseteq R_k^\Delta.$$

Therefore

$$t \in \Phi_\Delta(R_k^\Delta) = R_{k+1}^\Delta \subseteq R_\omega^\Delta.$$

Thus R_ω^Δ is a fixed point.

Finally, if X is any fixed point of Φ_Δ , then

$$P^\Delta \subseteq X.$$

Induction gives

$$R_k^\Delta \subseteq X \quad (k < \omega),$$

and hence

$$R_\omega^\Delta \subseteq X.$$

Therefore R_ω^Δ is the least fixed point. $\square$

Define

$$\text{Ret}_n(\Delta) := R_\omega^\Delta = \text{lfp}(\Phi_\Delta),$$

and

$$\text{Sc}_n(\Delta) := V_n \setminus \text{Ret}_n(\Delta).$$

Definition 16. *A complete proof tree is Δ -active when it contains no vertex in Z_V and no proof edge in Z_E .*

Write

$$\mathfrak{T}_n^\Delta(t)$$

for the family of Δ -active complete proof trees rooted at t .

Theorem 13 (Retention characterization). *For every $t \in V_n^{\text{thm}}$,*

$$t \in \text{Ret}_n(\Delta) \iff \mathfrak{T}_n^\Delta(t) \neq \emptyset. \quad (12)$$

Proof. Suppose first that $t \in \text{Ret}_n(\Delta)$. Choose the least k for which $t \in R_k^\Delta$. Since R_0^Δ contains only primitive support, $k > 0$. By construction there exists an active hyperedge

$$e = (S_e, t, \pi_e)$$

with

$$S_e \subseteq R_{k-1}^\Delta.$$

Induction on k supplies active complete proof trees for every theorem vertex in S_e . Attaching them below e gives an active complete proof tree for t .

Conversely, let T be a Δ -active complete proof tree for t . Induct on its height. All primitive leaves belong to R_0^Δ . If the support of an internal proof edge lies in R_k^Δ , its target lies in R_{k+1}^Δ . Therefore the root belongs to $\text{Ret}_n(\Delta)$. $\square$

8.4 Maximality

Definition 17. A set

$$W \subseteq V_n \setminus Z_V$$

is a Δ -grounded subsystem when

$$P^\Delta \subseteq W$$

and every

$$t \in W \cap V_n^{\text{thm}}$$

possesses a Δ -active complete proof tree all of whose vertices belong to W .

Proposition 14 (Maximal grounded retention). *The set $\text{Ret}_n(\Delta)$ is a Δ -grounded subsystem, and every Δ -grounded subsystem W satisfies*

$$W \subseteq \text{Ret}_n(\Delta).$$

Proof. Every surviving primitive belongs to $\text{Ret}_n(\Delta)$. For each retained theorem, Theorem 13 supplies an active complete proof tree whose recursively generated vertices are retained. Hence $\text{Ret}_n(\Delta)$ is grounded.

If W is another grounded subsystem, every theorem in W has an active complete proof tree. By Theorem 13 its root belongs to $\text{Ret}_n(\Delta)$. Thus $W \subseteq \text{Ret}_n(\Delta)$. $\square$

Corollary 4 (Minimal removal). *The scission set*

$$\text{Sc}_n(\Delta)$$

is inclusion-minimal among removal sets whose complements are Δ -grounded subsystems containing all surviving primitive support.

Proof. If W is any such retained subsystem, then

$$W \subseteq \text{Ret}_n(\Delta)$$

by Proposition 14. Taking complements in V_n gives

$$\text{Sc}_n(\Delta) \subseteq V_n \setminus W.$$

$\square$

8.5 Essential Dependence

For a complete proof tree T , let

$$\text{PrimSupp}(T)$$

be its primitive leaf set and define

$$\mathcal{S}_n(t) := \{\text{PrimSupp}(T) : T \in \mathfrak{T}_n(t)\}.$$

For $p \in V_n^{\text{prim}}$, write

$$p \prec_{\text{ess}} t$$

when

$$p \in \bigcap_{S \in \mathcal{S}_n(t)} S.$$

Corollary 5 (Singleton invalidation). *For*

$$\Delta_p = (\{p\}, \emptyset),$$

one has

$$t \in \text{Sc}_n(\Delta_p) \iff p \prec_{\text{ess}} t. \quad (13)$$

Proof. The theorem token t survives exactly when some complete proof tree avoids p . By definition this fails exactly when p occurs in every primitive support set of t . $\square$

More generally, for $Z \subseteq V_n^{\text{prim}}$,

$$t \in \text{Sc}_n((Z, \emptyset))$$

if and only if

$$S \cap Z \neq \emptyset \quad \forall S \in \mathcal{S}_n(t).$$

Thus multiple-premise invalidation is a hitting-set problem on the family of certified primitive proof supports.

8.6 Foundation-Safe Scission

Let

$$\mathbf{P}_{\text{prot}} = (\Sigma_{\text{prot}}, A_{\text{prot}})$$

be the protected presentation.

For the scission theorem we make explicit the identity-adequacy condition:

$$\text{IdTh}(\mathbf{P}_{\text{prot}}) = \text{IdTh}(\mathbf{P}_0). \quad (14)$$

Let

$$V_n^{\text{prot}} \subseteq V_n^{\text{prim}}$$

be the tokens representing the transported protected primitive assertions.

$\square$

Definition 18. A scission datum $\Delta = (Z_V, Z_E)$ is foundation-safe when

$$Z_V \cap V_n^{\text{prot}} = \emptyset.$$

It is signature-stable when every symbol required by re-

tained support remains well defined after the corresponding primitive withdrawals.

Let

$$\mathbf{P}_n^\Delta = (\Sigma_n^\Delta, A_n^\Delta)$$

be the repaired presentation obtained by deleting the revoked nonprotected primitive assertions and restricting the signature only as required by signature stability.

Theorem 15 (Minimal Scission Preservation). *Assume:*

1. $\mathbf{P}_n$ is an anchored certified stage satisfying

$$\text{IdTh}(\mathbf{P}_n) = \text{IdTh}(\mathbf{P}_0);$$

2. (14) holds;

3. Δ is foundation-safe and signature-stable.

Then:

- (i) A theorem token survives scission if and only if it possesses at least one Δ -active complete proof tree.
- (ii) Every theorem lacking such a proof is removed.
- (iii) The retained support $\text{Ret}_n(\Delta)$ is maximal among grounded systems compatible with Δ , and $\text{Sc}_n(\Delta)$ is minimal among the corresponding removal sets.
- (iv) Every theorem possessing an independent certified proof avoiding the invalidated support is retained.
- (v) The protected identity theory is preserved:

$$\text{IdTh}(\mathbf{P}_n^\Delta) = \text{IdTh}(\mathbf{P}_n). \quad (15)$$

- (vi) The repaired presentation remains anchor-nonvacuous:

$$\text{Mod}_{\text{anc}}(\mathbf{P}_n^\Delta) \neq \emptyset.$$

Proof. Assertions (i) and (ii) are Theorem 13. Assertion (iii) follows from Proposition 14 and Corollary 4. Assertion (iv) is the forward direction of (12).

For (v), foundation safety implies that all transported protected primitive assertions remain in the repaired presentation. Hence

$$\text{IdTh}(\mathbf{P}_{\text{prot}}) \subseteq \text{IdTh}(\mathbf{P}_n^\Delta).$$

Since scission removes assumptions,

$$A_n^\Delta \subseteq A_n,$$

and therefore

$$\text{IdTh}(\mathbf{P}_n^\Delta) \subseteq \text{IdTh}(\mathbf{P}_n).$$

Using

$$\text{IdTh}(\mathbf{P}_{\text{prot}}) = \text{IdTh}(\mathbf{P}_0) = \text{IdTh}(\mathbf{P}_n)$$

forces equality throughout, proving (15).

For (vi), choose

$$\mathfrak{M} \in \text{Mod}_{\text{anc}}(\mathbf{P}_n).$$

The repaired presentation is weaker, apart from a possible signature restriction. The corresponding reduct of $\mathfrak{M}$ therefore satisfies A_n^Δ and retains the same anchor reduct. $\square$

8.7 Pre-Admission and Post-Admission Control

The two principal constructions of this part govern opposite temporal regimes:

$\text{Front}_n(\theta)$: uncertified support before assertoric admission, $\text{Sc}_n(\Delta)$: invalidated support after prior admission.

(16)

Together with Theorem 10, they give the compact error-control principle

missing support is exposed, uncertified support cannot be laundered, and invalidated support can be excised minimally.
--

(17)

The remaining task is to combine these results with ordinary certified growth, realization closure, formal cross-domain interpretation, and relative semantic soundness. This is the content of the master theorem in Part III.

9 Regulated Development

We now combine ordinary certified growth, quarantine, and theoretical scission.

For a certified research state write

$$\mathfrak{X} = (\mathfrak{C}_{\text{ref}}, \mathbf{P}, \mathcal{R}, \mathcal{K}, Q).$$

Define its certified projection by

$$\text{CertPart}(\mathfrak{X}) := (\mathfrak{C}_{\text{ref}}, \mathbf{P}, \mathcal{R}, \mathcal{K}). \quad (18)$$

The quarantine component is excluded because historical retention of a failed candidate does not alter certified mathematical content.

9.1 Three transition types

A regulated development step is one of the following.

Certified update. If

$$\text{Cert}(E) = 1,$$

then

$$\mathfrak{X} \mapsto \mathcal{U}(\mathfrak{X}, E).$$

Quarantine. If

$$\text{Cert}(E) = 0,$$

then

$$\mathfrak{X} \mapsto \mathcal{Q}(\mathfrak{X}, E).$$

By Theorem 1, this leaves $\text{CertPart}(\mathfrak{X})$ unchanged.

Scission. For a foundation-safe, signature-stable invalidation datum

$$\Delta = (Z_V, Z_E),$$

the transition

$$\mathfrak{X} \mapsto \mathfrak{S}_\Delta(\mathfrak{X})$$

replaces the theorem-support system by the retained fixed point

$$\text{Ret}(\Delta)$$

defined in Part II.

Definition 19 (Regulated history). A finite regulated history is a sequence

$$\mathfrak{X}^{[0]}, \mathfrak{X}^{[1]}, \dots, \mathfrak{X}^{[N]} \quad (19)$$

in which each successor transition is a certified update, quarantine, or foundation-safe signature-stable scission.

Definition 20 (Protected-normal history). A regulated history is protected-normal when no transition invalidates or replaces

$$\mathfrak{C}_{\text{ref}}, \quad \Sigma_{\text{prot}}, \quad A_{\text{prot}}, \quad \mathcal{L}_{\text{id}}.$$

A failure of protected structure is therefore not an ordinary scission event. It triggers foundational re-analysis and lies outside the theorem below.

10 Formal Translation

The certification architecture also permits abstract mathematics to be instantiated in another domain without identifying the two underlying systems.

Let

$$\mathbf{M} = (\Sigma_M, A_M)$$

be an abstract bridge presentation and

$$\mathbf{D} = (\Sigma_D, A_D)$$

a domain presentation.

Definition 21 (Interpretation). An interpretation

$$\tau : \mathbf{M} \rightsquigarrow \mathbf{D}$$

is a translation of formulas such that

$$\mathbf{M} \vdash \varphi \implies \mathbf{D} \vdash \tau(\varphi)$$

for every theorem φ in the specified transferable fragment.

Proposition 16 (Formal theorem transfer). If

$$\tau : \mathbf{M} \rightsquigarrow \mathbf{D}$$

is a valid interpretation and

$$\mathbf{M} \vdash \varphi,$$

then

$$\mathbf{D} \vdash \tau(\varphi).$$

Proof. This is the defining proof-preservation property of the interpretation. $\square$

The proposition concerns formal domain mathematics only. If $\mathcal{X}_D$ is a real external system, the stronger assertion that $\mathbf{D}$ adequately represents $\mathcal{X}_D$ requires a distinct domain-evidence certificate.

Consequently,

$$\text{formal theorem transfer} \not\Rightarrow \text{empirical domain adequacy.} \quad (20)$$

Likewise, a common abstract theorem does not imply physical identity:

theorem transfer $\neq$ mechanism transfer $\neq$ ontological identification.

10.1 Grounded Theorems

For a state $\mathfrak{X}$, write

$$\text{Ground}_{\mathfrak{X}}(\theta)$$

when $\theta \in \mathcal{K}$ possesses at least one complete proof tree whose primitive support consists entirely of currently certified tokens appropriate to the scope of θ .

Thus

$$\text{Ground}_{\mathfrak{X}}(\theta)$$

excludes both hidden quarantine provenance and undisclosed open hypotheses.

11 The Master Theorem

For a chronological state $\mathfrak{X}^{[j]}$, write

$$\text{Front}^{[j]}(\theta)$$

for the certification frontier computed from the certified presentation, proof routes, and certification data current at that state.

Theorem 17 (Certified Growth and Scission). *Let*

$$\mathfrak{X}^{[0]}, \dots, \mathfrak{X}^{[N]}$$

be a protected-normal finite regulated history. Assume:

1. $\mathfrak{X}^{[0]}$ *is anchored;*
2. *every certified update satisfies the certification rules of Sections 4–6 and the concordance condition of Definition 3;*
3. *every assertoric theorem admitted to the theorem registry satisfies the certified-ancestry condition of Section 7;*
4. *every scission transition uses the proof-support hypergraph and retention operator of Section 8;*
5. *the protected identity-adequacy condition*

$$\text{IdTh}(\mathbf{P}_{\text{prot}}) = \text{IdTh}(\mathbf{P}^{[0]})$$

holds.

Then, for every $0 \leq j \leq N$, the following statements hold.

(i) **Protected identity is invariant:**

$$\text{IdTh}(\mathbf{P}^{[j]}) = \text{IdTh}(\mathbf{P}^{[0]}). \quad (21)$$

The Canonical Molecular Anchor remains fixed and

$$\text{Mod}_{\text{anc}}(\mathbf{P}^{[j]}) \neq \emptyset.$$

(ii) **Failed candidates do not alter certified content.** *For every quarantine transition,*

$$\text{CertPart}(\mathfrak{X}^{[j+1]}) = \text{CertPart}(\mathfrak{X}^{[j]}). \quad (22)$$

(iii) **Every retained assertoric theorem is grounded:**

$$\theta \in \mathcal{K}^{[j]} \implies \text{Ground}_{\mathfrak{X}^{[j]}}(\theta). \quad (23)$$

(iv) **Absence of a premise-complete frontier member blocks assertoric promotion.** *If a claim θ has route-external readiness and*

$$\emptyset \notin \text{Front}^{[j]}(\theta),$$

then no recorded route supports assertoric certification of θ . For each

$$\Gamma \in \text{Front}^{[j]}(\theta),$$

the corresponding route supports only

$$\left(\bigwedge \Gamma \right) \rightarrow \theta.$$

(v) **Independent premise certification can close a route.** *If all members of a frontier deficit Γ subsequently acquire the certification required by their logical status and all other route conditions remain valid, then the re-evaluated deficit of that route is empty.*

(vi) **The target theorem cannot certify its own missing premise.** *For*

$$R \in \Gamma_n(\mathfrak{r}; \theta),$$

the fact that certification of R would close $\mathfrak{r}$ is not sufficient to certify R .

(vii) **Formal external translation is controlled.** *A theorem may be transferred through a valid interpretation as in Proposition 16; no empirical claim about a real external system follows without the separately required domain certificate.*

(viii) **Scission is dependency-minimal.** *For every scission datum Δ , a theorem survives exactly when it has a Δ -active complete proof tree. The retained subsystem is maximal among Δ -grounded systems and the removed subsystem is minimal relative to the same invalidation datum.*

(ix) **Independent branches survive.** *If an invalidated branch supports one proof of Q , but another certified proof of Q avoids that branch, then Q remains in the certified theorem registry.*

Thus ordinary growth, conditional realization, formal domain translation, quarantine, and post-admission repair may coexist without silent replacement of the protected foundation.

Proof. We proceed by induction on the chronological index j . For the initial state the protected claims hold by hypothesis. Suppose they hold at stage j .

If the next transition is a certified update, protected identity, anchor fixity, and anchor-compatible non-vacuity follow from Theorem 4. Newly admitted assertoric theorems have certified ancestry by hypothesis, so they are grounded.

If the next transition is quarantine, the certified projection is unchanged by Theorem 1. Hence all protected and grounded certified content persists.

If the next transition is scission, foundation safety preserves every protected primitive token. By Theorem 15, the repaired presentation remains anchor-nonvacuous and has the same protected identity theory. The same theorem and Theorem 13 show that precisely those theorem tokens possessing surviving grounded proof trees remain. This establishes (i)–(iii) inductively.

For (iv), apply Theorem 7. Absence of the empty set from

the frontier means that every recorded route has at least one uncertified admission-requiring premise. Proposition 8 then gives (??).

For (v), independent certification removes the corresponding members from the route deficit by (7). If every deficit member is removed, the route becomes premise-complete.

For (vi), premise certification is defined from the status and independent support of the premise, not from the desirability or consequence of the theorem that uses it. Therefore theorem dependence cannot itself change a premise from uncertified to certified.

Statement (vii) is Proposition 16 together with the distinction (20).

Statement (viii) follows from Theorem 13, Proposition 14, and Corollary 4.

Finally, (ix) is immediate from the retention characterization: an independent Δ -active complete proof tree places the theorem root in

$$\text{Ret}(\Delta).$$

□

Corollary 6 (Strict growth is permitted). *The conclusions of Theorem 17 are compatible with*

$$\text{Th}(\mathbf{P}^{[0]}) \subseteq \text{Th}(\mathbf{P}^{[j]})$$

for one or more stages j .

Proof. The master theorem fixes IdTh rather than the complete consequence theory. New consequences outside the protected identity fragment are therefore permitted. □

Thus protected invariance is not full theoretical conservativity.

11.1 Three Consequence Types

The master theorem accommodates three logically distinct forms of growth.

Mathematical extension. A new certified construction enlarges the formal mathematics:

$$\mathbf{P}_n \longrightarrow \mathbf{P}_{n+1}.$$

Realization consequence. For a specified context ρ , certified contextual premises may yield

$$\mathbf{P}_n \langle \rho \rangle \vdash Q_{\text{chr}}(\rho).$$

If every recorded route retains at least one uncertified admission-requiring realization premise, equivalently

$$\emptyset \notin \text{Front}_n(Q_{\text{chr}}(\rho)),$$

then the capability claim remains conditional relative to the recorded route family.

External translation. An abstract theorem may be interpreted in another formal domain:

$$\mathbf{M} \rightsquigarrow \mathbf{D}.$$

This establishes a formal target-domain theorem but not empirical adequacy of the target model.

These three operations share mathematical structure but not epistemic status.

12 Relative Soundness

The preceding results concern certification structure. To infer semantic validity, one must state the metatheoretic assumptions.

Definition 22 (Correct certification environment). *A certification environment is correct when:*

1. *the background proof calculus is sound;*
2. *accepted proof certificates are valid derivations;*
3. *definitions and signature translations are correctly typed;*
4. *provenance records contain the primitive support actually used;*
5. *concordance certificates correctly establish the claimed protected preservation;*
6. *accepted realization and domain certificates satisfy their declared external evidentiary relations;*
7. *the external anchor data are correct for the intended reference system.*

Theorem 18 (Relative soundness). *Assume a correct certification environment. If θ is an assertoric theorem retained in a regulated state and $\mathbf{B}_\theta$ is its certified scope presentation, then*

$$\mathbf{B}_\theta \vdash \theta,$$

and consequently, by soundness,

$$\mathbf{B}_\theta \models \theta.$$

Proof. By Theorem 17, θ has a grounded complete certified proof. Correctness of its proof certificate gives

$$\mathbf{B}_\theta \vdash \theta.$$

Soundness of the underlying calculus gives

$$\mathbf{B}_\theta \models \theta.$$

□

For a realization or external empirical claim, applicability to a physical system remains conditional on the correctness of the corresponding external adequacy judgment.

The theorem therefore does not establish infallibility of experimental science, proof verification, or certification itself.

13 Generator Independence

Let a generated candidate record be

$$\widehat{E} = (g, E, \Xi),$$

where g records its source, E its formal content, and Ξ its certificate data.

The source may be a human mathematician, an AI system, a theorem prover, or a hybrid process.

Definition 23. *Two generated records*

$$\widehat{E}_1 = (g_1, E, \Xi), \quad \widehat{E}_2 = (g_2, E, \Xi)$$

are certification-equivalent when their formal candidate content and all evidence relevant to the certification gates are identical.

Theorem 19 (Generator independence). *Certification-equivalent candidates have identical certification status:*

$$\text{Cert}(\widehat{E}_1) = \text{Cert}(\widehat{E}_2). \quad (24)$$

Proof. Each certification predicate is, by construction, a function of the candidate's formal mathematical content, protected-concordance data, proof provenance, and any applicable realization or domain evidence. The generator label g occurs in none of these predicates. Therefore all applicable gate values agree. $\square$

Corollary 7 (Source neutrality). *A mathematically invalid candidate does not acquire admissibility because it was human-generated, and a mathematically valid candidate does not lose admissibility because it was AI-generated.*

13.1 Open Generation, Constrained Admission

The preceding theorem permits a precise interpretation of AI-assisted mathematical generation.

Let

$$\mathfrak{G} : \mathcal{I} \longrightarrow \mathcal{G}_n$$

be any candidate generator. No assumption is made that $\mathfrak{G}$ is sound.

It may generate

$$E_1, E_2, \dots$$

containing correct, incorrect, incomplete, or unsupported mathematics.

The structural consequence is only:

$$\text{Cert}(E) = 0 \implies E \text{ produces no certified ordinary extension.} \quad (25)$$

Together with Non-Propagation, this yields:

$$\text{an uncertified generated premise cannot enter an assertoric certified theorem merely through repeated downstream reuse.} \quad (26)$$

If a false-positive certification nevertheless occurs, scission provides the post-admission repair mechanism.

Accordingly, the framework does *not* prove that a generative system cannot hallucinate mathematics. It establishes the weaker structural separation

$$\text{candidate-generation space} \neq \text{certified-theory space.} \quad (27)$$

14 Consequences

The frontier construction has a further consequence when a theorem depends on contextual physical premises.

Suppose

$$\Gamma = \{R_1, \dots, R_k\} \in \text{Front}_n(Q).$$

Then Part II yields the conditional theorem

$$R_1 \wedge \dots \wedge R_k \longrightarrow Q.$$

This makes

$$R_1, \dots, R_k$$

possible targets for independent derivation or experiment.

The direction of logical motivation is therefore permitted:

$$Q \rightsquigarrow_{\text{motivation}} \text{test } R_i.$$

The direction of evidentiary justification is not:

$$Q \not\Rightarrow R_i.$$

Hence a proof may identify which physical assertion would close a capability theorem without supplying evidence that the assertion is true.

14.1 Mathematical frontier compression

A new proof may reduce empirical burden.

For example, suppose initially

$$\Gamma_n(\mathbf{r}; Q) = \{R_1, R_2, R_3\}.$$

If later certified mathematics proves

$$R_1 \wedge R_2 \vdash R_3,$$

then, along the revised route,

$$R_3$$

need no longer be independently certified.

Similarly, discovery of an alternative route r' may give

$$\Gamma_n(r'; Q) \subseteq \Gamma_n(r; Q).$$

Thus stronger mathematics can compress a realization frontier without weakening the standard for empirical admission.

14.2 Two Temporal Boundaries

Certification frontiers and theoretical scission address complementary problems:

$$\begin{array}{ll} \text{Front}_n(\theta) & : \text{missing support before admission,} \\ \text{Sc}_n(\Delta) & : \text{invalidated support after admission.} \end{array} \quad (28)$$

The first prevents unsupported promotion.

The second repairs a support structure that previously passed certification but later lost that status.

Non-Propagation connects the two: material removed or quarantined after scission cannot silently return as an unmarked premise of later assertoric proofs.

14.3 Limits of the Result

The theory has four immediate limitations.

First, operational frontiers are relative to the proof routes that have actually been supplied. They do not assert that no undiscovered proof has a smaller deficit.

Second, the construction does not solve general theorem search.

Third, certification is only as reliable as the proof, provenance, concordance, and evidentiary procedures used to implement it.

Fourth, invalidation of the protected foundation itself is outside ordinary scission and requires foundational revision.

These limitations are compatible with the main result. The theory is a calculus of regulated admission and repair, not an oracle for mathematical or empirical truth.

15 Conclusion

We have given a compact formal theory for the growth of a mathematical presentation constrained by an external protected reference.

The principal constructions are:

restricted protected concordance,
route-specific certification deficits,
the frontier antichain $\text{Front}_n(\theta)$,

provenance-labeled ancestral support,

and

the fixed-point scission operators $\text{Ret}_n(\Delta)$, $\text{Sc}_n(\Delta)$.

Together they yield the structural chain

$$\begin{array}{l} \text{Cert}_n(E) = 0 \implies \text{no certified admission,} \\ \text{Cert}_n(E) = 1 \implies \text{certified ordinary growth,} \\ \emptyset \notin \text{Front}_n(\theta) \implies \text{no premise-complete recorded route,} \\ \emptyset \in \text{Front}_n(\theta) \implies \text{a premise-complete recorded route exists,} \\ \text{ExtReady}_n(\theta) = 1 \wedge \emptyset \in \text{Front}_n(\theta) \\ \implies \text{a certificate-complete recorded route exists,} \\ \text{support invalidation} \implies \text{maximal grounded retention.} \end{array} \quad (29)$$

The last implication is stronger than chronological rollback. A theorem is removed because its proof support has failed, not because it was derived after another theorem that failed.

The resulting theory therefore permits strict mathematical extension while maintaining a stable criterion of identity. It permits conditional physical or external interpretations without promoting their open assumptions. It also permits later correction without discarding independently supported mathematics.

Applied to the Chrysene Formalism, the result may be summarized as

$$\begin{array}{l} \text{Canonical Molecular Anchor} \\ + \text{open candidate generation} \\ + \text{protected concordance} \\ + \text{proof provenance} \\ + \text{certification frontiers} \\ + \text{dependency-aware scission} \\ \implies \\ \text{regulated mathematical growth.} \end{array} \quad (30)$$

The central requirement is consequently not that every proposed extension be correct. It is that the formal system preserve a rigorous distinction among proposal, conditional consequence, certified theorem, contextual realization, external application, and later withdrawal.

In particular,

$$\boxed{\text{generation is not certification.}} \quad (31)$$

A Synthesis Figures

The following diagrams are not used as premises in any proof. They summarize the logical architecture developed in the main text. In particular, they distinguish pre-admission certification, conditional realization, and post-admission scission.

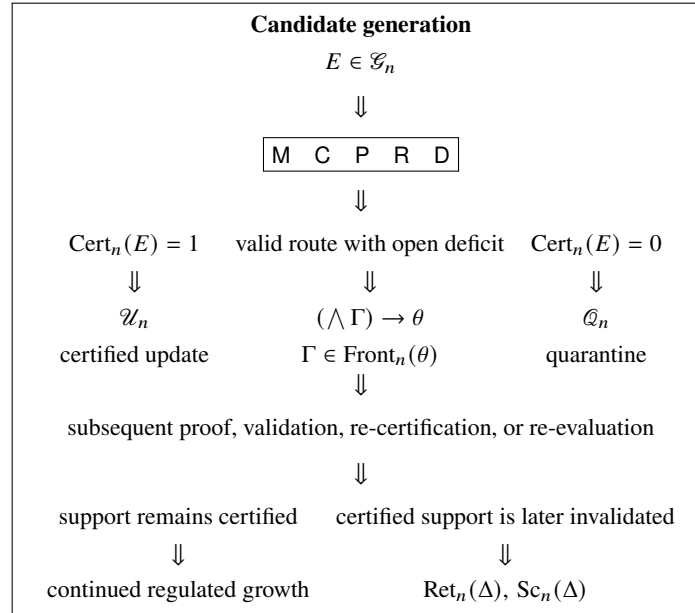


Figure 1: Certification architecture of the condensed theory. Candidate generation is logically prior to certification. A passing candidate may modify the certified state; a failed candidate is quarantined; a valid route with an open deficit remains conditional. If previously certified support is later invalidated, the retained theory is recomputed by theoretical scission.

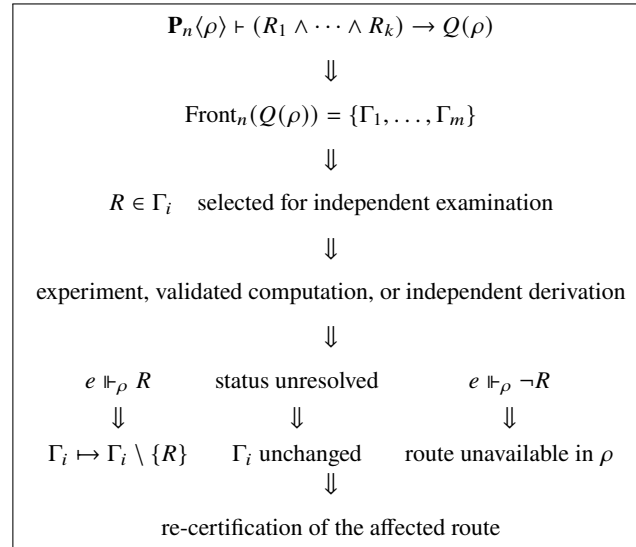


Figure 2: Certification-frontier interpretation for a contextual realization. The proof identifies which premise would close a route, but the desired conclusion supplies no evidentiary support for that premise. Independent evidence may contract the deficit, leave it unchanged, or render the route physically unavailable in the stated context.

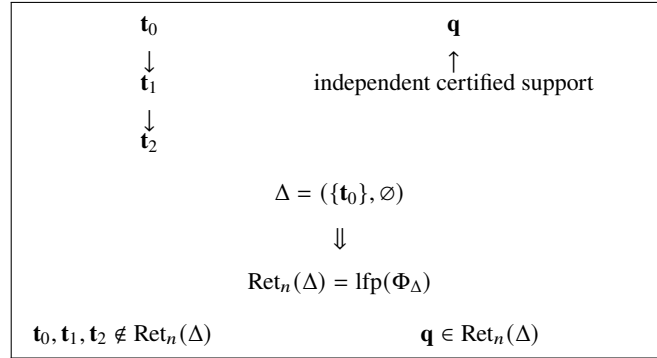


Figure 3: Dependency-aware theoretical scission. If every proof of $\mathbf{t}_1$ uses the invalidated token $\mathbf{t}_0$, and every proof of $\mathbf{t}_2$ uses $\mathbf{t}_1$, that branch is removed. The independently grounded theorem $\mathbf{q}$ survives. The operation therefore follows proof support rather than chronological order.

B Relation to Adjacent Formal Traditions

The constructions of the paper combine several established ideas. None of the individual ingredients—formal derivability, conservative extension, proof certificates, truth maintenance, provenance, or machine-checked theorem proving—is claimed here as a new mathematical discipline. The distinctive object is their combination around an externally anchored presentation with a protected identity fragment, route-relative certification deficits, typed realization conditions, provenance-sensitive non-propagation, and fixed-point scission.

B.1 Formal consequence and restricted conservativity

The distinction between primitive assumptions, derivability, and semantic consequence is standard proof-theoretic and model-theoretic practice; see, for example, Gentzen and Enderton [Gen35, End01]. The present construction retains this distinction by representing a theory through a presentation

$$\mathbf{P} = (\Sigma, A)$$

rather than identifying it with its complete deductive closure.

The concordance condition is related to conservative and definitional extension but is deliberately weaker than full conservativity. Only a designated identity fragment

$$\mathcal{L}_{\text{id}}$$

is held conservative under ordinary growth. Consequently, new theorems involving previously available objects remain admissible even when they are expressible in an older language. The purpose is to preserve theory identity without imposing theorem-level stasis.

B.2 Fixed-point semantics of grounded retention

The scission operator uses a monotone map

$$\Phi_{\Delta} : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$$

and identifies retained support with its least fixed point. This lies within the general fixed-point tradition represented by Tarski [Tar55].

The role of the construction here is specialized. It prevents a collection of mutually supporting theorem vertices from surviving merely because they form a cycle. Retention requires finite grounding in surviving certified primitive support.

B.3 Belief revision, truth maintenance, and diagnosis

AGM belief revision studies rational change to accepted belief sets under expansion, contraction, and revision [AGM85, Gär88].

Truth-maintenance systems instead make dependency information operational: Doyle's TMS and de Kleer's assumption-based TMS explicitly track justification structure so that belief status can change when assumptions change [Doy79, dK86].

Theoretical scission is close in spirit but has a narrower mathematical objective. It retains explicit alternative proof objects and asks:

which theorem vertices still possess a grounded certified proof?

It therefore operates on proof provenance rather than only on a deductively closed belief state.

Reiter's model-based diagnosis is also relevant [Rei87]. Its use of minimal hitting sets to characterize diagnostic alternatives is structurally related to the observation in the present paper that invalidating a theorem with several primitive proof-support sets amounts to hitting every surviving support set. The present use concerns proof survival rather than diagnosis of physical components.

B.4 Provenance

Provenance methods record how a result depends upon its sources. Semiring provenance provides a particularly influential formal example in database theory [GKT07].

Here provenance is attached directly to logical support tokens. Thus

$$[p; s_1] \neq [p; s_2]$$

may hold despite equality of the underlying formula. A proposition used as a quarantined assumption, an explicit hypothesis, a realization premise, and an independently proved theorem therefore constitutes four distinct certification objects. This additional typing is what permits Non-Propagation to distinguish rediscovery from laundering of an uncertified justification.

B.5 Proof certificates and independent checking

The certificate-carrying aspect of the present theory has a direct neighbor in work on foundational proof certificates. Chihani, Miller, and Renaud developed a framework in which heterogeneous proof evidence can be elaborated and checked against a comparatively small trusted proof kernel [CMR13]. More recent work on proof-producing automated reasoning similarly emphasizes independently checkable proof objects rather than trust in the procedure that generated them [BBC⁺23].

The present certificate bundle is broader in scope. A mathematical proof certificate is only one component. Concordance, provenance, realization, and external-domain adequacy are logically distinct conditions and cannot be repaired by mathematical proof validity alone.

B.6 Interactive theorem proving and formal libraries

Modern proof assistants provide an important practical realization of the separation between proof generation and proof checking. Lean and Lean 4 implement small-kernel interactive theorem-proving architectures [dMKA⁺15, dMU21], while mathlib demonstrates the feasibility of maintaining a large community-developed corpus of machine-checked mathematics [The20].

These systems do not by themselves solve the problem addressed here. A proof assistant can establish that a formal sentence follows in its formal environment; it does not automatically establish that a formalization preserves an externally specified molecular identity or that an empirical realization premise is physically true. This distinction motivates the separate concordance and realization layers of the present construction.

B.7 AI-assisted formal theorem proving

The rapid development of machine-assisted theorem generation makes the separation between candidate generation and certified proof especially relevant. LeanDojo couples language-model theorem proving with fine-grained information about proof premises and formal interaction with Lean [YSG⁺23]. AlphaGeometry combines learned proposal generation with symbolic deduction and produces formally checkable geometric proofs [TWL⁺24]. AlphaProof subsequently demonstrated reinforcement-learning-based search for formal proofs in Lean at olympiad level [HMS⁺26].

Recent agentic systems reinforce the same architectural distinction. APOLLO, for example, uses Lean feedback to identify and repair failed subproofs generated by language models before the repaired proof is reverified [OFM25a]. These developments show why high-capacity generation and rigorous checking should be treated as different operations.

They also expose a further distinction important here:

machine-verified proof $\neq$ verified faithfulness of the formalization.

Recent analyses of formal theorem-proving benchmarks have identified substantial discrepancies between informal problem statements and their formal counterparts, emphasizing that proof correctness inside a formal system and correspondence to the intended statement are separate questions [OFM25b]. The protected concordance layer of the present theory addresses an analogous issue for an externally anchored mathematical formalism.

B.8 Position of the present construction

The closest neighboring traditions can therefore be summarized as

proof theory	→	formal derivability,
conservative extension	→	controlled theory growth,
proof certificates	→	independently checkable proofs,
truth maintenance	→	dependency-sensitive revision,
diagnosis	→	minimal hitting-set structure,
provenance	→	source-sensitive dependency,
formal theorem proving	→	machine-checkable derivation.

The present paper combines these around the additional data

$$(\mathfrak{C}_{\text{ref}}, \mathcal{L}_{\text{id}}, \text{Front}_n, \text{AncSupp}, \text{Ret}_n, \text{Sc}_n).$$

The resulting architecture distinguishes two failure regimes:

pre-admission	:	frontier, quarantine, and non-propagation,
post-admission	:	dependency-aware scission.

Its distinctive claim is therefore not that proof checking, belief revision, provenance, or fixed-point reasoning is new. Rather, it is that these mechanisms can be assembled into a single theory of *externally anchored certified mathematical growth* in which protected identity, conditional realization, proof provenance, and later withdrawal are represented simultaneously.

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