

# Stable Transfer Geometry of the Substituted Chrysene Network: $\mathcal{H}^\infty$ Transformation Metrics, Dissipative Well-Posedness, and Survival Under Persistent Disturbance

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## ABSTRACT

We construct a family of distances on the admissible domain of the Chrysene Formalism and determine which metric axioms each satisfies. The central object is the unitary-orbit  $\mathcal{H}^\infty$  distance between the transfer realizations of two structural forms, together with its path closure over admissible transformations; it is an extended pseudometric, and its zero set is exactly the relation of transfer isomorphism, which is shown to be logically independent of isomorphism of the underlying molecular graphs. The construction requires a well-posedness result: the worst-case norm of a transfer operator built from the directional Hamiltonian is generically infinite, and we prove that it is finite precisely when the active region carries no mode that is energetically embedded in the continuum of a second fibration yet orthogonal to every channel into it. The dissipation making this possible is not postulated but computed, as the coupling-weighted density of states of that fibration, and the norm it removes from the admissible sector is shown to be transferred rather than destroyed. Two statements follow: a radius within which a form may be transformed and remain robustly stable, and a common-storage condition under which arbitrary admissible switching among forms preserves a uniform gain bound. Frozen-form stability is shown by counterexample not to imply the latter. The choice of the worst-case over the mean-square norm is justified by four independent arguments, including an explicit resonance family inside the axioms' own data; a self-adjoint perturbation class for which the worst-case norm is instead the wrong measure is identified, and an exact margin supplied for it. Invariance of the admissible domain is shown to be a peak-gain statement not controlled by the worst-case norm, so that survival, the conjunction of robust stability with that invariance, is certified here in two separate halves and by no single certificate. An explicit witness realization with closed-form values is given, the formalism's own explicit model being shown to have infinite gain. Fourteen non-implications, twelve new deferred targets, a concordance table recording five corrections to the prior corpus, and a self-identified defect ledger are stated.

**Keywords:** Chrysene Formalism;  $\mathcal{H}^\infty$  control; transfer geometry; pseudometric; noncommutative geometry; limiting absorption principle; embedded eigenvalue; switched systems; dwell time; structured singular value; robust stability; admissible domain; deferred theorem program.

## 1 Introduction

### 1.1 The problem

Axiom A12 of the Chrysene Formalism supplies the admissible domain  $\Omega_{\text{adm}}$  with a single distance, the extended admissible path distance  $d_C^*$ , built from a weight function that the axiom posits and does not derive [Sch26c]. Axiom A10 separately supplies a transfer operator, built from a directional Hamiltonian, which measures how a disturbance entering one terminal of a motif emerges at another. The two structures have not been related. The applied papers of Volume 1, Number 1 use a worst-case operator norm pervasively — it is the single most frequent technique in that corpus [Sch26b, Tab. 2] — but no paper establishes that the resulting quantity is finite, that it

satisfies any metric axiom, or that it controls the behaviour it is invoked to control.

### 1.2 The objective

The objective of this paper is to construct a mathematical object over the admissible transformations of the substituted chrysene network — transitions by which the network passes from one structural form to another — with respect to which the persistence of the network under persistent, unmodeled external disturbance, and its capacity to compensate for such disturbance by transformation, can be stated precisely and, under explicitly listed realization hypotheses, proved.

Three modes of persistence are distinguished. In the first, a node transforms into a structurally distinct form while the net-

work's transfer structure is preserved up to a controlled deviation: the reference instance is cleavage of the C1–C12a bond under oxidative conditions, followed by closure of the five-membered ring (C1, C2, C3, O<sup>(3)</sup>, C $\alpha$ <sup>(3)</sup>) through the ether oxygen of a 2-phosphoethoxy substituent, the ring being pendant with attachment path C3–C4–C4a into the residual tricyclic system, together with its image (C7, C8, C9, O<sup>(9)</sup>, C $\alpha$ <sup>(9)</sup>) under the framework automorphism (Definition 2.28, Proposition 2.31; the accompanying oxidation and valence accounting is Remark 2.29). In the second, a form is compatible with its environment, including a nucleated environment state that persists after the form has moved on (Section 8.1). In the third, a component leaving the surviving network is transferred into a second fibration rather than deleted (Section 4.5).

A second objective is served by the first. The mathematics developed here uses the molecular content of Axioms A1–A3 nowhere; what it uses is the fibrated, sequence-structured, admissibility-constrained structure of Axioms A4–A12. Making that explicit (Section 10) is what would allow the same stability analysis to be carried to other networks of the same formal type.

Throughout, *survival* denotes the conjunction of two statements that this paper certifies separately: robust stability of the form against the environment (Theorem 5.9) and invariance of the admissible domain under the disturbance (Theorem 8.13). Neither implies the other (Proposition 9.1(iii)), and no single certificate for the conjunction is supplied here (Remark 9.6).

### 1.3 What is established

1. A criterion for finiteness of the worst-case norm (Theorem 4.17), and the identification of the dissipation that makes it possible with the coupling-weighted density of states of the second fibration (Proposition 4.13), whose norm accounting is conservative (Proposition 4.20).
2. An extended pseudometric  $d_{\mathcal{H}^\infty}$  on  $\Omega_{\text{adm}}$  whose zero set is transfer isomorphism (Propositions 5.6 and 5.7), a bottleneck ultrametric companion (Proposition 5.14), and an honest quasi-metric treatment of the directional attenuation quantity the corpus uses informally (Proposition 5.18).
3. A robust-stability radius for a single form (Theorem 5.9), and, under Hypothesis 6.2, a common-storage condition under which arbitrary admissible switching among forms preserves a uniform gain bound (Theorem 6.13, Corollary 6.16), together with a counterexample showing the second does not follow from the first (Proposition 6.6).
4. A justification of the worst-case over the mean-square norm by four independent arguments (Sections 7.1–7.4), and an exact margin for the self-adjoint perturbation class on which the worst-case norm is itself conservative (Theorem 7.15).
5. The separation of the survival claim into robust stability and invariance of  $\Omega_{\text{adm}}$ , the latter being a peak-gain statement

(Theorem 8.13) that the worst-case norm provably does not control (Proposition 8.15).

6. A witness realization with closed-form values throughout (Appendix A), the formalism's own explicit model being shown to have infinite gain (Remark 4.6).

### 1.4 What is not claimed

No axiom is introduced (Remark 2.8); no deferred target of [Sch26c] is discharged (Table 2); no result of this paper licenses an inference to physical scission (Proposition 9.1(i), (ii)); no chemical, energetic or kinetic claim is asserted or used (Remark 2.24); and no existing paper of the corpus is certified by the portability result of Section 10 (Remark 10.7). The paper's own defects, including one error found in the proof of its central theorem, are recorded in Table 6.

### 1.5 Logical position

Every numbered statement carries one of four status labels (Definition 2.7): axiomatic, derived realization theorem, target, or empirical instantiation claim. Every result that uses a hypothesis beyond A1–A12 names that hypothesis, and every hypothesis is registered in Section 2.6. The molecular reading of any statement is confined to remarks explicitly marked as empirical instantiation claims, and no such remark is used in a proof. This paper falls within the framework-revision scope of [Sch26b, §4.5] and is written to the orthogonal-constraint review bar of [Sch26b, §4.3].

### 1.6 Organization

Section 2 fixes conventions, status labels, the hypothesis register, and the locant dictionary. Section 3 defines forms, transfer isomorphism, and the reduction of a defect form to a composite realization. Section 4 settles well-posedness. Section 5 constructs the transformation geometry and the robust-stability radius. Section 6 treats survival across transformations. Section 7 justifies the choice of norm. Section 8 completes the survival claim and situates the remaining candidate quantities. Section 9 states what is not established, and Section 10 what transfers to other networks. Appendix A is the witness realization, Appendix B the concordance and defect ledger, and Appendix C the established literature the paper draws on.

## 2 Conventions, Logical Status, and the Hypothesis Register

This section fixes notation, isolates the two frequency variables that the sequel keeps distinct, defines the four logical status labels used throughout, introduces the substrate-free structure over which all later theorems are proved, and registers, by name,

every realization hypothesis on which a later statement depends. No axiom is introduced here or anywhere in this paper; see Remark 2.8.

## 2.1 Notation

Throughout,  $\mathfrak{C} = (V_{\mathfrak{C}}, E_{\mathfrak{C}}, \iota, \lambda)$  denotes the molecular reduct of Axiom A1,  $\mathfrak{Q}_{\mathfrak{C}}$  the quasi-local algebra of Axiom A6, and

$$\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3 \quad (1)$$

the sequence algebra of Axiom A7, generated by  $\mathfrak{Q}_{\mathfrak{C}}$  together with the translation unitaries  $\{u_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{Z}^3}$  subject to the covariance relation  $u_{\mathbf{m}} P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}$ . We write  $(\mathfrak{A}_{\mathfrak{C}}^{\circ}, \mathcal{H}_{\omega}, D_{\mathfrak{C}})$  for the spectral triple of Axiom A9,  $\omega$  for the state of Axiom A7, and  $(X, j_{\text{str}}, \mathcal{E}_0, w, \{C_j\}, \{\tau_j\})$  for the admissibility datum of Axiom A12, with admissible domain  $\Omega_{\text{adm}}$ , admissible transition relation  $\mathcal{E}_{\text{adm}}$ , and extended admissible path distance  $d_{\mathfrak{C}}^*$  as in [Sch26c].

For Hilbert spaces  $\mathcal{K}, \mathcal{H}$  we write  $\mathcal{B}(\mathcal{K}, \mathcal{H})$  for the bounded operators from  $\mathcal{K}$  to  $\mathcal{H}$ ,  $\mathcal{B}(\mathcal{H})$  for  $\mathcal{B}(\mathcal{H}, \mathcal{H})$ ,  $\sigma(\cdot)$  for spectrum,  $\rho(\cdot)$  for resolvent set,  $\sigma_{\max}(\cdot)$  for the largest singular value, and  $A^*$  for the adjoint. Operator-valued functions are always understood in the strong sense unless stated otherwise. The symbol  $\oplus$  denotes an orthogonal direct sum of Hilbert spaces and, for operators, the corresponding block-diagonal operator.

## 2.2 The two frequency variables

Two distinct conjugate variables occur in the corpus and are not interchangeable. The first is conjugate to the Hamiltonian evolution of Axioms A10–A11; the second is conjugate to the lattice translation action  $\alpha$  of Axioms A6–A7. We fix both.

**Definition 2.1** (Energy variable and causal half-plane). Let  $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$ . The *energy variable* is the complex spectral parameter  $z \in \mathbb{C}_+$  appearing in the resolvent  $R_B(z) = (z - H_B)^{-1}$  of the directional Hamiltonian of Axiom A10, with the causal convention  $e^{izt}$ ,  $\text{Im } z > 0$ , of Axiom A11. Its boundary value is written  $E \in \mathbb{R}$ .

**Definition 2.2** (Quasi-momentum and the spatial torus). Let  $\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$ . The *quasi-momentum variable* is the parameter  $\theta \in \mathbb{T}^3$  dual to the translation group  $\mathbb{Z}^3$  of Axiom A6, under which a translation-covariant operator on  $\ell^2(\mathbb{Z}^3) \otimes \mathcal{K}$  decomposes as a direct integral over  $\mathbb{T}^3$  of its fibre symbols.

**Definition 2.3** (Joint frequency domain and  $\mathcal{H}^{\infty}$  norm). Write  $\varsigma := (z, \theta) \in \mathbb{C}_+ \times \mathbb{T}^3$ . For an operator-valued map  $T : \mathbb{C}_+ \times \mathbb{T}^3 \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{H})$  that is analytic in  $z$  for each fixed

$\theta$  and continuous in  $\theta$ , set

$$\|T\|_{\mathcal{H}^{\infty}} := \sup_{\theta \in \mathbb{T}^3} \sup_{z \in \mathbb{C}_+} \sigma_{\max}(T(z, \theta)). \quad (2)$$

The space of such  $T$  with  $\|T\|_{\mathcal{H}^{\infty}} < \infty$  is denoted  $\mathcal{H}^{\infty}(\mathbb{C}_+ \times \mathbb{T}^3; \mathcal{B}(\mathcal{K}, \mathcal{H}))$ . When  $T$  does not depend on  $\theta$  the second supremum is vacuous and (2) reduces to the classical  $H^{\infty}$  norm on  $\mathbb{C}_+$ ; when  $T$  does not depend on  $z$  it reduces to a supremum over the spatial torus alone.

*Remark 2.4* (Reconciliation of the monograph conventions). The evaluation of an  $H^{\infty}$  norm on the unit circle  $\partial\mathbb{D}$  in [Sch26f, Def. 2.1.3, Ch. 4] and its evaluation on a half-plane in [Sch26c] are, under Definition 2.3, statements about different variables and are therefore not in conflict. The unit-circle variable of [Sch26f] is the component  $\theta_z$  of  $\theta$  conjugate to translation along the stacking direction; the half-plane variable of [Sch26c] is the energy variable  $z$  of Definition 2.1. Neither is a time-frequency variable conjugate to the sequence index of Axiom A7: by [Sch26c, Rem. 14] the sequence order is not a physical time. The convention (2) follows the treatment of spatially invariant systems in [BPD02], where the spatial transform variable is carried alongside, and not identified with, the temporal one.

## 2.3 Disturbance classes

**Definition 2.5** (Disturbance classes). Let  $\mathcal{K}$  be a Hilbert space and let  $w : \mathbb{R}_{\geq 0} \rightarrow \mathcal{K}$  be measurable. We distinguish:

- (i) the *finite-energy class*  $\mathcal{W}_2 := L^2(\mathbb{R}_{\geq 0}; \mathcal{K})$ , with norm  $\|w\|_2$ ;
- (ii) the *persistent class*  $\mathcal{W}_{\text{pow}}$  of  $w$  with finite mean power

$$\|w\|_{\text{pow}}^2 := \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|w(t)\|^2 dt < \infty; \quad (3)$$

- (iii) the *amplitude-bounded class*  $\mathcal{W}_{\infty} := L^{\infty}(\mathbb{R}_{\geq 0}; \mathcal{K})$ , with norm  $\|w\|_{\infty}$ .

*Remark 2.6* (Operational reading of “chaotic environment”). The phrase *chaotic environment*, used informally throughout [Sch26f] and the applied papers of Volume 1, Number 1, is read in this paper exclusively as membership of the disturbance in  $\mathcal{W}_{\text{pow}}$  or  $\mathcal{W}_{\infty}$ , that is, as persistence and boundedness without a model, and never as an assertion about positive Lyapunov exponents, ergodicity, or mixing of an underlying dynamical system. No such assertion is made or used. In particular  $\mathcal{W}_{\text{pow}} \not\subseteq \mathcal{W}_2$ : a persistent disturbance

is not of finite energy, which is one of the two structural reasons that the quantity of Definition 2.3, rather than an  $H^2$  quantity, is the relevant one (Section 7).

## 2.4 Logical status

Every numbered statement in this paper carries exactly one of four status labels, placed immediately after its number. The first three are those of [Sch26c, Defs. 1, 3, 5]; the fourth is that of [Sch26c, Def. 6].

**Definition 2.7** (Status labels). Let  $\mathcal{A} = \{A1, \dots, A12\}$ , let  $\mathfrak{R} \in \text{Ext}_{12}(\mathbb{C}^{\text{chr}})$  be a specified full realization, and let  $H_{\mathfrak{R}}$  denote an explicitly stated collection of realization hypotheses in the sense of [Sch26c, Def. 3]. A statement  $P$  is labelled

- [A] *axiomatic* if  $\mathcal{A} \models P$ ; that is,  $P$  is proved from the twelve axioms alone;
- [D] a *derived realization theorem* if  $\mathcal{A} \cup H_{\mathfrak{R}} \models P$  and  $\mathcal{A} \not\models P$ , and a complete proof from  $\mathcal{A} \cup H_{\mathfrak{R}}$  is given in this paper, with  $H_{\mathfrak{R}}$  named explicitly in the statement;
- [T] a *target* if  $P$  is asserted in the form  $\mathcal{A} \cup H_{\mathfrak{R}} \models P$  with  $H_{\mathfrak{R}}$  identified but the implication not discharged here;
- [E] an *empirical instantiation claim* if  $P$  asserts that a physically realized system satisfies a hypothesis or conclusion of the formalism, so that  $P$  is not a mathematical consequence of  $\mathcal{A} \cup H_{\mathfrak{R}}$  under any assignment and is discharged, if at all, by measurement.

**Remark 2.8** (No thirteenth axiom). This paper introduces no axiom. Every structure below is either a definition internal to  $\mathcal{A}$ , or a realization hypothesis in the sense of [Sch26c, Def. 3]. The restriction is deliberate: an additional axiom would reopen the non-vacuity theorem [Sch26c, Thm. 61] and the acyclicity of the axiom ladder [Sch26c, Prop. 39], neither of which is re-established here.

**Remark 2.9** (Status of statements carried over from the monographs). Several results of [Sch26d, Sch26f, Sch26e] are restated in this paper in modified form. A restatement is labelled by its status *here*, and the relation to the original is recorded in the concordance table of Appendix B rather than in the statement itself. No statement of [Sch26d, Sch26f, Sch26e] is imported with its original status.

## 2.5 Fibrated transfer networks

The theorems of Sections 4–8 use none of the molecular content of Axioms A1–A3. To make the range of their applicability explicit, and to permit the transfer recorded in Section 10, we isolate the structure they do use.

**Definition 2.10** (Fibrated transfer network). A *fibrated transfer network* of rank  $d \in \mathbb{N}$  is a tuple

$$\mathfrak{R} = (\Lambda, \Sigma_{\text{sp}}, \varsigma_{\text{sp}}, \mathcal{S}, \mathcal{T}, \{\mathcal{H}_s\}_{s \in \mathcal{S}}, \{G_s\}_{s \in \mathcal{S}}, \mathcal{D}) \quad (4)$$

in which

- (N1)  $\Lambda \cong \mathbb{Z}^d$  is the *site lattice*, carrying the translation action of  $\mathbb{Z}^d$ ;
- (N2)  $\Sigma_{\text{sp}}$  is a finite *species alphabet* and  $\varsigma_{\text{sp}} : \Lambda \rightarrow \Sigma_{\text{sp}}$  a *species labelling*, the pair  $(\Lambda, \varsigma_{\text{sp}})$  being called a *population* of  $\Lambda$ ;
- (N3)  $\mathcal{S}$  is a nonempty set of *forms* and  $\mathcal{T} \subseteq \mathcal{S} \times \mathcal{S}$  a *transition relation*, not assumed symmetric;
- (N4)  $\{\mathcal{H}_s\}$  is a family of Hilbert spaces indexed by  $\mathcal{S}$ , and each  $G_s$  is an operator-valued map  $\mathbb{C}_+ \times \mathbb{T}^d \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{Y})$  with common input space  $\mathcal{K}$  and common output space  $\mathcal{Y}$ , analytic in  $z$  and continuous in  $\theta$ , called the *transfer realization of the form  $s$* ;
- (N5)  $\mathcal{D}$  is a *dissipation datum*: a distinguished decomposition  $\mathcal{H}_s = \mathcal{H}_s^{\text{pri}} \oplus \mathcal{H}_s^{\text{sec}}$  for each  $s$ , together with coupling operators between the two summands.

The network is *admissible* when  $\mathcal{S}$  carries constraint functionals and  $\mathcal{T}$  is restricted to pairs of states satisfying them, in the manner of Axiom A12.

**Remark 2.11** (Role of Definition 2.10). Definition 2.10 is a bookkeeping device, not a generalization claim. Nothing in it asserts that a given physical or informational network is a fibrated transfer network; such an assertion is, in every instance, a statement of type [E]. What Definition 2.10 does assert is the converse direction used in Section 10: any system for which the data (N1)–(N5) are exhibited inherits the conclusions of Sections 4–8 verbatim, because those proofs use no other data. The reference molecule enters only through Hypothesis 2.23 below, and only in statements labelled [E].

## 2.6 The hypothesis register

Each hypothesis below is a realization hypothesis in the sense of [Sch26c, Def. 3] and is referred to in later statements by its name. No statement of this paper uses an unnamed hypothesis.

**Hypothesis 2.12** ( $H_{\text{fib}}$ , state fibration). There is an assignment  $x \mapsto \mathcal{H}_x$  of a separable Hilbert space to each  $x \in \Omega_{\text{adm}}$ , and for each  $(x, y) \in \mathcal{E}_{\text{adm}}$  a distinguished isometry of the terminal spaces of the associated motif with  $\mathcal{H}_x$  and  $\mathcal{H}_y$  respectively.

**Hypothesis 2.13** ( $H_{\text{lab}}$ , edge labelling). There is a map assigning to each  $(x, y) \in \mathcal{E}_{\text{adm}}$  a directional motif  $(\alpha, N)$  in the sense of Axiom A10, together with its bounded self-adjoint Hamiltonian and coupling operators.

**Hypothesis 2.14** ( $H_{\text{form}}$ , common channels). All forms of the network under consideration carry transfer realizations with a common input space  $\mathcal{K}$  and a common output space  $\mathcal{Y}$ , so that  $G_x$  and  $G_y$  are comparable as elements of the same operator space for all  $x, y$ .

**Hypothesis 2.15** ( $H_{\text{sec}}$ , second fibration). There is a second  $\mathbb{Z}^3$ -fibrated network, disjoint from the first as a site set, together with a bounded coupling operator from each primary site space into it. The second network is the range of the handoff map of Section 4 and is not assumed to be preserved by the constraint functionals of Axiom A12.

**Hypothesis 2.16** ( $H_{\text{sp}}$ , cipher population). The second network of Hypothesis 2.15 carries the species alphabet  $\Sigma_{\text{sp}} = \{\text{p}, \text{d}\}$ , whose two letters correspond to the primary attachment set  $A_{\mathbb{C}}^{\text{pri}} = \{v_3, v_6, v_9, v_{12}\}$  and the dual-cipher locant set  $A_{\mathbb{C}}^{\text{dual}} = \{v_2, v_5, v_8, v_{11}\}$  of Axiom A1, and a species labelling  $\varsigma_{\text{sp}} : \Lambda \rightarrow \Sigma_{\text{sp}}$  in the sense of (N2). Each letter carries its own site transfer datum.

*Remark 2.17* (Two senses of “cipher”). Hypothesis 2.16 labels *sites of a lattice* by which attachment set the occupying unit uses. This is not the dual-cipher structure of Axiom A8, which is an assignment of commuting subalgebras  $\mathfrak{M}^{\text{pri}}$  and  $\mathfrak{M}^{\text{dual}} = (\mathfrak{M}^{\text{pri}})'$  acting on  $\mathcal{H}_{\omega}$ . The two are kept distinct throughout, and no result below identifies them. In particular, by [Sch26c, Thm. 15] every element of  $\mathfrak{M}^{\text{pri}}$  commutes with every element of  $\mathfrak{M}^{\text{dual}}$ , so a coupling between the two sectors of Axiom A8 cannot be an element of either; any such coupling is a further hypothesis and is not assumed here. Accordingly the network of Hypothesis 2.15 is called the *second fibration* throughout, and never the dual-cipher fibration: the adjective is reserved for the locant set  $A_{\mathbb{C}}^{\text{dual}}$  of Axiom A1, for the species letter d that Hypothesis 2.16 attaches to it, and for the commutant  $\mathfrak{M}^{\text{dual}}$  of Axiom A8. Naming a third object by it would transport a name across

the axiom ladder. Which species populates the second fibration is recorded where it is used, in Hypothesis 4.9, and not in the name of the sector.

**Hypothesis 2.18** ( $H_{\text{cov}}$ , translation covariance). The site transfer data of Hypothesis 2.16 are covariant for the  $\mathbb{Z}^3$  action of Axiom A6 on the sublattice on which  $\varsigma_{\text{sp}}$  is periodic, so that the direct-integral decomposition of Definition 2.2 applies on that sublattice.

**Hypothesis 2.19** ( $H_{\text{env}}$ , environment interconnection). There is a set  $\Omega_{\text{env}}$  of environment states, and for each  $e \in \Omega_{\text{env}}$  a causal operator  $E_e$  on the channel spaces of Hypothesis 2.14, such that the feedback interconnection of  $G_x$  with  $E_e$  is well posed for the pairs  $(x, e)$  considered.

**Hypothesis 2.20** ( $H_{\text{nuc}}$ , nucleation and persistence). There is a distinguished set  $\mathcal{N}$  of *nucleation data* and a partial map  $(x, e) \mapsto e'$  carrying a form and an environment state to a nucleated environment state, defined when  $x$  presents an element of  $\mathcal{N}$ . The nucleated state  $e'$  is retained under subsequent transitions of the form coordinate; this retention is the *persistence* property and is a hypothesis, not a conclusion.

**Hypothesis 2.21** ( $H_{\text{rec}}$ , reciprocity). The transfer data of a given form are reciprocal: in a fixed basis the associated resolvent kernel is complex symmetric, equivalently the realization is invariant under transposition of its input and output channels.

**Hypothesis 2.22** ( $H_{\text{out}}$ , constraint–output compatibility). The constraint functionals  $C_j$  of Axiom A12 factor through the output channel of Hypothesis 2.14: there are functions  $c_j$  with  $C_j(x) = c_j(y_x)$ , where  $y_x$  is the output signal of  $G_x$ . Without  $H_{\text{out}}$  no gain bound of any kind constrains membership in  $\Omega_{\text{adm}}$ .

**Hypothesis 2.23** ( $H_{\text{chem}}$ , chemical instantiation). The reference molecular reduct  $\mathbb{C}^{\text{chr}}$  and the rearrangement data of Section 2.7 instantiate Hypotheses 2.12–2.21 under a stated atom mapping.

*Remark 2.24* (Status of  $H_{\text{chem}}$ ). Hypothesis 2.23 is of type [E] and is used in no proof. It appears only in remarks that interpret a mathematical conclusion molecularly, and every

such remark is marked [E]. In particular no theorem of this paper becomes false, or loses a hypothesis, if  $H_{\text{chem}}$  fails.

## 2.7 Locants and the two rearrangement data

This subsection fixes the correspondence between IUPAC chrysene locants and the vertex notation of [Sch26c, App. A], and records the two rearrangements referred to in Sections 3 and 8. Everything in this subsection is combinatorial: it concerns labelled graphs, and asserts nothing about energetics, mechanism, or feasibility.

**Definition 2.25** (Reference locant notation). Write  $Ck$  for the IUPAC locant  $k \in \{1, \dots, 12\}$  of the chrysene carbon framework and  $Cka$ ,  $Ckb$  for its fusion carbons, so that the vertex set of the reference framework is

$$V(\mathfrak{C}^{\text{chr}}) = \{C1, \dots, C12\} \cup \{C4a, C4b, C6a, C10a, C10b, C12a\}, \quad (5)$$

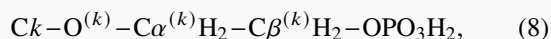
of cardinality 18, with the four rings

$$\begin{aligned} R_1 &= (C1, C2, C3, C4, C4a, C12a), \\ R_2 &= (C4a, C4b, C10b, C11, C12, C12a), \\ R_3 &= (C4b, C5, C6, C6a, C10a, C10b), \\ R_4 &= (C6a, C7, C8, C9, C10, C10a). \end{aligned} \quad (6)$$

The primary attachment set of Axiom A1 is  $\{C3, C6, C9, C12\}$  and the dual-cipher locant set is  $\{C2, C5, C8, C11\}$ . The identification of (5)–(6) with the vertex notation  $v_1, \dots, v_{12}$ ,  $w_1, \dots, w_6$  of [Sch26c, App. A.5] is fixed by  $v_k \leftrightarrow Ck$  together with the adjacency certificate given there, which determines the fusion vertices uniquely:

$$\begin{aligned} w_1 &= C12a, & w_2 &= C10b, & w_3 &= C4a, \\ w_4 &= C10a, & w_5 &= C4b, & w_6 &= C6a. \end{aligned} \quad (7)$$

**Definition 2.26** (2-Phosphoethoxy substituent). A locant  $Ck$  with  $k \in \{3, 6, 9, 12\}$  is said to carry a 2-phosphoethoxy substituent when its attachment datum is the chain



with  $O^{(k)}$  the ether oxygen bonded to  $Ck$  and  $C\alpha^{(k)}$ ,  $C\beta^{(k)}$  the two chain carbons, the terminal phosphate being written as the free acid and any salt form being admitted.

**Remark 2.27** (Nomenclature). [E] The chain (8) is a two-carbon phosphoethoxy group and is not a polyethylene

glycol; the granted specification [Sch25] distinguishes hydroxylalkoxy from hydroxylpolyethyleneoxy substituents, and (8) is of the former class. Where solubility of the polycyclic core is at issue, it is the four terminal phosphates and not a polyether chain that is the relevant feature. No solubility claim is made or used in this paper.

**Definition 2.28** (Rearrangement datum). A *rearrangement datum* is a quadruple  $\mathcal{R} = (b^-, \text{ox}, b^+, \gamma)$  consisting of a bond  $b^-$  of the reference framework to be deleted, an *oxidation record*  $\text{ox}$  assigning to each endpoint of  $b^-$  the oxygen substituent it acquires, a pair  $b^+$  of atoms to be joined, and the cycle  $\gamma$  of the resulting graph containing  $b^+$ . We record two:

$$\mathcal{R}^{(3)} = \left( \{C1, C12a\}, (C1:\text{oxo}, C12a:\text{hydroxy}), \{C\alpha^{(3)}, C1\}, (C1, C2, C3, O^{(3)}, C\alpha^{(3)}) \right), \quad (9)$$

$$\mathcal{R}^{(9)} = \left( \{C7, C6a\}, (C7:\text{oxo}, C6a:\text{hydroxy}), \{C\alpha^{(9)}, C7\}, (C7, C8, C9, O^{(9)}, C\alpha^{(9)}) \right). \quad (10)$$

In each case  $\gamma$  is a cycle of length five containing one oxygen atom, closed by the newly formed carbon–carbon bond  $b^+$ .

**Remark 2.29** (Valence ledger). [E] The rewrite of Definition 2.28 is not bond-count preserving, and the following accounting is what makes (9) describe a single molecule. Deletion of  $b^-$  under oxidative conditions leaves C1 as a carbonyl carbon and C12a as an oxygenated aryl carbon; in the aromatic tautomer the latter carries a hydroxyl, so that the residual fused system retains its aromaticity. Formation of  $b^+$  adds  $C\alpha^{(3)}$  across that carbonyl, so that in the product

- C1 is  $sp^3$ , bonded to C2,  $C\alpha^{(3)}$ , a hydroxyl and a hydrogen;
- $C\alpha^{(3)}$  has lost one hydrogen and is written  $C\alpha^{(3)}H$  at every point of use after (9), in place of the  $C\alpha H_2$  of (8);
- C12a carries a hydroxyl and remains within the residual fused system.

The five-membered ring is therefore not saturated by the rewrite alone; the word is not used of it. This accounting follows the transformation shown schematically in [Sch25, Fig. 6, sheet 19], and its status is that of Remark B.7: it is an empirical instantiation claim, and no energetic, kinetic or mechanistic assertion is made or used.

**Remark 2.30** (Connectivity after  $\mathcal{R}^{(3)}$ ). Deletion of  $\{C1, C12a\}$  destroys the cycle  $R_1$  of (6) and leaves the three remaining rings  $R_2, R_3, R_4$  intact; the fragment  $C1-C2-C3-C4$  remains attached to the residual system through the bonds  $C3-C4$  and  $C4-C4a$ . The cycle  $\gamma$  of (9) therefore shares no edge with  $R_2, R_3, R_4$ : it is pendant, and its attachment path into the residual fused system is  $C3-C4-C4a$ , the carbon  $C4$  being a spacer belonging to neither  $\gamma$  nor  $R_2$ . The exocyclic chain  $C\beta^{(3)}H_2OPO_3H_2$  is carried by the ring atom  $C\alpha^{(3)}$ , adjacent to the ring oxygen  $O^{(3)}$ . The attachment data at  $C6, C9, C12$  are unaffected. After both  $\mathcal{R}^{(3)}$  and  $\mathcal{R}^{(9)}$ , the residual fused system is  $R_2 \cup R_3$ .

**Proposition 2.31** (Symmetry relation of the two rearrangements). [A] Let  $\sigma$  be the permutation of  $V(\mathfrak{C}^{\text{chr}})$  determined by the transpositions

$$\begin{aligned} C1 \leftrightarrow C7, \quad C2 \leftrightarrow C8, \quad C3 \leftrightarrow C9, \\ C4 \leftrightarrow C10, \quad C5 \leftrightarrow C11, \quad C6 \leftrightarrow C12, \\ C4a \leftrightarrow C10a, \quad C4b \leftrightarrow C10b, \quad C6a \leftrightarrow C12a. \end{aligned} \quad (11)$$

Then  $\sigma$  is a graph automorphism of the reference carbon framework; it preserves the primary attachment set and the dual-cipher locant set of Axiom A1 setwise; and, when extended to the substituent atoms of (8) by  $O^{(3)} \leftrightarrow O^{(9)}$ ,  $C\alpha^{(3)} \leftrightarrow C\alpha^{(9)}$ ,  $C\beta^{(3)} \leftrightarrow C\beta^{(9)}$ , it carries  $\mathcal{R}^{(3)}$  to  $\mathcal{R}^{(9)}$ .

*Proof.* By (6) the edge set of the framework is

$$\begin{aligned} E = \{ & C1C2, C2C3, C3C4, C4C4a, \\ & C4aC12a, C12aC1, C4aC4b, C4bC10b, \\ & C10bC11, C11C12, C12C12a, C4bC5, \\ & C5C6, C6C6a, C6aC10a, C10aC10b, \\ & C6aC7, C7C8, C8C9, C9C10, C10C10a \}, \end{aligned} \quad (12)$$

of cardinality 21, consistent with 18 vertices and four independent cycles. Applying (11) edgewise: the first six edges of (12) map to  $C7C8, C8C9, C9C10, C10C10a, C10aC6a, C6aC7$ , all in  $E$ ; the next five map to  $C10aC10b, C10bC4b, C4bC5, C5C6, C6C6a$ , all in  $E$ ; and the remaining ten are the images of these under  $\sigma^{-1} = \sigma$ . Hence  $\sigma(E) = E$  and  $\sigma$  is an automorphism. It carries  $\{C3, C6, C9, C12\}$  and  $\{C2, C5, C8, C11\}$  to themselves. Finally  $\sigma(\{C1, C12a\}) = \{C7, C6a\}$ ,  $\sigma(\{C\alpha^{(3)}, C1\}) = \{C\alpha^{(9)}, C7\}$ , and  $\sigma$  carries the cycle of (9) to that of (10), which is the assertion.  $\square$

**Remark 2.32** (What Proposition 2.31 does and does not give). Proposition 2.31 is a statement about labelled graphs and is independent of Axioms A4–A12. It licenses treating  $\mathcal{R}^{(3)}$

and  $\mathcal{R}^{(9)}$  as a single orbit for combinatorial purposes only. It does not follow that the two rearrangements have equal transfer realizations: the realization of a form depends on the data of Hypotheses 2.12–2.13, which are not required to be  $\sigma$ -equivariant, and by [Sch26c, Rem. 32 and display (444)] a geometric symmetry of  $\mathfrak{C}$  does not by itself induce a symmetry of transfer.  $\sigma$ -equivariance of the realization is an additional hypothesis, invoked where needed and never assumed silently.

**Remark 2.33** (Graph-theoretic status of the rearranged framework). [A] The graph obtained from  $\mathfrak{C}^{\text{chr}}$  by  $\mathcal{R}^{(3)}$  contains a cycle of length five and a residual tricyclic system, and is therefore not a four-ring polyhex; it does not belong to the comparison class  $\mathcal{B}_4$  and is not a model of Axiom A1. Consequently a rearranged unit is treated throughout as a structural state  $x \in X$  in the sense of Axiom A12, equivalently as a local defect of the background network of Axiom A6, and never as an alternative molecular reduct. In particular [Sch26c, Thm. 36] is not applicable to it, and no conclusion of Axioms A1–A3 is asserted of it.

**Remark 2.34** (Atom accounting; the word “isomorphic”). [E] The passage from  $\mathfrak{C}^{\text{chr}}$  with substituent (8) to the rearranged framework of Definition 2.28 is not a constitutional isomerization: by Remark 2.29 the rewrite adds oxygen at both endpoints of  $b^-$  and alters the hydrogen count at  $C1$  and  $C\alpha^{(3)}$ , so that reagents are consumed and the product does not share the molecular formula of the starting structure. The transformation is shown schematically in [Sch25, Fig. 6, sheet 19]; what each instrument of the patent record does and does not cover is stated in Remark B.7, and no reagent set is specified in this paper. No energetic, kinetic, or mechanistic claim is made here, and none is used below. The term *isomorphic*, where it is applied in this paper to a pair of forms, therefore never denotes graph isomorphism or constitutional isomerism; it denotes transfer isomorphism in the sense of Definition 3.6 of Section 3, namely vanishing of the unitary-orbit distance of Section 5.

**Remark 2.35** (Nucleation data, and the limits of the term “pocket”). [E] The element of  $\mathcal{N}$  presented by a rearranged form under Hypothesis 2.20 is, in the intended molecular reading, the five-membered ring of (9) together with its exocyclic phosphate; alternative nucleation data admitted by Hypothesis 2.20 include a coordinated metal ion and a defect of the network of Axiom A6. The term *vestigial*, or *ghost*, *pocket* for a persistent nucleated environment state follows the usage of the provisional filings listed in [Sch26f, p. 212], in particular the filing directed to pharmacologi-

cal chaperones and biomimetic ligands targeting vestigial scaffold binding sites. That citation supplies provenance for the term and not a chemical definition of a pocket: no residue geometry, cavity volume, or catalytic residue set is supplied here, and none is used. That any nucleation datum nucleates a catalytic site is an assertion of type [E], conditional in addition on the Identification Hypothesis  $\Phi$  of [Sch26b], and it enters no proof.

### 3 Forms, Admissible Transformations, and Network Realizations

This section fixes the objects between which the geometry of Section 5 measures distance. Three things are established. First, the notion of a *form* and of the transfer realization it carries (Section 3.1). Second, the equivalence relation of *transfer isomorphism*, which is the only sense in which two forms are called isomorphic anywhere in this paper, and its logical independence from isomorphism of the underlying graphs (Section 3.2). Third, the reduction of a form carrying a local rearrangement to a composite realization in the sense of Definition 4.1, which is what licenses the standing finite-dimensionality assumption of Section 4 (Section 3.3).

#### 3.1 Forms and their realizations

**Definition 3.1** (Form). A *form* is an element  $x \in \Omega_{\text{adm}}$  of the admissible domain of Axiom A12. Two forms  $x, y$  are *directly transformable* if  $(x, y) \in \mathcal{E}_{\text{adm}}$ ; a finite  $\mathcal{E}_{\text{adm}}$ -path from  $x$  to  $y$  is a *transformation route*, and  $P_{\text{adm}}(x, y)$  denotes the set of such routes, as in Axiom A12.

**Definition 3.2** (Form realization). Assume Hypotheses 2.12, 2.13 and 2.14. A *realization* of the form  $x$  is a map

$$G_x : \mathbb{C}_+ \times \mathbb{T}^3 \longrightarrow \mathcal{B}(\mathcal{K}, \mathcal{Y}), \quad (13)$$

analytic in the energy variable and continuous in the quasi-momentum variable of Definitions 2.1 and 2.2, with input space  $\mathcal{K}$  and output space  $\mathcal{Y}$  common to all forms under consideration. The realization is *stable* if  $\|G_x\|_{\mathcal{H}^\infty} < \infty$  in the sense of Definition 2.3. We write  $\mathcal{G}$  for the set of stable realizations of forms of  $\Omega_{\text{adm}}$ .

**Remark 3.3** (Relation to the composite realization). Definition 3.2 is stated at the level of the transfer function alone. The composite realization  $\mathfrak{G}_x$  of Definition 4.1 is an instance: its transfer function (28) satisfies (13), and it is stable precisely under the criterion of Theorem 4.17. Nothing below uses the internal data of  $\mathfrak{G}_x$  except through  $G_x$ ;

this is what makes the geometry of Section 5 independent of the choice of internal coordinates and, by Remark 2.11, transferable to any fibrated transfer network.

**Hypothesis 3.4** ( $H_{\text{ch}}$ , finite channels).  $\dim \mathcal{K} = k < \infty$  and  $\dim \mathcal{Y} = q < \infty$ .

**Lemma 3.5** (Gain as induced norm). [D] Assume Hypothesis 3.4 and let  $G_x$  be stable and independent of  $\theta$ . Then  $\|G_x\|_{\mathcal{H}^\infty}$  equals the norm of the associated convolution operator  $L^2(\mathbb{R}_{\geq 0}; \mathcal{K}) \rightarrow L^2(\mathbb{R}_{\geq 0}; \mathcal{Y})$ ,

$$\|G_x\|_{\mathcal{H}^\infty} = \sup \left\{ \|y\|_2 : w \in \mathcal{W}_2, \|w\|_2 \leq 1 \right\}, \quad (14)$$

where  $y$  is the output produced by the disturbance  $w$ . For  $\theta$ -dependent  $G_x$  the same identity holds with  $L^2$  taken over  $\mathbb{R}_{\geq 0} \times \mathbb{Z}^3$ .

*Proof.* By Lemma 4.4 and stability,  $G_x$  belongs to the Hardy class  $H^\infty(\mathbb{C}_+; \mathcal{B}(\mathcal{K}, \mathcal{Y}))$  and has nontangential boundary values a.e. on  $\mathbb{R}$ , with  $\|G_x\|_{\mathcal{H}^\infty} = \text{ess sup}_{E \in \mathbb{R}} \sigma_{\max}(G_x(E))$  by the maximum principle for bounded analytic operator-valued functions. The Plancherel theorem identifies the convolution operator with multiplication by  $G_x(E)$  on  $L^2(\mathbb{R}; \mathcal{K})$ , whose norm is the essential supremum of  $\sigma_{\max}(G_x(E))$ ; see [ZDG96, Ch. 4]. The  $\theta$ -dependent case follows by applying the same argument fibrewise in the direct-integral decomposition of Definition 2.2 and taking the supremum over  $\theta$ , which is legitimate because the fibres are mutually orthogonal.  $\square$

#### 3.2 Transfer isomorphism

**Definition 3.6** (Transfer isomorphism). Two realizations  $G_x, G_y \in \mathcal{G}$  are *transfer isomorphic*, written  $G_x \simeq G_y$ , if there exist unitaries  $U \in \mathcal{U}(\mathcal{Y})$  and  $V \in \mathcal{U}(\mathcal{K})$ , independent of  $(z, \theta)$ , with

$$U G_x(z, \theta) V = G_y(z, \theta) \quad \text{for all } (z, \theta) \in \mathbb{C}_+ \times \mathbb{T}^3. \quad (15)$$

Forms  $x, y$  are transfer isomorphic when their realizations are.

**Proposition 3.7** (Elementary properties). [A]  $\simeq$  is an equivalence relation on  $\mathcal{G}$ , and  $G_x \simeq G_y$  implies  $\|G_x\|_{\mathcal{H}^\infty} = \|G_y\|_{\mathcal{H}^\infty}$ .

*Proof.* Reflexivity is  $U = \text{id}_{\mathcal{Y}}, V = \text{id}_{\mathcal{K}}$ ; symmetry follows by passing to  $U^*, V^*$ ; transitivity by composing the unitaries. The norm identity holds because  $\sigma_{\max}$  is invariant under left and right multiplication by unitaries, pointwise in  $(z, \theta)$ .  $\square$

**Proposition 3.8** (Independence of graph and transfer isomorphism). [D] *Neither of the following implications holds in general:*

- (i) *the molecular graphs underlying two forms are isomorphic  $\Rightarrow$  the forms are transfer isomorphic;*
- (ii) *the forms are transfer isomorphic  $\Rightarrow$  the molecular graphs underlying them are isomorphic.*

*Proof.* Both are refuted by explicit witnesses.

(i). Take  $\mathcal{K} = \mathcal{Y} = \mathbb{C}$ ,  $n = 1$ ,  $H_p = 0$ ,  $B = C = 1$ ,  $J = 0$ , and the wide-band self-energy  $\Sigma_\kappa(z) \equiv -i\kappa$  with  $\kappa > 0$ , which satisfies  $\text{Im } \Sigma_\kappa \leq 0$  and is therefore admissible in Definition 4.2. Then  $M_\kappa(z) = z + i\kappa$  and

$$G^{(\kappa)}(z) = \frac{1}{z + i\kappa}, \quad \|G^{(\kappa)}\|_{\mathcal{H}^\infty} = \frac{1}{\kappa}. \quad (16)$$

Two forms may carry the same molecular graph while Hypothesis 2.13 assigns them coupling data with  $\kappa_1 \neq \kappa_2$ , since the labelling of Hypothesis 2.13 is not determined by the graph. Their gains differ, so by Proposition 3.7 they are not transfer isomorphic.

(ii). Retain the data of (16) for the form  $x$ , and let  $y$  carry  $\mathcal{H}'_{\text{pri}} = \mathbb{C}^2$ ,  $H'_p = \text{diag}(0, \omega)$  with  $\omega \in \mathbb{R}$ ,  $B' = (1, 0)^T$ ,  $C' = (1, 0)$ ,  $J' = 0$ , and self-energy  $\Sigma'(z) = \text{diag}(-i\kappa, -i\kappa')$  with  $\kappa' > 0$ . Then  $M'(z) = \text{diag}(z + i\kappa, z - \omega + i\kappa')$  is diagonal and

$$G_y(z) = C'M'(z)^{-1}B' = \frac{1}{z + i\kappa} = G_x(z), \quad (17)$$

so  $G_x \simeq G_y$  with  $U = V = 1$ . The second site is present in the graph of  $y$  and absent from that of  $x$ , so the two graphs have different vertex counts and are not isomorphic.  $\square$

**Remark 3.9** (Consequences). Proposition 3.8(ii) is the formal reason that the word *isomorphic*, applied in this paper to a pair of forms, cannot be read as constitutional isomerism or graph isomorphism; see Remark 2.34. Part (i) is the formal reason that a geometric symmetry of the reference framework, such as the automorphism  $\sigma$  of Proposition 2.31, does not by itself identify the transfer behaviour of the two rearrangements  $\mathcal{R}^{(3)}$  and  $\mathcal{R}^{(9)}$ . Both parts are instances of the standing restraint that geometric anisotropy does not imply transfer anisotropy [Sch26b, Eq. (2)].

### 3.3 Defect forms and reduction to a composite realization

**Definition 3.10** (Background network and defect region). Let  $H_{\text{per}}$  denote the Hamiltonian of the unperturbed  $\mathbb{Z}^3$ -fibrated network of Axiom A6, translation-covariant and of

finite range in the sense of (31), acting on  $\mathcal{H}_{\text{net}} = \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^m$ . A form  $x$  is a *defect form* with *defect region*  $D \subset \mathbb{Z}^3$ ,  $|D| < \infty$ , if its Hamiltonian is

$$H^{(x)} = H_{\text{per}} + V_x, \quad V_x = V_x^*, \quad \text{rank } V_x < \infty, \quad (18)$$

with  $\text{ran } V_x \subseteq \mathcal{H}_{\text{def}} := \ell^2(D) \otimes \mathbb{C}^m$ .

**Remark 3.11** (Status of a rearranged unit). [A] By Remark 2.33 a unit carrying the rearrangement  $\mathcal{R}^{(3)}$  or  $\mathcal{R}^{(9)}$  of Definition 2.28 is not a model of Axiom A1 and is treated as a structural state. Definition 3.10 is the network-level form of that decision: a rearranged unit occupies finitely many sites and modifies  $H_{\text{per}}$  there, so the associated  $V_x$  has finite rank. No axiom is modified, and the background  $H_{\text{per}}$  is the one supplied by Axiom A6.

**Proposition 3.12** (Defect splitting). [D] *Let  $x$  be a defect form with defect region  $D$  and let  $P_{\text{pri}}$  be the orthogonal projection onto  $\mathcal{H}_{\text{def}}$ ,  $P_{\text{sec}} = I - P_{\text{pri}}$ ,  $\mathcal{H}_{\text{amb}} := \text{ran } P_{\text{sec}}$ . Then, with respect to  $\mathcal{H}_{\text{def}} \oplus \mathcal{H}_{\text{amb}}$ ,*

$$H^{(x)} = \begin{pmatrix} H_p & W \\ W^* & H_s \end{pmatrix}, \quad (19)$$

where  $H_p = P_{\text{pri}}H^{(x)}P_{\text{pri}}|_{\mathcal{H}_{\text{def}}}$  is a self-adjoint operator on a space of dimension  $n = m|D| < \infty$ ,  $W = P_{\text{pri}}H^{(x)}P_{\text{sec}}$  has finite rank because  $H_{\text{per}}$  has finite range, and  $H_s = P_{\text{sec}}H^{(x)}P_{\text{sec}}|_{\mathcal{H}_{\text{amb}}}$  differs from  $P_{\text{sec}}H_{\text{per}}P_{\text{sec}}|_{\mathcal{H}_{\text{amb}}}$  by an operator of finite rank. Consequently  $(\mathcal{H}_{\text{def}}, \mathcal{H}_{\text{amb}}, H_p, H_s, W, B, C, J)$  is a composite realization in the sense of Definition 4.1.

*Proof.* Self-adjointness of the diagonal blocks and the relation  $(P_{\text{pri}}HP_{\text{sec}})^* = P_{\text{sec}}HP_{\text{pri}}$  are immediate.  $\dim \mathcal{H}_{\text{def}} = m|D|$  since  $|D| < \infty$ . For  $W$ : by (31) the hopping range  $F$  is finite, so  $P_{\text{pri}}H_{\text{per}}P_{\text{sec}}$  is supported on the finitely many pairs  $(\mathbf{k}, \mathbf{k}')$  with  $\mathbf{k} \in D$ ,  $\mathbf{k}' \notin D$ ,  $\mathbf{k} - \mathbf{k}' \in F$ ; the contribution of  $V_x$  is of finite rank by hypothesis. For  $H_s$ : since  $\text{ran } V_x \subseteq \mathcal{H}_{\text{def}}$  we have  $P_{\text{sec}}V_xP_{\text{sec}} = 0$ , so in fact  $H_s = P_{\text{sec}}H_{\text{per}}P_{\text{sec}}|_{\mathcal{H}_{\text{amb}}}$  exactly. The remaining requirements of Definition 4.1 are the input, output and feedthrough operators, supplied by Hypothesis 2.13.  $\square$

**Remark 3.13** (Reading of Hypothesis 2.15). Proposition 3.12 shows that the ambient sector  $\mathcal{H}_{\text{amb}}$  of a defect form already contains the complement of the defect region within the primary network. Throughout Sections 4–5, the space  $\mathcal{H}_{\text{sec}}$  of Definition 4.1 is to be read as the orthogonal sum of that complement with the second fibration of Hypotheses 2.15 and 4.9. Both summands are translation-covariant up to

finite rank, so Lemma 3.14 below applies to each; the band data (35) then receive one contribution from each summand, and  $\Gamma$  is the sum. No result of Section 4 is altered by this reading, since its proofs use only  $\Gamma \geq 0$  and the regularity supplied by Proposition 4.13.

**Lemma 3.14** (Limiting absorption under a finite-rank modification). [D] *Let  $H_0$  satisfy the hypotheses of Proposition 4.13 on the interval  $I$ , let  $V = V^*$  have finite rank with  $\text{ran } V \subseteq \mathcal{H}_{\text{def}}$ , and set  $H_1 = H_0 + V$ . Then there is a closed set  $\mathcal{Z} \subset I$  of Lebesgue measure zero such that for every  $\lambda \in I \setminus \mathcal{Z}$  the limit  $P_{\text{pri}}(H_1 - \lambda - i0)^{-1}P_{\text{pri}}$  exists in norm and depends continuously on  $\lambda$  on compact subsets of  $I \setminus \mathcal{Z}$ .*

*Proof.* Write  $R_j(z) = (z - H_j)^{-1}$ . The second resolvent identity gives

$$R_1(z) = R_0(z)(I - VR_0(z))^{-1}, \quad (20)$$

whenever the inverse exists. Since  $\text{ran } V \subseteq \mathcal{H}_{\text{def}}$  and  $\dim \mathcal{H}_{\text{def}} < \infty$ , the operator  $I - VR_0(z)$  acts as the identity on  $\mathcal{H}_{\text{def}}^\perp$  up to a finite-rank correction, and its invertibility is equivalent to that of the finite matrix  $A(z) := I_{\mathcal{H}_{\text{def}}} - P_{\text{pri}}VR_0(z)P_{\text{pri}}$ . By Proposition 4.13 applied with coupling  $P_{\text{pri}}$ , the map  $z \mapsto P_{\text{pri}}R_0(z)P_{\text{pri}}$  is analytic on  $\mathbb{C}_+$  and extends continuously to  $I$ ; hence  $d(z) := \det A(z)$  is analytic on  $\mathbb{C}_+$ , continuous on  $\mathbb{C}_+ \cup I$ , and  $d(z) \rightarrow 1$  as  $|z| \rightarrow \infty$  in  $\mathbb{C}_+$ , so  $d \neq 0$ . Let  $\mathcal{Z} := \{\lambda \in I : d(\lambda) = 0\}$ , a closed subset of  $I$ . If  $\mathcal{Z}$  had positive Lebesgue measure then, by Privalov's theorem, the bounded analytic function  $d$  would vanish identically on  $\mathbb{C}_+$ , contradicting  $d \rightarrow 1$ ; hence  $|\mathcal{Z}| = 0$ . For  $\lambda \in I \setminus \mathcal{Z}$ ,  $A(\lambda)$  is invertible, and by continuity  $A(z)^{-1}$  converges in norm as  $z \rightarrow \lambda$  nontangentially from  $\mathbb{C}_+$ ; (20) then gives the assertion, with continuity on compact subsets of  $I \setminus \mathcal{Z}$  following from that of  $P_{\text{pri}}R_0P_{\text{pri}}$  and of  $A^{-1}$ .  $\square$

**Remark 3.15** (Sharpness). The exceptional set  $\mathcal{Z}$  of Lemma 3.14 cannot be removed: its points are exactly the energies at which the finite-rank modification produces a state that the ambient continuum does not absorb, and these are precisely the embedded modes excluded by condition (40). When  $H_{\text{per}}$  and  $V$  are real-analytic in  $\theta$  the function  $d$  is real-analytic on  $I$  and  $\mathcal{Z}$  is finite; the measure-zero statement above is what the hypotheses of Proposition 4.13 give without that additional assumption.

### 3.4 Cascades

**Lemma 3.16** (Dressed cascade). [A] *Let  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$  and*

$$H = \begin{pmatrix} H_1 & V_{12} & 0 \\ V_{12}^* & H_2 & V_{23} \\ 0 & V_{23}^* & H_3 \end{pmatrix}, \quad (21)$$

*and for  $z \in \mathbb{C}_+$  put*

$$\begin{aligned} g_3(z) &:= (z - H_3)^{-1}, \\ \tilde{g}_2(z) &:= (z - H_2 - V_{23}g_3(z)V_{23}^*)^{-1}, \\ \tilde{g}_1(z) &:= (z - H_1 - V_{12}\tilde{g}_2(z)V_{12}^*)^{-1}. \end{aligned} \quad (22)$$

*Then all three inverses exist, and the  $(1, 3)$  block of  $(z - H)^{-1}$  is*

$$[(z - H)^{-1}]_{13} = \tilde{g}_1(z) V_{12} \tilde{g}_2(z) V_{23} g_3(z). \quad (23)$$

*Proof.* Existence of the inverses follows as in Lemma 4.4: each of the three operators in (22) has imaginary part at least  $(\text{Im } z)I$ , by induction from  $g_3$  upward, using  $\text{Im}(VgV^*) \leq 0$  for  $\text{Im } g \leq 0$ . Formula (23) is block Gaussian elimination on  $(z - H)$ : eliminating the third block row and column replaces  $H_2$  by  $H_2 + V_{23}g_3V_{23}^*$  and contributes the right factor  $V_{23}g_3$ ; eliminating the resulting second block row and column replaces  $H_1$  by  $H_1 + V_{12}\tilde{g}_2V_{12}^*$  and contributes the factor  $V_{12}\tilde{g}_2$ .  $\square$

**Remark 3.17** (Why the bare product fails). The factors in (23) are the *dressed* propagators (22), not the bare resolvents  $(z - H_j)^{-1}$ : the propagator of an intermediate block is modified by the block downstream of it. Consequently the submultiplicative estimate

$$\|[(z - H)^{-1}]_{13}\| \leq \|\tilde{g}_1V_{12}\| \|\tilde{g}_2V_{23}\| \|g_3\| \quad (24)$$

is available only in terms of dressed quantities. This is the precise obstruction to defining a distance between forms by concatenating bare edge quantities, and the reason that the geometry of Section 5 is built from a quantity that satisfies the triangle inequality directly rather than from a product bound converted by a logarithm.

## 4 Dissipative Well-Posedness and the Second Fibration

The quantity  $\|\cdot\|_{\mathcal{H}^\infty}$  of Definition 2.3 is not finite for an arbitrary realization built from Axiom A10 data. This section determines exactly when it is finite. The obstruction is located in Section 4.2, the mechanism that removes it is constructed in Section 4.3, and the criterion is proved in Section 4.4. Sections 4.5 and 4.6 record what the mechanism does and does not license: the loss of norm from the admissible sector is a transfer into the second

fibration and not an annihilation, and finiteness of the gain is incompatible with exact preservation of the admissible sector.

## 4.1 The composite realization

We recall the objects of Section 3 in the form used below.

**Definition 4.1** (Composite realization). Assume Hypotheses 2.12, 2.13, 2.14 and 2.15. A *composite realization* of a form  $x \in \Omega_{\text{adm}}$  is the datum

$$\mathfrak{G}_x = (\mathcal{H}_{\text{pri}}, \mathcal{H}_{\text{sec}}, H_p, H_s, W, B, C, J) \quad (25)$$

in which  $\mathcal{H}_{\text{pri}}$  is a finite-dimensional Hilbert space,  $\dim \mathcal{H}_{\text{pri}} = n$ , carrying the active region of the form;  $H_p = H_p^* \in \mathcal{B}(\mathcal{H}_{\text{pri}})$ ;  $\mathcal{H}_{\text{sec}}$  is the state space of the second fibration of Hypothesis 2.15 and  $H_s = H_s^* \in \mathcal{B}(\mathcal{H}_{\text{sec}})$ ; the coupling  $W \in \mathcal{B}(\mathcal{H}_{\text{sec}}, \mathcal{H}_{\text{pri}})$  has finite rank; and  $B \in \mathcal{B}(\mathcal{K}, \mathcal{H}_{\text{pri}})$ ,  $C \in \mathcal{B}(\mathcal{H}_{\text{pri}}, \mathcal{Y})$ ,  $J \in \mathcal{B}(\mathcal{K}, \mathcal{Y})$  are the input, output and feedthrough operators of the motif datum of Axiom A10. The *composite Hamiltonian* is the bounded self-adjoint operator

$$H = \begin{pmatrix} H_p & W \\ W^* & H_s \end{pmatrix} \quad \text{on } \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}, \quad (26)$$

and  $P_{\text{pri}}, P_{\text{sec}}$  denote the orthogonal projections onto the two summands.

**Definition 4.2** (Self-energy and form transfer function). For  $z \in \rho(H_s)$  set

$$\Sigma(z) := W(z - H_s)^{-1}W^* \in \mathcal{B}(\mathcal{H}_{\text{pri}}), \quad M(z) := z - H_p - \Sigma(z). \quad (27)$$

The *form transfer function* of  $\mathfrak{G}_x$  is

$$G_x(z) := C M(z)^{-1} B + J, \quad (28)$$

defined for those  $z \in \mathbb{C}_+$  at which  $M(z)$  is invertible.

**Remark 4.3** (Boundedness of the constituents). [A]  $H_p$ ,  $H_s$  and  $W$  are bounded by Axiom A10, so  $H$  of (26) is bounded and self-adjoint and  $\sigma(H)$  is a compact subset of  $\mathbb{R}$ . Consequently the difficulty addressed in this section is not unboundedness of the Hamiltonian: it is the behaviour of  $M(z)^{-1}$  as  $z$  approaches the real axis.

**Lemma 4.4** (Dissipativity). [A] For every  $z \in \mathbb{C}_+$ ,

$$\text{Im } \Sigma(z) \leq 0 \quad \text{and} \quad \text{Im } M(z) \geq (\text{Im } z) I_{\mathcal{H}_{\text{pri}}}. \quad (29)$$

In particular  $M(z)$  is invertible on  $\mathbb{C}_+$  with  $\|M(z)^{-1}\| \leq$

$(\text{Im } z)^{-1}$ , and  $G_x$  is analytic on  $\mathbb{C}_+$ .

*Proof.* Write  $R := (z - H_s)^{-1}$  and  $z = E + i\eta$  with  $\eta > 0$ . Since  $H_s = H_s^*$ ,

$$\frac{1}{2i}(R - R^*) = \frac{1}{2i}R((\bar{z} - H_s) - (z - H_s))R^* = -\eta RR^*, \quad (30)$$

so  $\text{Im } R = -\eta RR^* \leq 0$  and therefore  $\text{Im } \Sigma(z) = W(\text{Im } R)W^* \leq 0$ . Hence  $\text{Im } M(z) = \eta I - \text{Im } \Sigma(z) \geq \eta I$ . If  $M(z)\psi = 0$  then  $\eta\|\psi\|^2 \leq \langle \psi, \text{Im } M(z)\psi \rangle = \text{Im} \langle \psi, M(z)\psi \rangle = 0$ , so  $\psi = 0$ ; invertibility and the bound follow from the numerical-range estimate  $\|M(z)^{-1}\| \leq (\inf \sigma(\text{Im } M(z)))^{-1}$ .  $\square$

## 4.2 The conservative obstruction

**Proposition 4.5** (No dissipation from a pure-point second sector). [D] Assume  $H_s$  has purely point spectrum with eigenvalues  $\{\lambda_k\}$  and spectral projections  $\{Q_k\}$ ; this holds in particular whenever  $\mathcal{H}_{\text{sec}}$  is finite-dimensional. Then for every  $E \in \mathbb{R} \setminus \overline{\{\lambda_k\}}$  the limit  $\Sigma(E + i0)$  exists and is self-adjoint, so  $\Gamma(E) := -\text{Im } \Sigma(E + i0) = 0$ . If moreover some  $E_0$  in that set is an eigenvalue of  $H_p + \Sigma(E_0)$  whose eigenvector  $\psi$  satisfies  $C \text{adj}(M(E_0))B \neq 0$ , then  $\|G_x\|_{\mathcal{H}^\infty} = \infty$ .

*Proof.* For  $E$  outside the closure of the point spectrum,  $\Sigma(E + i\eta) = \sum_k WQ_kW^*/(E + i\eta - \lambda_k)$  converges in norm as  $\eta \downarrow 0$  to  $\sum_k WQ_kW^*/(E - \lambda_k)$ , which is self-adjoint because each  $WQ_kW^*$  is self-adjoint and each coefficient is real. Hence  $\Gamma(E) = 0$ . For the second assertion,  $M(E_0)$  is singular by hypothesis, and  $M$  is norm-continuous at  $E_0$  from  $\mathbb{C}_+$ ; since  $\dim \mathcal{H}_{\text{pri}} = n < \infty$  we may write  $M(z)^{-1} = \text{adj } M(z)/\det M(z)$  with  $\det M(E_0) = 0$  and  $\text{adj } M(E_0) \neq 0$  whenever  $\text{rank } M(E_0) = n - 1$ . Then  $\|CM(z)^{-1}B\| \rightarrow \infty$  as  $z \rightarrow E_0$  in  $\mathbb{C}_+$ , because the numerator tends to  $C \text{adj}(M(E_0))B \neq 0$  while the denominator tends to 0. The supremum in (2) is therefore infinite.  $\square$

**Remark 4.6** (The explicit model). [A] The explicit twelve-axiom model  $\mathfrak{R}_{\text{ex}}$  of [Sch26c, App. B.7] realizes Axiom A10 with the transfer operator  $T(z) = 1/z$ , which is the case  $n = 1$ ,  $H_p = 0$ ,  $W = 0$ ,  $B = C = 1$ ,  $J = 0$  of Definition 4.1. Proposition 4.5 applies with  $E_0 = 0$  and gives  $\|T\|_{\mathcal{H}^\infty} = \infty$ . Thus the formalism's own witness realization does not have finite gain, and no non-vacuity argument for the constructions of this paper may be based on it; a separate witness is supplied in Appendix A.

**Corollary 4.7** (Shifted finiteness). [D] For every  $\eta > 0$ ,  $\sup_{\text{Im } z \geq \eta} \|G_x(z)\| \leq \|C\| \|B\| \eta^{-1} + \|J\| < \infty$ .

*Proof.* Immediate from Lemma 4.4 and (28).  $\square$

**Remark 4.8** (Why Corollary 4.7 is not a resolution). The bound of Corollary 4.7 depends on the arbitrary parameter  $\eta$  and diverges as  $\eta \downarrow 0$ . A distance built from it would depend on a broadening scale that no axiom supplies, and would not be comparable across forms whose natural scales differ. Section 4.3 therefore produces dissipation from the structure of the second fibration rather than from a parameter.

### 4.3 The second fibration and its population

**Hypothesis 4.9** ( $\mathcal{H}_{\text{pop}}$ , homogeneous d-population). The second fibration of Hypothesis 2.15 carries the same site format as the primary network and is populated homogeneously by units of the dual-cipher species d of Hypothesis 2.16. Explicitly,  $\mathcal{H}_{\text{sec}} = \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^m$  with  $m < \infty$  the fibre dimension of the common site format, and  $H_s$  is covariant for the translation action of Axiom A6 and of finite range:

$$(H_s u)(\mathbf{k}) = \sum_{\mathbf{a} \in F} t_{\mathbf{a}} u(\mathbf{k} - \mathbf{a}), \quad t_{\mathbf{a}}^* = t_{-\mathbf{a}}, \quad (31)$$

with  $F \subset \mathbb{Z}^3$  finite and  $t_{\mathbf{a}} \in \mathcal{B}(\mathbb{C}^m)$ .

**Remark 4.10** (Effect of homogeneity). Hypothesis 4.9 makes the population of the second fibration uniform, so that the ordered products of site transfer data along a lattice direction are products of a single matrix and commute. The sequence dependence of Axiom A7, which is the noncommutativity of localization with translation [Sch26c, Thm. 8], is untouched by this and continues to hold; what does *not* arise under Hypothesis 4.9 is any dependence of the band set on the order of a mixed population, since the population is not mixed. A mixed population is admitted by Hypothesis 2.16 and is treated in Target 4.24.

**Definition 4.11** (Symbol and band functions). Under Hypothesis 4.9, let  $\mathcal{F}$  denote the Fourier transform  $\ell^2(\mathbb{Z}^3) \rightarrow L^2(\mathbb{T}^3)$  and  $\mu$  normalized Haar measure on  $\mathbb{T}^3$ . Then

$$\mathcal{F} H_s \mathcal{F}^* = \int_{\mathbb{T}^3}^{\oplus} h(\theta) d\mu(\theta), \quad h(\theta) = \sum_{\mathbf{a} \in F} e^{i\theta \cdot \mathbf{a}} t_{\mathbf{a}}, \quad (32)$$

with  $h(\theta) = h(\theta)^*$  a trigonometric polynomial in  $\theta$ . Its eigenvalues  $E_1(\theta) \leq \dots \leq E_m(\theta)$ , repeated by multiplicity, are the *band functions*, and  $P_j(\theta)$  denote the corresponding

spectral projections. A band function is *flat* if it is constant on  $\mathbb{T}^3$ . Likewise  $\widehat{W}(\theta) \in \mathcal{B}(\mathbb{C}^m, \mathcal{H}_{\text{pri}})$  denotes the symbol of the finite-rank coupling  $W$ , a trigonometric polynomial in  $\theta$ .

**Hypothesis 4.12** ( $\mathcal{H}_{\text{reg}}$ , band regularity and boundary continuity). Let  $\rho > 0$  satisfy  $\sigma(H_s) \subset (-\rho, \rho)$  and set

$$R := \rho + 1 + \|H_p\| + \|W\|^2, \quad I := [-R, R]. \quad (33)$$

Then: no flat band takes a value in  $I$ ; and the boundary-value map  $\lambda \mapsto \Sigma(\lambda + i0)$  extends continuously from  $\mathbb{C}_+$  to all of  $I$ , including the band edges. Off the bands  $\Sigma$  is real-analytic and self-adjoint, so that, writing  $\Sigma(\lambda + i0) = \Lambda(\lambda) - i\Gamma(\lambda)$  throughout  $I$ , one has  $\Gamma \geq 0$  on  $I$  and  $\Gamma = 0$  off the bands.

**Proposition 4.13** (Limiting absorption for the second fibration). [D] Assume Hypotheses 4.9 and 4.12. Then for every  $\lambda \in I$  the limit

$$\Sigma(\lambda + i0) := \lim_{\eta \downarrow 0} \Sigma(\lambda + i\eta) \quad (34)$$

exists in the norm of  $\mathcal{B}(\mathcal{H}_{\text{pri}})$ , the convergence is uniform on  $I$ , and, writing  $\Sigma(\lambda + i0) = \Lambda(\lambda) - i\Gamma(\lambda)$ , the maps  $\Lambda, \Gamma : I \rightarrow \mathcal{B}(\mathcal{H}_{\text{pri}})$  are continuous with  $\Gamma(\lambda) \geq 0$ ; at every  $\lambda \in I$  that is a regular value of each non-flat band function one has in addition the representation

$$\Gamma(\lambda) = \pi \sum_j \int_{E_j^{-1}(\lambda)} \frac{\widehat{W}(\theta) P_j(\theta) \widehat{W}(\theta)^*}{|\nabla E_j(\theta)|} d\mathcal{H}^2(\theta) \geq 0, \quad (35)$$

where  $\mathcal{H}^2$  is two-dimensional Hausdorff measure on the level surface, normalized consistently with  $\mu$ .

*Proof.* Fix  $u, v \in \mathcal{H}_{\text{pri}}$ . By (32) and the spectral decomposition of  $h(\theta)$ ,

$$\langle u, \Sigma(z)v \rangle = \sum_j \int_{\mathbb{T}^3} \frac{\langle u, \widehat{W} P_j \widehat{W}^* v \rangle}{z - E_j(\theta)} d\mu(\theta). \quad (36)$$

Each summand is the Cauchy transform of the pushforward measure  $\nu_{uv}^j := (E_j)_* \langle u, \widehat{W} P_j \widehat{W}^* v \rangle \mu$  on  $\mathbb{R}$ , so  $\langle u, \Sigma(z)v \rangle = \int_{\mathbb{R}} (z - \lambda)^{-1} d\nu_{uv}(\lambda)$  with  $\nu_{uv} = \sum_j \nu_{uv}^j$ . Under Hypothesis 4.12 no flat band contributes an atom in  $I$ , and the boundary values  $\Sigma(\lambda + i0)$  exist and are continuous on  $I$ , which is the first assertion; norm convergence follows from convergence of the sesquilinear forms because  $\dim \mathcal{H}_{\text{pri}} = n < \infty$ , and uniformity on  $I$  from continuity on a compact set.

Positivity holds at every  $\lambda \in I$ : by Lemma 4.4,  $\text{Im } \Sigma(z) \leq 0$  on  $\mathbb{C}_+$ , and the boundary value inherits the inequality, so  $\Gamma(\lambda) \geq 0$ .

For the representation (35), let  $\lambda \in I$  be a regular value of each non-flat band function. Then each level set  $E_j^{-1}(\lambda)$  is a compact  $C^1$  surface, the integrand is continuous on it, and the co-area formula gives

$$\frac{dv_{uv}^j}{d\lambda}(\lambda) = \int_{E_j^{-1}(\lambda)} \frac{\langle u, \widehat{W}P_j \widehat{W}^* v \rangle}{|\nabla E_j(\theta)|} d\mathcal{H}^2(\theta). \quad (37)$$

Since  $|\nabla E_j|$  is bounded below on a neighbourhood of that level set,  $v_{uv}$  has a continuous density  $n_{uv}$  near  $\lambda$ , and the Sokhotski–Plemelj theorem for Cauchy transforms of measures with continuous density gives

$$\lim_{\eta \downarrow 0} \langle u, \Sigma(\lambda + i\eta)v \rangle = \text{p.v.} \int_{\mathbb{R}} \frac{n_{uv}(\lambda')}{\lambda - \lambda'} d\lambda' - i\pi n_{uv}(\lambda), \quad (38)$$

which identifies  $\Gamma$  as in (35), the integrand being nonnegative because  $P_j(\theta) \geq 0$  implies  $\widehat{W}P_j \widehat{W}^* \geq 0$  pointwise.

At a critical value of a band function the level set need not be a  $C^1$  surface and (35) is unavailable; there the continuity clause of Hypothesis 4.12 and  $\Gamma \geq 0$  are the only properties used, and they are the only properties that Theorem 4.17 requires. By Sard's theorem the set of critical values is of Lebesgue measure zero.  $\square$

**Remark 4.14** (What is hypothesized rather than derived). The continuity of  $\lambda \mapsto \Sigma(\lambda + i0)$  on the whole of  $I$ , including the band edges, is a clause of Hypothesis 4.12 and is not derived from finite range and analyticity of the symbol on  $\mathbb{T}^d$ . At a critical value of a band function the representation (35) is unavailable, and only that continuity clause together with  $\Gamma \geq 0$  is used; see the restriction stated in Proposition 4.13. Deriving the continuity clause from structural hypotheses on the symbol, or exhibiting a symbol satisfying Hypothesis 4.9 for which it fails, is left open.

**Remark 4.15** (What the second fibration supplies). Proposition 4.13 is the point at which an infinite, translation-covariant population enters essentially. By Proposition 4.5 a finite or pure-point second sector yields  $\Gamma \equiv 0$ ; by (35) an absolutely continuous band yields  $\Gamma \geq 0$  and, at energies where the coupling meets the band,  $\Gamma \neq 0$ . The dissipation used throughout this paper is therefore not postulated: it is the density of states of the second fibration seen through the coupling  $W$ . By Remark 2.17,  $W$  is not an element of  $\mathfrak{M}^{\text{pri}}$  or  $\mathfrak{M}^{\text{dual}}$  and is not supplied by Axiom A8; it is the content of Hypothesis 2.15.

## 4.4 The exclusion criterion

**Hypothesis 4.16** ( $H_{\min}$ , non-degeneracy of the channel). Whenever  $M(\lambda)$  is singular for some  $\lambda \in I$ , its rank is  $n - 1$  and  $C \operatorname{adj}(M(\lambda))B \neq 0$ .

**Theorem 4.17** (Embedded-mode exclusion). [D] Assume Hypotheses 2.12, 2.13, 2.14, 2.15, 4.9, 4.12 and 4.16, and let  $\mathfrak{G}_x$  be a composite realization with  $\sigma(H_p) \subset \operatorname{int} I$ . Then

$$\|G_x\|_{\mathcal{H}^\infty} < \infty \quad (39)$$

if and only if there is no pair  $(\lambda, \psi)$  with  $\lambda \in \mathbb{R}$  and  $\psi \in \mathcal{H}_{\text{pri}} \setminus \{0\}$  satisfying

$$(\lambda - H_p - \Lambda(\lambda))\psi = 0 \quad \text{and} \quad \Gamma(\lambda)\psi = 0. \quad (40)$$

*Proof.* Write  $M(\lambda) := \lambda - H_p - \Lambda(\lambda) + i\Gamma(\lambda)$  for  $\lambda \in I$ , consistently with (27) and Proposition 4.13, so that  $M$  is continuous on  $\mathbb{C}_+ \cup I$  and analytic on  $\mathbb{C}_+$ .

*Step 1: reduction to  $I$ .* Let  $I = [-R, R]$  be as in (33). For real  $\lambda$  with  $|\lambda| > R$  we have  $\operatorname{dist}(\lambda, \sigma(H_s)) \geq |\lambda| - \rho > 1$ , so  $\|\Sigma(\lambda)\| \leq \|W\|^2$  by Definition 4.2, and therefore

$$\|M(\lambda)^{-1}\| \leq (|\lambda| - \|H_p\| - \|W\|^2)^{-1} < 1, \quad (41)$$

the middle quantity being positive by (33); the same bound holds for  $z \in \mathbb{C}_+$  with  $|z| > R$ , since  $\operatorname{Im} \Sigma(z) \leq 0$  only improves it. Hence no pair  $(\lambda, \psi)$  with  $|\lambda| > R$  can satisfy (40), and it suffices to control  $M$  on  $K := \{z \in \mathbb{C}_+ : |z| \leq R, \operatorname{Im} z \leq \eta_0\}$  for a fixed  $\eta_0 > 0$ , Lemma 4.4 covering  $\operatorname{Im} z > \eta_0$ . On  $K \cap \mathbb{R} \subseteq I$  the maps  $\Lambda, \Gamma$  are continuous by Hypothesis 4.12, including at the band edges and in the spectral gaps.

*Step 2: sufficiency.* Suppose no pair  $(\lambda, \psi)$  satisfies (40). Let  $\lambda \in I$  and suppose  $M(\lambda)\psi = 0$  for some  $\psi \neq 0$ . Taking the imaginary part of  $\langle \psi, M(\lambda)\psi \rangle = 0$  and using  $\Lambda(\lambda) = \Lambda(\lambda)^*$ ,  $\Gamma(\lambda) = \Gamma(\lambda)^*$  gives  $\langle \psi, \Gamma(\lambda)\psi \rangle = 0$ ; since  $\Gamma(\lambda) \geq 0$  by Proposition 4.13, this forces  $\Gamma(\lambda)^{1/2}\psi = 0$  and hence  $\Gamma(\lambda)\psi = 0$ . The real part then gives  $(\lambda - H_p - \Lambda(\lambda))\psi = 0$ , so  $(\lambda, \psi)$  satisfies (40), a contradiction. Therefore  $M(\lambda)$  is invertible for every  $\lambda \in I$ . The map  $\lambda \mapsto \|M(\lambda)^{-1}\|$  is continuous on the compact set  $I$ , hence bounded there, say by  $c_0$ . By continuity of  $M$  on  $\mathbb{C}_+ \cup I$  and compactness of  $K$ , there is  $\eta_1 \in (0, \eta_0]$  with  $\|M(z) - M(\operatorname{Re} z)\| \leq 1/(2c_0)$  whenever  $z \in K$  and  $\operatorname{Im} z \leq \eta_1$ ; a Neumann series then gives  $\|M(z)^{-1}\| \leq 2c_0$  on that region, while Lemma 4.4 gives  $\|M(z)^{-1}\| \leq \eta_1^{-1}$  on the remainder of  $K$ . With Step 1,  $\|M(z)^{-1}\|$  is bounded on  $\mathbb{C}_+$ , and (28) gives (39).

*Step 3: necessity.* Suppose  $(\lambda_0, \psi)$  satisfies (40). Then  $M(\lambda_0)\psi = (\lambda_0 - H_p - \Lambda(\lambda_0))\psi + i\Gamma(\lambda_0)\psi = 0$ , so  $\det M(\lambda_0) = 0$ . By Hypothesis 4.16,  $\operatorname{rank} M(\lambda_0) = n - 1$  and  $C \operatorname{adj}(M(\lambda_0))B \neq 0$ . Writing  $M(z)^{-1} = \operatorname{adj} M(z)/\det M(z)$

and using continuity of  $M$  at  $\lambda_0$  from  $\mathbb{C}_+$ ,

$$\lim_{z \rightarrow \lambda_0, z \in \mathbb{C}_+} \|CM(z)^{-1}B\| = \lim_{z \rightarrow \lambda_0} \frac{\|C \operatorname{adj} M(z)B\|}{|\det M(z)|} = \infty, \quad (42)$$

since the numerator converges to a nonzero limit and the denominator to 0. Hence  $\|G_x\|_{\mathcal{H}^\infty} = \infty$ .  $\square$

**Remark 4.18** (Reading of Theorem 4.17). Condition (40) states that  $\psi$  is an eigenvector of the shifted primary Hamiltonian which the second fibration does not see. Such a  $\psi$  is a mode of the active region that is energetically embedded in the continuum of the second network yet decoupled from it. Theorem 4.17 says that finiteness of the gain is exactly the absence of such modes: not the absence of resonances, which are permitted and merely large, but the absence of states that cannot hand off at all.

**Remark 4.19** (Chemical reading). [E] Under Hypothesis 2.23, condition (40) is the assertion that the rearranged unit of Definition 2.28 supports no mode at the relevant energies that is orthogonal to every channel into the second network. That an actual substituted chrysen assembly satisfies this is an empirical instantiation claim; it is used in no proof, and Theorem 4.17 is a statement about  $\mathfrak{G}_x$  alone.

## 4.5 Handoff is transfer, not deletion

**Proposition 4.20** (Sector balance). [A] Let  $H$  be as in (26) and  $\psi_t := e^{-iHt}\psi_0$ . Then for all  $t$ ,

$$\|P_{\text{pri}}\psi_t\|^2 + \|P_{\text{sec}}\psi_t\|^2 = \|\psi_0\|^2, \quad (43)$$

and

$$\frac{d}{dt} \|P_{\text{pri}}\psi_t\|^2 = 2 \operatorname{Im} \langle P_{\text{pri}}\psi_t, WP_{\text{sec}}\psi_t \rangle = -\frac{d}{dt} \|P_{\text{sec}}\psi_t\|^2. \quad (44)$$

*Proof.*  $H$  is bounded and self-adjoint, so  $e^{-iHt}$  is unitary and (43) holds. Differentiating  $\|P_{\text{pri}}\psi_t\|^2 = \langle \psi_t, P_{\text{pri}}\psi_t \rangle$  and using  $\dot{\psi}_t = -iH\psi_t$  gives  $\frac{d}{dt} \|P_{\text{pri}}\psi_t\|^2 = i \langle \psi_t, [H, P_{\text{pri}}]\psi_t \rangle$ . From (26),  $[H, P_{\text{pri}}]$  has only off-diagonal blocks, equal to  $-W$  and  $W^*$ , which yields the first equality in (44); the second follows by differentiating (43).  $\square$

**Remark 4.21** (Non-implication). [A] Proposition 4.20 is the precise sense in which the mechanism of Section 4.3 is a *transfer* of norm into the second fibration and not an annihilation: no norm is lost from  $\mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}$ . In particular neither  $\Gamma(\lambda) \neq 0$ , nor decay of  $\|P_{\text{pri}}\psi_t\|$ , nor infinitude of  $\|G_x\|_{\mathcal{H}^\infty}$  licenses the inference of a physical

scission event; that inference is excluded by the standing restraint  $d_C^*(x, y) = +\infty \not\Rightarrow$  autonomous physical scission of [Sch26b, Eq. (3)] and [Sch26c, Rem. 39], and nothing in this section weakens it.

## 4.6 The confinement–finiteness dichotomy

**Proposition 4.22** (Exact confinement excludes finite gain). [D] Assume Hypothesis 4.16 and suppose the admissible sector is exactly preserved, that is  $[H, P_{\text{pri}}] = 0$ , equivalently  $W = 0$ . Then  $\Sigma \equiv 0$ ,  $\Gamma \equiv 0$ , and  $\|G_x\|_{\mathcal{H}^\infty} = \infty$  whenever  $\sigma(H_p) \neq \emptyset$ , which holds since  $\dim \mathcal{H}_{\text{pri}} = n \geq 1$ .

*Proof.* By (26),  $[H, P_{\text{pri}}] = 0$  if and only if  $W = 0$ , whence  $\Sigma \equiv 0$  and  $M(z) = z - H_p$ . Any  $\lambda_0 \in \sigma(H_p)$  is real, and  $M(\lambda_0)$  is singular; Hypothesis 4.16 and Step 3 of the proof of Theorem 4.17 give  $\|G_x\|_{\mathcal{H}^\infty} = \infty$ .  $\square$

**Remark 4.23** (Consequence for the preservation criterion). Proposition 4.22 places the finiteness of the gain in direct opposition to exact invariance of the admissible sector: a form whose active region is perfectly confined, in the sense of the preservation criterion [Sch26c, Prop. 31], has infinite worst-case gain. Persistence under persistent disturbance, as developed in Sections 5–7, therefore cannot be obtained by strengthening confinement; it requires a channel out of the admissible sector, and Hypothesis 2.15 supplies exactly that channel while Proposition 4.20 keeps the norm accounted for. This is the formal content of the third survival mode.

## 4.7 Targets

**Theorem 4.24** (Target: mixed populations). [T] Let the second fibration carry a population  $\varsigma_{\text{sp}} : \mathbb{Z}^3 \rightarrow \{\mathfrak{p}, \mathfrak{d}\}$  that is not constant. Target: identify hypotheses on  $\varsigma_{\text{sp}}$  under which the conclusion of Proposition 4.13 persists. For  $\varsigma_{\text{sp}}$  periodic, the conclusion follows from Proposition 4.13 applied on the sublattice of periods with an enlarged fibre, and the band set then depends on the ordering of the population within a period; for  $\varsigma_{\text{sp}}$  aperiodic or random the statement is open, and in the random case it contains the question of absolutely continuous spectrum for the weakly disordered three-dimensional Anderson model, which is not resolved here and is not assumed anywhere in this paper.

**Theorem 4.25** (Target: extended active regions). [T] Theorem 4.17 assumes  $\dim \mathcal{H}_{\text{pri}} < \infty$ . Target: extend the criterion to an active region of infinite dimension, for which  $\operatorname{adj} M$  is unavailable and Hypothesis 4.16 must be replaced by a Fredholm condition on  $M(\lambda)$ .

**Theorem 4.26** (Target: scattering-theoretic form). [T] *Target: identify the scattering operator of the pair  $(H, H_p \oplus H_s)$ , in the Lax–Phillips formulation [LP67], with the characteristic function of a contraction in the sense of [SNF70], and show that the resulting characteristic function coincides with the boundary value of  $G_x$  up to a unitary factor. Under Hypotheses 4.9 and 4.12 the scattering operator is unitary on each band, so this Target concerns the identification and not the boundedness, which is Theorem 4.17.*

## 5 The $\mathcal{H}^\infty$ Transformation Geometry

Section 4 determined when a single form has finite gain. This section measures the distance between forms. The construction proceeds in three stages: a distance between realizations that is a genuine pseudometric without any logarithm (Section 5.1); its path closure over admissible transformations, which is the transformation metric of the title (Section 5.2); and the robustness statement that gives the metric its meaning, namely that a ball in this metric is a set of forms guaranteed to survive a disturbance of given size (Section 5.3). Two auxiliary structures follow: a bottleneck closure, which answers a different survival question (Section 5.4), and an intra-form attenuation quantity, which is genuinely only a quasi-pseudometric (Section 5.5).

### 5.1 The unitary-orbit distance

**Definition 5.1** (Unitary-orbit distance). Assume Hypothesis 3.4. For  $G_x, G_y \in \mathcal{G}$  set

$$\delta_u(G_x, G_y) := \inf_{\substack{U \in \mathcal{U}(\mathcal{Y}) \\ V \in \mathcal{U}(\mathcal{K})}} \|UG_xV - G_y\|_{\mathcal{H}^\infty}. \quad (45)$$

**Proposition 5.2** ( $\delta_u$  is a pseudometric, and the infimum is attained). [D] *Assume Hypothesis 3.4. Then on  $\mathcal{G}$ :*

- (i) *the infimum in (45) is attained;*
- (ii)  *$\delta_u \geq 0$ ,  $\delta_u(G_x, G_x) = 0$ , and  $\delta_u$  is symmetric;*
- (iii)  *$\delta_u$  satisfies the triangle inequality;*
- (iv)  *$\delta_u(G_x, G_y) = 0$  if and only if  $G_x \simeq G_y$  in the sense of Definition 3.6.*

*Proof.* (i) The map  $(U, V) \mapsto \|UG_xV - G_y\|_{\mathcal{H}^\infty}$  is continuous on  $\mathcal{U}(\mathcal{Y}) \times \mathcal{U}(\mathcal{K})$ : for  $U, U'$  and  $V, V'$ ,

$$\|UG_xV - U'G_xV'\|_{\mathcal{H}^\infty} \leq \|G_x\|_{\mathcal{H}^\infty} (\|U - U'\| + \|V - V'\|), \quad (46)$$

using unitary invariance of  $\sigma_{\max}$  and  $\|G_x\|_{\mathcal{H}^\infty} < \infty$ . Under Hypothesis 3.4 the unitary groups are compact, so the infimum is attained.

(ii) Nonnegativity and vanishing on the diagonal are clear. For symmetry, unitary invariance of the norm gives

$$\|UG_xV - G_y\|_{\mathcal{H}^\infty} = \|G_x - U^*G_yV^*\|_{\mathcal{H}^\infty}, \quad (47)$$

and  $(U, V) \mapsto (U^*, V^*)$  is a bijection of the index set, so the two infima coincide.

(iii) Let  $(U_1, V_1)$  attain  $\delta_u(G_x, G_y)$  and  $(U_2, V_2)$  attain  $\delta_u(G_y, G_z)$ . Then

$$\begin{aligned} \delta_u(G_x, G_z) &\leq \|U_2U_1G_xV_1V_2 - G_z\|_{\mathcal{H}^\infty} \\ &\leq \|U_2(U_1G_xV_1 - G_y)V_2\|_{\mathcal{H}^\infty} \\ &\quad + \|U_2G_yV_2 - G_z\|_{\mathcal{H}^\infty} \\ &= \delta_u(G_x, G_y) + \delta_u(G_y, G_z). \end{aligned} \quad (48)$$

(iv) If  $G_x \simeq G_y$  then the witnessing pair gives  $\delta_u = 0$ . Conversely if  $\delta_u = 0$  then by (i) the infimum is attained at some  $(U, V)$ , so  $UG_xV = G_y$  identically, which is (15).  $\square$

**Proposition 5.3** (Gain is 1-Lipschitz). [A] *For all  $G_x, G_y \in \mathcal{G}$ ,*

$$|\|G_x\|_{\mathcal{H}^\infty} - \|G_y\|_{\mathcal{H}^\infty}| \leq \delta_u(G_x, G_y). \quad (49)$$

*Proof.* For any admissible  $(U, V)$ ,  $\|G_y\|_{\mathcal{H}^\infty} \leq \|UG_xV\|_{\mathcal{H}^\infty} + \|UG_xV - G_y\|_{\mathcal{H}^\infty} = \|G_x\|_{\mathcal{H}^\infty} + \|UG_xV - G_y\|_{\mathcal{H}^\infty}$ . Taking the infimum over  $(U, V)$  gives  $\|G_y\|_{\mathcal{H}^\infty} - \|G_x\|_{\mathcal{H}^\infty} \leq \delta_u(G_x, G_y)$ , and the symmetric argument with  $x, y$  interchanged, together with Proposition 5.2(ii), gives (49).  $\square$

**Remark 5.4** (No logarithm is required). Proposition 5.2(iii) is an additive triangle inequality obtained directly, without converting a submultiplicative bound. This is the structural reason for measuring between *realizations of forms* rather than between endpoints of a transfer path: by Remark 3.17 the latter yields only a product bound in dressed quantities, whose logarithm is additive along a fixed chain but does not compare distinct chains. The quantity  $\delta_u$  compares forms, and the path structure enters only in Section 5.2, where it carries reachability rather than the triangle inequality.

### 5.2 The transformation metric

**Definition 5.5** (Edge cost and transformation metric). For  $(x, y) \in \mathcal{E}_{\text{adm}}$  set  $c(x, y) := \delta_u(G_x, G_y)$  when both  $G_x$  and  $G_y$  lie in  $\mathcal{G}$ , and  $c(x, y) := +\infty$  otherwise, so that an edge incident to a form of infinite gain is assigned infinite cost. For a transformation route  $\gamma = (x = x_0, x_1, \dots, x_N = y) \in$

$P_{\text{adm}}(x, y)$  set  $L_c(\gamma) := \sum_{i=1}^N c(x_{i-1}, x_i)$ , and define

$$d_{\mathcal{H}^\infty}(x, y) := \begin{cases} \inf_{\gamma \in P_{\text{adm}}(x, y)} L_c(\gamma), & P_{\text{adm}}(x, y) \neq \emptyset, \\ +\infty, & P_{\text{adm}}(x, y) = \emptyset. \end{cases} \quad (50)$$

**Proposition 5.6** (Basic properties). [D]  $d_{\mathcal{H}^\infty}$  is an extended pseudometric on  $\Omega_{\text{adm}}$ . Moreover

$$\delta_u(G_x, G_y) \leq d_{\mathcal{H}^\infty}(x, y) \quad \text{for all } x, y \in \Omega_{\text{adm}}, \quad (51)$$

and  $d_{\mathcal{H}^\infty}(x, y) = +\infty$  precisely when  $x$  and  $y$  lie in distinct connected components of the graph  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$ .

*Proof.* Nonnegativity is clear, and  $d_{\mathcal{H}^\infty}(x, x) = 0$  via the empty route. Symmetry holds because  $\mathcal{E}_{\text{adm}}$  is symmetric by Axiom A12 and  $c$  is symmetric by Proposition 5.2(ii), so route reversal is a bijection  $P_{\text{adm}}(x, y) \rightarrow P_{\text{adm}}(y, x)$  preserving  $L_c$ . The triangle inequality follows by concatenation of routes, with the convention  $r + \infty = \infty$ . For (51), iterate Proposition 5.2(iii) along any route to obtain  $\delta_u(G_x, G_y) \leq L_c(\gamma)$ , then take the infimum; when no route exists the inequality is trivial. The last assertion is immediate from (50).  $\square$

**Proposition 5.7** (When  $d_{\mathcal{H}^\infty}$  separates points). [D] Suppose  $c(e) \geq c_0 > 0$  for every  $e \in \mathcal{E}_{\text{adm}}$ . Then  $d_{\mathcal{H}^\infty}(x, y) \geq c_0$  for all  $x \neq y$ , so  $d_{\mathcal{H}^\infty}$  is an extended metric. Without that hypothesis,  $d_{\mathcal{H}^\infty}$  induces a metric on the quotient  $\Omega_{\text{adm}}/\sim$ , where  $x \sim y$  iff  $d_{\mathcal{H}^\infty}(x, y) = 0$ , and each equivalence class consists of forms that are pairwise transfer isomorphic.

*Proof.* The first assertion holds because every nonempty route has at least one edge. For the second,  $\sim$  is an equivalence relation by Proposition 5.6, and the induced function on the quotient is a metric by the standard quotient construction for pseudometrics. If  $d_{\mathcal{H}^\infty}(x, y) = 0$  then  $\delta_u(G_x, G_y) = 0$  by (51), hence  $G_x \simeq G_y$  by Proposition 5.2(iv).  $\square$

**Remark 5.8** (What the zero set records). By Proposition 5.7, the forms at  $d_{\mathcal{H}^\infty}$ -distance zero from  $x$  are exactly those reachable from  $x$  by admissible transformations along which the realization is unchanged up to constant unitaries. This is the precise content of the informal requirement that a transformation *maintain the stability of the network*: it is not a requirement on the molecular graph, which changes (Remark 2.33), but on the transfer behaviour, which does not.

### 5.3 The survival radius

**Theorem 5.9** ([Robust-stability radius]). [D] Assume Hypotheses 3.4 and 2.19, let  $x \in \Omega_{\text{adm}}$  with  $\gamma_x := \|G_x\|_{\mathcal{H}^\infty} < \infty$ , and let  $\delta > 0$ . Then for every  $y \in \Omega_{\text{adm}}$  with

$$d_{\mathcal{H}^\infty}(x, y) < \frac{1}{\delta} - \gamma_x, \quad (52)$$

and every causal environment operator  $\Delta$  with  $\|\Delta\|_{\mathcal{H}^\infty} \leq \delta$ , the feedback interconnection of  $G_y$  with  $\Delta$  is well posed and internally stable.

*Proof.* By Propositions 5.3 and 5.6,

$$\|G_y\|_{\mathcal{H}^\infty} \leq \gamma_x + \delta_u(G_x, G_y) \leq \gamma_x + d_{\mathcal{H}^\infty}(x, y) < \frac{1}{\delta}, \quad (53)$$

so  $\|G_y\|_{\mathcal{H}^\infty} \|\Delta\|_{\mathcal{H}^\infty} < 1$ . Both  $G_y$  and  $\Delta$  are stable and causal, so the small-gain theorem [ZDG96, Thm. 9.1] applies and yields well-posedness and internal stability of the interconnection.  $\square$

**Corollary 5.10** (Robust-stability balls). [D] Under the hypotheses of Theorem 5.9, the set of forms robustly stable against every environment perturbation of size at most  $\delta$  contains the open  $d_{\mathcal{H}^\infty}$ -ball about  $x$  of radius  $\delta^{-1} - \gamma_x$ , which is nonempty precisely when  $\gamma_x < \delta^{-1}$ .

**Remark 5.11** (Interpretation, and its limits). Theorem 5.9 is the sense in which  $d_{\mathcal{H}^\infty}$  is a *stable* transfer geometry: a form may be transformed anywhere within a metric ball whose radius is set by the disturbance level, and the transformed form still survives. Two limits must be stated. First, the conclusion is robust stability of each form separately; it is not a statement about the process of switching between forms, which is addressed in Section 6 and for which frozen-form stability is insufficient. Second, by Proposition 4.22 a form with  $\gamma_x < \infty$  necessarily fails exact confinement, so the balls of Corollary 5.10 consist of forms that do exchange norm with the ambient sector; by Proposition 4.20 that exchange is a transfer and not a loss.

**Remark 5.12** (Admissibility is not certified). Theorem 5.9 bounds an  $L^2$ -induced gain (Lemma 3.5) and therefore certifies nothing about the pointwise constraints  $C_j < \tau_j$  defining  $\Omega_{\text{adm}}$ . Without Hypothesis 2.22 no gain bound constrains membership in  $\Omega_{\text{adm}}$  at all, and even with it the relevant quantity is a peak gain rather than  $\|\cdot\|_{\mathcal{H}^\infty}$ ; see Section 8. The two halves of the survival claim are kept apart throughout for this reason.

## 5.4 The bottleneck closure

**Definition 5.13** (Bottleneck transformation distance). With  $c$  as in Definition 5.5, set  $L_c^\vee(\gamma) := \max_{1 \leq i \leq N} c(x_{i-1}, x_i)$  for a route  $\gamma$ , with  $L_c^\vee = 0$  for the empty route, and

$$d_{\mathcal{H}^\infty}^\vee(x, y) := \inf_{\gamma \in P_{\text{adm}}(x, y)} L_c^\vee(\gamma), \quad (54)$$

with the value  $+\infty$  when no route exists.

**Proposition 5.14** (Ultrametric property and comparison). [D]  $d_{\mathcal{H}^\infty}^\vee$  is an extended pseudo-ultrametric on  $\Omega_{\text{adm}}$ .

$$d_{\mathcal{H}^\infty}^\vee(x, z) \leq \max\{d_{\mathcal{H}^\infty}^\vee(x, y), d_{\mathcal{H}^\infty}^\vee(y, z)\} \quad \text{for all } x, y, z, \quad (55)$$

and  $d_{\mathcal{H}^\infty}^\vee \leq d_{\mathcal{H}^\infty}$  pointwise.

*Proof.* Symmetry and vanishing on the diagonal are as in Proposition 5.6. For (55), let  $\gamma_1$  be a route from  $x$  to  $y$  and  $\gamma_2$  from  $y$  to  $z$ ; their concatenation  $\gamma$  is a route from  $x$  to  $z$  with  $L_c^\vee(\gamma) = \max\{L_c^\vee(\gamma_1), L_c^\vee(\gamma_2)\}$ . Taking infima over  $\gamma_1$  and  $\gamma_2$  independently gives (55), and (55) implies the triangle inequality. Finally  $L_c^\vee(\gamma) \leq L_c(\gamma)$  for every route, whence  $d_{\mathcal{H}^\infty}^\vee \leq d_{\mathcal{H}^\infty}$ .  $\square$

**Proposition 5.15** (Widest-route margin). [D] Let  $x, y \in \Omega_{\text{adm}}$  and  $\varepsilon > 0$ . There exists a transformation route from  $x$  to  $y$  along which every single transformation satisfies  $c(e) \leq d_{\mathcal{H}^\infty}^\vee(x, y)$  if the infimum in (54) is attained, and  $c(e) \leq d_{\mathcal{H}^\infty}^\vee(x, y) + \varepsilon$  in general; and no route exists along which every transformation satisfies  $c(e) < d_{\mathcal{H}^\infty}^\vee(x, y)$ .

*Proof.* The first assertion is the definition of the infimum. For the second, if such a route existed its bottleneck value would be strictly less than  $d_{\mathcal{H}^\infty}^\vee(x, y)$ , contradicting (54).  $\square$

**Remark 5.16** (Which survival question each closure answers).  $d_{\mathcal{H}^\infty}$  bounds the cumulative deviation of the realization along a transformation route and is the quantity entering the survival radius of Theorem 5.9.  $d_{\mathcal{H}^\infty}^\vee$  bounds the deviation incurred at the single worst transformation and is the quantity relevant when the constraint is that no individual transformation may exceed a fixed tolerance. The two are not comparable in the reverse direction: a route of many small steps has small  $d_{\mathcal{H}^\infty}^\vee$  and possibly large  $d_{\mathcal{H}^\infty}$ . A path functional of the form  $J(\gamma) = \sup_i \|S(x_i)\|$ , as used in [Sch26f, Def. 4.1.1], is a bottleneck functional in this sense and is therefore governed by Proposition 5.14 rather than by Proposition 5.6; in particular it satisfies an ultrametric, not merely a metric, triangle inequality, and the two should not

be substituted for one another.

## 5.5 Intra-form attenuation

The preceding constructions compare forms. The following compares sites within a fixed form, and is recorded because the corpus uses such a quantity informally; it is included with its defects stated, not as an alternative to  $d_{\mathcal{H}^\infty}$ .

**Definition 5.17** (Attenuation cost). Fix a form with a distinguished finite set of ports. For an ordered pair  $(a, b)$  of ports let  $t_{ba}(E)$  denote the corresponding block of the scattering matrix, and set

$$c_{\text{att}}(a, b) := -\log \sup_{E \in \mathbb{R}} \|t_{ba}(E)\|, \quad (56)$$

with the value  $+\infty$  when the supremum is 0. The *attenuation quasi-distance*  $d_{\text{att}}$  is the path closure of  $c_{\text{att}}$  over directed port paths.

**Proposition 5.18** (Quasi-pseudometric, and when it is symmetric). [D]  $d_{\text{att}}$  is an extended quasi-pseudometric: it is nonnegative, vanishes on the diagonal, and satisfies the triangle inequality, but is not symmetric in general. It is symmetric under either of the following:

- (i) Hypothesis 2.21, under which the scattering matrix is complex symmetric, so  $t_{ab} = t_{ba}^\top$  and the norms agree;
- (ii) losslessness together with exactly two ports, for which unitarity of the scattering matrix forces  $\|t_{ab}(E)\| = \|t_{ba}(E)\|$  for every  $E$ .

*Proof.* Nonnegativity holds because passivity gives  $\|t_{ba}(E)\| \leq 1$ , so the logarithm in (56) is nonpositive. The triangle inequality is the definition of a path closure. (i) If  $S(E) = S(E)^\top$  then  $t_{ab} = t_{ba}^\top$  and  $\|t_{ba}^\top\| = \|t_{ba}\|$ . (ii) For a unitary  $2 \times 2$  block matrix  $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ , the relations  $S^*S = SS^* = I$  give  $t^*t = I - r^*r$  and  $t'(t')^* = I - rr^*$ , and  $r^*r$  and  $rr^*$  have the same nonzero eigenvalues, whence  $\|t\| = \|t'\|$ . Asymmetry in general is exhibited in Example 5.19.  $\square$

**Example 5.19** (A lossless nonreciprocal three-port). [D] Let  $\phi \in \mathbb{R}$ , put  $\tau := e^{i\phi/3}$ , and let

$$H_\phi = \begin{pmatrix} 0 & \tau & \bar{\tau} \\ \bar{\tau} & 0 & \tau \\ \tau & \bar{\tau} & 0 \end{pmatrix}, \quad (57)$$

a three-site ring carrying flux  $\phi$ , each site coupled to its own

lead with common rate  $g > 0$ , so that

$$S_\phi(E) = I - 2ig(E - H_\phi + ig)^{-1}. \quad (58)$$

Then  $S_\phi(E)$  is unitary for every real  $E$ , so the scatterer is lossless. For  $\phi = 0$  the matrix  $H_\phi$  is real symmetric, Hypothesis 2.21 holds, and with  $g = \frac{1}{2}$

$$\sup_E |S_0]_{21}| = \sup_E |S_0]_{12}| = \frac{2}{3}. \quad (59)$$

For  $\phi = \pi/2$  and  $g = \frac{1}{2}$ , by contrast,

$$\sup_E |S_{\pi/2}]_{21}| = \frac{12}{13}, \quad \sup_E |S_{\pi/2}]_{12}| = \frac{1}{\sqrt{3}}, \quad (60)$$

the first supremum attained at  $E = 0$  and the second at  $E = E_* \approx -1.65831$ , so that

$$c_{\text{att}}(1, 2) = \log \frac{13}{12} \approx 0.0800, \quad c_{\text{att}}(2, 1) = \frac{1}{2} \log 3 \approx 0.5493. \quad (61)$$

Hence  $c_{\text{att}}$ , and with it  $d_{\text{att}}$ , is not symmetric.

*Remark 5.20* (Losslessness does not restore symmetry). Example 5.19 is lossless, so the asymmetry of (60) is not produced by absorption. Clause (ii) of Proposition 5.18 is restricted to two ports for exactly this reason: for three or more ports a unitary scattering matrix may be strongly nonreciprocal, which is the circulator effect. The correct statement is therefore that symmetry of  $d_{\text{att}}$  requires reciprocity, not merely conservation.

**Proposition 5.21** (Resonance degeneracy). [D] *Let a single level  $\varepsilon \in \mathbb{R}$  be coupled to two leads with rates  $\Gamma_L, \Gamma_R > 0$ , so that*

$$t(E) = \frac{i\sqrt{\Gamma_L\Gamma_R}}{E - \varepsilon + \frac{i}{2}(\Gamma_L + \Gamma_R)}. \quad (62)$$

*Then  $\sup_E |t(E)| = 2\sqrt{\Gamma_L\Gamma_R}/(\Gamma_L + \Gamma_R)$ , which equals 1 if and only if  $\Gamma_L = \Gamma_R$ . In that case  $c_{\text{att}} = 0$  for a pair of distinct ports, so  $d_{\text{att}}$  is a pseudo-quasi-distance and not a quasi-distance.*

*Proof.* The modulus of (62) is maximized at  $E = \varepsilon$ , giving the stated value; by the arithmetic–geometric mean inequality it is at most 1, with equality iff  $\Gamma_L = \Gamma_R$ . Then (56) gives  $c_{\text{att}} = -\log 1 = 0$  between distinct ports.  $\square$

*Remark 5.22* (Why  $d_{\text{att}}$  is not the paper’s metric). Propositions 5.18 and 5.21 record two independent defects of  $d_{\text{att}}$ : it need not be symmetric, and it fails to separate ports joined by a balanced resonance. Neither defect afflicts  $d_{\mathcal{H}^\infty}$ ,

which is symmetric by Proposition 5.6 and separates forms exactly up to transfer isomorphism by Proposition 5.7. The quantity  $d_{\text{att}}$  is retained only because it is the rigorous form of the attenuation language used in the corpus, and it is used nowhere in Sections 6–8.

## 5.6 Relation to the metrics already present in the formalism

*Remark 5.23* (Comparison with  $d_C^*$ ).  $d_{\mathcal{H}^\infty}$  and the extended admissible path distance  $d_C^*$  of Axiom A12 are path closures over the same relation  $\mathcal{E}_{\text{adm}}$  but with different edge data:  $d_C^*$  uses the axiom’s weight function  $w$ ,  $d_{\mathcal{H}^\infty}$  uses  $c = \delta_u$ . Consequently the two share their infinite values,  $d_{\mathcal{H}^\infty}(x, y) = +\infty \iff d_C^*(x, y) = +\infty$ , both being equivalent to disconnection in  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$ . No inequality holds between them in either direction without a hypothesis relating  $w$  to  $c$  edgewise; if  $c_1 w(e) \leq c(e) \leq c_2 w(e)$  for constants  $0 < c_1 \leq c_2$  then  $c_1 d_C^* \leq d_{\mathcal{H}^\infty} \leq c_2 d_C^*$ , and otherwise nothing. In particular  $d_{\mathcal{H}^\infty}$  is not a candidate for the spectral–path metric reconstruction target of [Sch26c, Thm. 44], which concerns  $d_C^*$  and the spectral distance of Axiom A9.

*Remark 5.24* (Independence from the spectral distance). The Dirac operator  $D_{\mathcal{G}}$  of Axiom A9 and the directional Hamiltonian of Axiom A10 are logically independent objects, and neither is constructed from the other [Sch26c, Prop. 27, Rem. 29]. Since  $d_{\mathcal{H}^\infty}$  is built from A10 data alone, no relation between  $d_{\mathcal{H}^\infty}$  and the spectral distance of  $D_{\mathcal{G}}$  is asserted here; establishing one is recorded as a Target in Section 9.

## 6 Survival Across Transformations

Theorem 5.9 certifies each form separately. It says nothing about the passage between forms, and this section shows that the gap is real: a family of forms every one of which is individually robust can fail to survive the act of transforming among them (Section 6.2). Two sufficient conditions are then established. The first, a common storage functional, gives a gain bound valid under arbitrary admissible switching (Section 6.3); its structural form (Corollary 6.16) is satisfied exactly when the handoff into the second fibration is uniform over forms, so the channel that makes each form well posed in Section 4 is also what makes switching safe. The second is a dwell-time condition (Section 6.4), which trades that uniformity for a lower bound on the time between transformations.

## 6.1 Time-domain realizations and switching signals

**Definition 6.1** (Laplace variable). Set  $s := -iz$ , so that the causal half-plane  $\mathbb{C}_+$  of Definition 2.1 is carried onto the open right half-plane  $\{\operatorname{Re} s > 0\}$ , and put  $\widehat{G}_x(s) := G_x(is)$ .

**Hypothesis 6.2** ( $H_{ss}$ , common state space). There is a finite-dimensional Hilbert space  $\mathfrak{X}$  such that every form  $x$  under consideration carries a realization  $(A_x, B_x, C_x, D_x)$  with  $A_x \in \mathcal{B}(\mathfrak{X})$ ,  $B_x \in \mathcal{B}(\mathcal{K}, \mathfrak{X})$ ,  $C_x \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$ ,  $D_x \in \mathcal{B}(\mathcal{K}, \mathcal{Y})$ , such that  $\sigma(A_x) \subset \{\operatorname{Re} s < 0\}$  and

$$\widehat{G}_x(s) = C_x(sI - A_x)^{-1}B_x + D_x. \quad (63)$$

**Remark 6.3** (Scope of Hypothesis 6.2). Hypothesis 6.2 holds precisely when the self-energy  $\Sigma_x$  of Definition 4.2 is rational, which is the Markovian case: the wide-band self-energy  $\Sigma \equiv -i\kappa$  used in the proof of Proposition 3.8 is of this kind, with  $\widehat{G}(s) = -i(s + \kappa)^{-1}$  and  $A = -\kappa$ . The half-line self-energy  $g_s$  of Proposition A.4 is algebraic but not rational, so the witness of Appendix A does not satisfy Hypothesis 6.2; extending the results of this section to a non-rational self-energy is Target 6.26. Each frozen form retains its  $\mathcal{H}^\infty$  gain under either description, since (63) changes the representation and not the function.

**Definition 6.4** (Admissible switching signal). An *admissible switching signal* is a right-continuous, piecewise constant map  $\sigma : [0, \infty) \rightarrow \Omega_{\text{adm}}$  with locally finitely many discontinuities  $0 < t_1 < t_2 < \dots$ , such that  $(\sigma(t_k^-), \sigma(t_k)) \in \mathcal{E}_{\text{adm}}$  for every  $k$ . The set of admissible switching signals with  $\sigma(0) = x$  is written  $\Sigma_{\text{adm}}[x]$ . The *switched system* is

$$\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + B_{\sigma(t)}w(t), \quad y(t) = C_{\sigma(t)}\xi(t) + D_{\sigma(t)}w(t), \quad (64)$$

with state transition operator  $\Phi_\sigma(t, r)$ , and its *switched gain* is

$$\gamma_{\text{sw}}[x] := \sup_{\sigma \in \Sigma_{\text{adm}}[x]} \sup_{0 \neq w \in \mathcal{W}_2} \frac{\|y\|_2}{\|w\|_2}, \quad \xi(0) = 0. \quad (65)$$

**Remark 6.5** (Admissible switching respects Axiom A12). Definition 6.4 permits a transition only along an edge of  $\mathcal{E}_{\text{adm}}$ , so a switching signal never leaves  $\Omega_{\text{adm}}$  and never crosses  $\partial_C \Omega_{\text{adm}}$ . A trajectory of (64) is therefore a motion in the state variable  $\xi$  along a transformation route in the sense of Definition 3.1, and the routes of Section 5.2 are exactly the possible value sequences of  $\sigma$ .

## 6.2 Frozen-form stability is not survival

**Proposition 6.6** (Switching defeats frozen-form stability).

[D] *There exist two forms  $x_a, x_b$  with  $(x_a, x_b) \in \mathcal{E}_{\text{adm}}$ , satisfying Hypothesis 6.2 on  $\mathfrak{X} = \mathbb{C}^2$ , such that  $\|\widehat{G}_{x_a}\|_{\mathcal{H}^\infty}$  and  $\|\widehat{G}_{x_b}\|_{\mathcal{H}^\infty}$  are finite and both  $A_{x_a}, A_{x_b}$  are Hurwitz, yet*

$$\sup_{\sigma \in \Sigma_{\text{adm}}[x_a]} \|\Phi_\sigma(t, 0)\| \geq C e^{\beta t} \quad (t \geq 0) \quad (66)$$

for constants  $C > 0$  and  $\beta > 0$ . In particular the switched system is not uniformly asymptotically stable.

*Proof.* Take

$$A_a = \begin{pmatrix} -1 & -100 \\ 1 & -1 \end{pmatrix}, \quad A_b = A_a^T, \quad (67)$$

each with spectrum  $\{-1 \pm 10i\}$ , hence Hurwitz, so each frozen realization is exponentially stable and, with any bounded  $B, C, D$ , has finite  $\mathcal{H}^\infty$  norm. Let  $T := \frac{3}{20}$  and let  $\sigma$  be the periodic signal taking the value  $x_a$  on  $[2kT, (2k+1)T)$  and  $x_b$  on  $[(2k+1)T, (2k+2)T)$ . Its monodromy operator over one period is  $\mathcal{M} := e^{A_b T} e^{A_a T}$ , whose spectral radius is

$$\rho(\mathcal{M}) = 73.7184 \dots > 1. \quad (68)$$

Hence  $\|\mathcal{M}^k\| \geq \rho(\mathcal{M})^k$ , and with  $t = 2kT$  this gives (66) with  $\beta = \log \rho(\mathcal{M})/(2T) = 14.33 \dots$ . The numerical value (68) is verified in Remark 6.27.  $\square$

**Remark 6.7** (Consequence for the gain). If in addition  $(A_x, B_x)$  is reachable and  $(C_x, A_x)$  observable for the forms of Proposition 6.6, then  $\gamma_{\text{sw}}[x_a] = \infty$ : the exponentially growing mode is excited by some input and seen by the output, so no finite bound in (65) can hold. This is the standard switched-system phenomenon [Lib03, Ch. 2], and it is the precise sense in which Theorem 5.9 is not a survival theorem. A form may be transformed anywhere inside the ball of Corollary 5.10 and remain individually robust, while the sequence of transformations destroys the network.

**Remark 6.8** (Why the witness does not exhibit this). The matrices (67) satisfy  $A + A^T \not\leq 0$ ; they are strongly non-normal. By Proposition 6.15 below, the composite realizations of Section 4 in their Markovian reduction cannot produce such behaviour, because they are dissipative with respect to the ambient inner product. Proposition 6.6 is therefore a statement about fibrated transfer networks in the sense of Definition 2.10 in general, and not a claim that the chrysene network realizes it; whether (67) can be realized with a common second fibration is settled negatively by

Theorem 6.11.

**Hypothesis 6.9** ( $H_{\text{shr}}$ , shared fibration with clean termination). All forms under consideration act on a common active-region state space  $\mathfrak{X}$  and couple into a single common second fibration  $\mathcal{H}_{\text{sec}}$ , the fibration terminating and resuming at the boundary of each form without modification, so that the coupling of the form  $x$  is an operator  $W_x \in \mathcal{B}(\mathcal{H}_{\text{sec}}, \mathfrak{X})$  and its Markovian reduction has self-energy  $\Sigma_x \equiv -\frac{i}{2}\Gamma_x$  with  $\Gamma_x := W_x W_x^* \geq 0$ .

**Proposition 6.10** (Shared fibration forces the dissipative form). [D] Under Hypothesis 6.9, each form admits the realization  $(A_x, B_x, -iC_x, 0)$  of Hypothesis 6.2 with

$$A_x = -iH_{p,x} - \frac{1}{2}\Gamma_x, \quad \text{hence} \quad A_x + A_x^* = -\Gamma_x \leq 0. \quad (69)$$

*Proof.* With  $\Sigma_x \equiv -\frac{i}{2}\Gamma_x$  and  $s = -iz$ ,

$$is - H_{p,x} + \frac{i}{2}\Gamma_x = i(s + iH_{p,x} + \frac{1}{2}\Gamma_x) = i(sI - A_x) \quad (70)$$

with  $A_x$  as in (69), so  $\widehat{G}_x(s) = -iC_x(sI - A_x)^{-1}B_x$ . Since  $H_{p,x}$  is self-adjoint,  $(-iH_{p,x}) + (-iH_{p,x})^* = 0$ , and  $\Gamma_x = W_x W_x^* \geq 0$  is self-adjoint, giving  $A_x + A_x^* = -\Gamma_x \leq 0$ .  $\square$

**Theorem 6.11** (Switching cannot destabilize a shared-fibration family). [D] Under Hypothesis 6.9,  $\|\Phi_\sigma(t, r)\| \leq 1$  for every admissible switching signal  $\sigma$  and all  $t \geq r \geq 0$ . In particular no family of forms sharing a second fibration exhibits the exponential growth of Proposition 6.6, and the matrices (67) are not realizable under Hypothesis 6.9.

*Proof.* Proposition 6.10 supplies the hypothesis of Proposition 6.15, which gives the contraction bound. For the last assertion,  $A_a + A_a^T$  of (67) has eigenvalues  $\{-101, 97\}$  and is therefore not negative semidefinite, so by (69) no  $\Gamma_a \geq 0$  represents it.  $\square$

### 6.3 Common storage

**Definition 6.12** (Common storage functional). Assume Hypothesis 6.2 and let  $\gamma > 0$ . A positive semidefinite  $P \in \mathcal{B}(\mathfrak{X})$  is a *common storage functional at level  $\gamma$*  if, writing  $V(\xi) := \langle \xi, P\xi \rangle$ , for every form  $x$  under consideration, every  $\xi \in \mathfrak{X}$  and every  $w \in \mathcal{K}$ ,

$$2\text{Re}\langle P\xi, A_x\xi + B_x w \rangle \leq \gamma^2 \|w\|^2 - \|C_x\xi + D_x w\|^2. \quad (71)$$

**Theorem 6.13** (Uniform gain under arbitrary admissible switching). [D] Assume Hypothesis 6.2 and suppose a common storage functional  $P$  at level  $\gamma$  exists. Then for every  $x \in \Omega_{\text{adm}}$ ,

$$\gamma_{\text{sw}}[x] \leq \gamma. \quad (72)$$

*Proof.* Fix  $\sigma \in \Sigma_{\text{adm}}[x]$  with switching times  $0 = t_0 < t_1 < \dots$ , let  $w \in \mathcal{W}_2$  and let  $\xi$  solve (64) with  $\xi(0) = 0$ . On each interval  $(t_k, t_{k+1})$  the signal  $\sigma$  is constant, so (71) applies with  $x = \sigma(t_k)$  and gives

$$\frac{d}{dt}V(\xi(t)) \leq \gamma^2 \|w(t)\|^2 - \|y(t)\|^2 \quad \text{a.e. on } (t_k, t_{k+1}). \quad (73)$$

The state  $\xi$  is absolutely continuous on  $[0, \infty)$  and  $V$  is a fixed continuous quadratic form, so  $t \mapsto V(\xi(t))$  is absolutely continuous on compact intervals, with no jump at any  $t_k$ ; this is the only place where commonality of  $P$  is used. Integrating (73) over  $[0, T]$  and summing over the finitely many subintervals gives

$$V(\xi(T)) - V(\xi(0)) \leq \gamma^2 \int_0^T \|w\|^2 - \int_0^T \|y\|^2. \quad (74)$$

Since  $V(\xi(0)) = 0$  and  $V \geq 0$ , we obtain  $\int_0^T \|y\|^2 \leq \gamma^2 \int_0^T \|w\|^2$  for every  $T$ , and letting  $T \rightarrow \infty$  gives (72) after taking the supremum over  $w$  and over  $\sigma$ .  $\square$

**Remark 6.14** (Existence of  $P$ ). For a single form, existence of  $P > 0$  satisfying (71) is equivalent to  $\|\widehat{G}_x\|_{\mathcal{H}^\infty} < \gamma$  by the bounded real lemma [ZDG96, Ch. 13]. For a family of forms the requirement is that one  $P$  serve them all, which is a strictly stronger condition than  $\sup_x \|\widehat{G}_x\|_{\mathcal{H}^\infty} < \gamma$ : by Proposition 6.6 the latter can hold while  $\gamma_{\text{sw}} = \infty$ , in which case no common  $P$  exists. Feasibility of the family (71) in the single unknown  $P$  is a linear matrix inequality problem, hence decidable by convex programming for a finite set of forms.

**Proposition 6.15** (Dissipative forms are contractive under any switching). [A] Suppose  $A_x + A_x^* \leq 0$  for every form  $x$  under consideration. Then  $\|\Phi_\sigma(t, r)\| \leq 1$  for every admissible  $\sigma$  and all  $t \geq r \geq 0$ .

*Proof.* On an interval where  $\sigma \equiv x$ ,  $\frac{d}{dt}\|\xi\|^2 = \langle \xi, (A_x + A_x^*)\xi \rangle \leq 0$ , so  $\|e^{A_x \tau}\| \leq 1$  for  $\tau \geq 0$ . The transition operator is a finite product of such contractions, hence a contraction.  $\square$

**Corollary 6.16** (Structural consequences of a shared fibration). [D] Assume Hypothesis 6.9. Then:

(i) unconditionally, the switched homogeneous system is non-expansive and admissible switching never destabilizes the family, whatever the individual couplings  $W_x$ ;

(ii) if moreover there are constants  $\gamma_0, b, c, d$  with

$$\Gamma_x \geq \gamma_0 I, \quad \|B_x\| \leq b, \quad \|C_x\| \leq c, \quad \|D_x\| \leq d, \quad (75)$$

then  $P = I$  is a common storage functional at level

$$\gamma := \left( \frac{b^2}{\gamma_0 - 2c^2} + 2d^2 \right)^{1/2}, \quad (76)$$

and  $\gamma_{\text{sw}}[x] \leq \gamma$  for every  $x \in \Omega_{\text{adm}}$ , uniformly over all admissible switching.

*Proof.* (i) is Theorem 6.11. For (ii), by Proposition 6.10 we have  $A_x + A_x^* = -\Gamma_x$ , so

$$2 \operatorname{Re} \langle \xi, A_x \xi + B_x w \rangle = -\langle \xi, \Gamma_x \xi \rangle + 2 \operatorname{Re} \langle \xi, B_x w \rangle. \quad (77)$$

For any  $\alpha > 0$ ,  $2 \operatorname{Re} \langle \xi, B_x w \rangle \leq \alpha \|\xi\|^2 + \alpha^{-1} b^2 \|w\|^2$ , and  $\|C_x \xi + D_x w\|^2 \leq 2c^2 \|\xi\|^2 + 2d^2 \|w\|^2$ . Hence the left side of (71) plus  $\|C_x \xi + D_x w\|^2$  is at most

$$(-\gamma_0 + \alpha + 2c^2) \|\xi\|^2 + (\alpha^{-1} b^2 + 2d^2) \|w\|^2. \quad (78)$$

Choosing  $\alpha := \gamma_0 - 2c^2 > 0$  annihilates the first bracket and leaves exactly  $\gamma^2 \|w\|^2$  with  $\gamma$  as in (76). Theorem 6.13 then applies.  $\square$

**Remark 6.17** (The same channel serves both purposes). Condition (75) requires the absorption  $\Gamma_x$  to be bounded below uniformly in the form. By Proposition 4.13,  $\Gamma_x$  is the coupling-weighted density of states of the second fibration, and by Proposition 4.22 its nonvanishing is exactly the failure of exact confinement. Corollary 6.16 therefore states that survival under transformation is purchased by the same mechanism that purchases well-posedness of a single form in Section 4: a handoff channel into the second fibration, open uniformly along the whole transformation route. By Proposition 4.20 the norm so transferred is conserved, not destroyed. This is the formal content of the survival claim in its third mode.

## 6.4 Dwell time

When no common storage functional exists, stability can still be recovered by restricting how often transformations occur.

**Definition 6.18** (Average dwell time). For  $\sigma \in \Sigma_{\text{adm}}[x]$  let  $N_\sigma(t, r)$  denote the number of switching times in  $(r, t]$ .

The signal  $\sigma$  has average dwell time  $\tau_a > 0$  with chatter bound  $N_0 \geq 0$  if

$$N_\sigma(t, r) \leq N_0 + \frac{t - r}{\tau_a} \quad \text{for all } t \geq r \geq 0. \quad (79)$$

**Theorem 6.19** (Dwell-time stability; proof sketch with citation). [D] Assume Hypothesis 6.2 and suppose that for each form  $x$  there is  $P_x > 0$  and  $\lambda > 0$  with

$$2 \operatorname{Re} \langle P_x \xi, A_x \xi \rangle \leq -2\lambda \langle \xi, P_x \xi \rangle, \quad P_x \leq \mu P_y \text{ for all } x, y, \quad (80)$$

with  $\mu \geq 1$ . Then for every admissible  $\sigma$  with average dwell time  $\tau_a > \log \mu / (2\lambda)$  the switched homogeneous system is uniformly exponentially stable, with decay rate at least  $\lambda - \log \mu / (2\tau_a) > 0$ .

*Proof sketch.* Put  $V_{\sigma(t)}(\xi) = \langle \xi, P_{\sigma(t)} \xi \rangle$ . Between switching times the first inequality of (80) gives  $V_{\sigma(t)}(\xi(t)) \leq e^{-2\lambda(t-t_k)} V_{\sigma(t_k)}(\xi(t_k))$ ; at each switching time the second gives a jump by a factor at most  $\mu$ . Composing over  $N_\sigma(t, 0)$  switches and using (79),

$$V_{\sigma(t)}(\xi(t)) \leq \mu^{N_0} e^{-2\lambda t + N_\sigma(t, 0) \log \mu} V_{\sigma(0)}(\xi(0)), \quad (81)$$

whose exponent is at most  $-2(\lambda - \log \mu / (2\tau_a))t + N_0 \log \mu$ . The full argument, including the corresponding  $\mathcal{L}_2$  gain bound, is [HM99, Thm. 3.2]; see also [Lib03, Ch. 3].  $\square$

**Remark 6.20** (Correspondence with the action cost of the monographs). The two constants of (80) have direct counterparts in [Sch26f, App. B]: the jump factor  $\mu$  is the analogue of the action cost  $\Delta F_{\text{sciss}}$  charged for a structural rearrangement, and  $\lambda$  the analogue of the rate at which the available gradient is dissipated between rearrangements. In that reading, Theorem 6.19 is the precise form of the assertion that the cost of a transformation must be paid out of the dissipation occurring between transformations, with  $\tau_a > \log \mu / (2\lambda)$  the exact exchange rate. The correspondence is an interpretation of [Sch26f], not a derivation of it: (80) is a hypothesis on the realizations, and no thermodynamic content is claimed here.

## 6.5 Compensating transformation policies

**Definition 6.21** (Compensating policy). Assume Hypothesis 2.19. A compensating transformation policy is a map  $\pi : \Omega_{\text{env}} \rightarrow \Omega_{\text{adm}}$  such that  $(\pi(e), \pi(e')) \in \mathcal{E}_{\text{adm}} \cup \{(u, u) : u \in \Omega_{\text{adm}}\}$  whenever the environment passes from  $e$  to  $e'$ . The policy is  $\delta$ -compensating at  $x$  if  $\|E_e\|_{\mathcal{H}^\infty} \leq \delta$  implies  $d_{\mathcal{H}^\infty}(x, \pi(e)) < \delta^{-1} - \|G_x\|_{\mathcal{H}^\infty}$  for every  $e \in \Omega_{\text{env}}$ .

**Proposition 6.22** (Policies survive). [D] Assume Hypothesis 6.2. Let  $\pi$  be  $\delta$ -compensating at  $x$ , and suppose in addition that the forms in the range of  $\pi$  admit a common storage functional at level  $\gamma$  with  $\gamma\delta < 1$ . Then every trajectory of (64) generated by  $\pi$  remains in  $\Omega_{\text{adm}}$ , each visited form is robustly stable against the environment operator in force, and the interconnection is stable under the switching the policy itself induces.

*Proof.* Each visited form lies in the ball of Corollary 5.10 by the definition of  $\delta$ -compensation, giving the second assertion via Theorem 5.9. The first holds because  $\pi$  takes values in  $\Omega_{\text{adm}}$  and moves only along  $\mathcal{E}_{\text{adm}}$  (Remark 6.5). For the third, the switching signal induced by  $\pi$  is admissible, so Theorem 6.13 bounds the induced  $\mathcal{W}_2 \rightarrow \mathcal{W}_2$  gain of the switched, and hence time-varying, plant by  $\gamma$ . With  $\|\Delta\|_{\mathcal{W}_2 \rightarrow \mathcal{W}_2} \leq \|\Delta\|_{\mathcal{H}^\infty} \leq \delta$  for the time-invariant environment factor and  $\gamma\delta < 1$ , the input–output form of the small-gain theorem, which requires only that both factors be causal and finite-gain and not that either be time-invariant [Lib03, Ch. 5], gives well-posedness and stability of the interconnection.  $\square$

*Remark 6.23* (What a policy does not supply). Proposition 6.22 presupposes that a  $\delta$ -compensating policy exists; it does not construct one, and no result of this paper guarantees existence. In particular the range of a policy must be contained in a  $d_{\mathcal{H}^\infty}$ -ball whose radius shrinks as  $\delta$  grows, so for sufficiently violent environments Corollary 5.10 gives an empty ball and no policy can succeed. That threshold,  $\delta = \|G_x\|_{\mathcal{H}^\infty}^{-1}$ , is the sharpest statement this paper makes about the limits of survival by transformation.

## 6.6 Targets

**Theorem 6.24** (Target: does  $d_{\mathcal{H}^\infty}$  control the jump factor?). [T] Target: identify hypotheses under which the constant  $\mu$  of (80) for an edge  $(x, y) \in \mathcal{E}_{\text{adm}}$  is bounded by a function of the edge cost  $c(x, y) = \delta_u(G_x, G_y)$  of Definition 5.5, monotone and equal to 1 at  $c = 0$ . Such a bound would convert Theorem 6.19 into a statement about the transformation geometry of Section 5 rather than about the realizations, and would make  $d_{\mathcal{H}^\infty}$  the controlling quantity for survival across transformations as well as within them. No such bound is established here;  $\delta_u$  compares transfer functions, while  $\mu$  compares storage functionals, and two realizations of the same transfer function may have arbitrarily different storage functionals.

**Theorem 6.25** (Target: optimal compensating policies). [T] Target: establish existence of a policy minimizing

$\sup_e d_{\mathcal{H}^\infty}(x, \pi(e))$  over  $\delta$ -compensating policies, and characterize it as the solution of a dynamic programming recursion on the graph  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$  with edge cost  $c$ .

**Theorem 6.26** (Target: non-Markovian forms). [T] Target: extend Theorem 6.13 and Theorem 6.19 to forms whose self-energy is not rational, so that Hypothesis 6.2 fails, as is the case for the witness realization of Appendix A. The natural route is a storage functional on the composite state space  $\mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}$  rather than on  $\mathfrak{X}$ , for which the conservative structure (26) gives  $A + A^* = 0$  and Proposition 6.15 is available but uninformative, the dissipation being carried by spreading in  $\mathcal{H}_{\text{sec}}$  rather than by a negative definite term.

*Remark 6.27* (Independent numerical verification). The constants of Proposition 6.6 were checked by direct computation of the monodromy operator: for (67) with  $T = \frac{3}{20}$  the eigenvalues of  $e^{A_b T} e^{A_a T}$  are  $7.4447 \times 10^{-3}$  and  $73.7185$ , giving  $\rho(\mathcal{M}) = 73.7185$  and growth rate  $\beta = 14.3342$  per unit time, while  $\|e^{A_a \cdot 2T}\| = 1.4212$  confirms that a single frozen form contracts over the same interval. The eigenvalues of  $A_a + A_a^\top$  are  $\{-101, 97\}$ , confirming that the hypothesis of Proposition 6.15 fails, as Remark 6.8 asserts.

## 7 Why the Worst-Case Norm and Not the Mean-Square Norm

Every construction of Sections 4–6 is built on the quantity  $\|\cdot\|_{\mathcal{H}^\infty}$  of Definition 2.3. The corpus also uses, in several places, a mean-square quantity. This section establishes that the two are not interchangeable for the purpose at hand, and does so in four independent ways: the mean-square norm is generically infinite on Axiom A10 data (Section 7.1); smallness in it does not control the worst-case norm, by an explicit family inside the formalism (Section 7.2); the robust-stability theorem that motivates the whole construction has no mean-square analogue, and this can be stated quantitatively (Section 7.3); and the two norms are the induced norms of different problems (Section 7.4). Section 7.5 then states the one case in which  $\|\cdot\|_{\mathcal{H}^\infty}$  is itself the wrong measure, and supplies the right one.

### 7.1 The mean-square norm is generically infinite

**Definition 7.1** (Mean-square norm). For  $G$  as in Definition 3.2, independent of  $\theta$  and with boundary values on  $\mathbb{R}$ , set

$$\|G\|_{\mathcal{H}^2}^2 := \frac{1}{2\pi} \int_{\mathbb{R}} \|G(E)\|_{\mathbb{F}}^2 dE, \quad (82)$$

$\|\cdot\|_{\mathbb{F}}$  denoting the Hilbert–Schmidt norm.

**Proposition 7.2** (Feedthrough obstruction). [A] *If the feedthrough operator  $J$  of Definition 4.1 is nonzero, then  $\|G\|_{\mathcal{H}^2} = \infty$ .*

*Proof.* By Remark 4.3 and Definition 4.2,  $\|\Sigma(z)\| \rightarrow 0$  and  $\|M(z)^{-1}\| \rightarrow 0$  as  $|z| \rightarrow \infty$ , so  $G(E) \rightarrow J$  as  $|E| \rightarrow \infty$  by (28). Hence there is  $R > 0$  with  $\|G(E)\|_F \geq \frac{1}{2}\|J\|_F > 0$  for  $|E| \geq R$ , and the integral (82) diverges.  $\square$

**Remark 7.3** (Scope of Proposition 7.2). Axiom A10 admits  $J \neq 0$  and stipulates that  $J$  may vanish only under an explicitly specified physical model [Sch26c, A10]. For such data the mean-square norm is not merely large but undefined as a finite quantity, whereas  $\|G\|_{\mathcal{H}^\infty} \leq \|J\| + \|C\|\|B\| \sup_z \|M(z)^{-1}\|$  remains finite under Theorem 4.17. The obstruction is therefore structural and not an artefact of a particular realization: it is present at the level of the axiom.

## 7.2 Mean-square smallness does not control the worst case

**Proposition 7.4** (A narrow-band resonance family inside Axiom A10). [D] *For  $\gamma \in (0, 1)$  let  $\mathcal{H}_{\text{pri}} = \mathbb{C}$ ,  $H_p = \varepsilon_0 \in \mathbb{R}$ ,  $\Sigma_\gamma \equiv -i\gamma$ ,  $C = 1$ ,  $J = 0$  and  $B = B_\gamma := \gamma^{3/4}$ , so that*

$$T_\gamma(z) = \frac{\gamma^{3/4}}{z - \varepsilon_0 + i\gamma} \quad (83)$$

*is a composite realization in the sense of Definition 4.1. Then*

$$\|T_\gamma\|_{\mathcal{H}^\infty} = \gamma^{-1/4} \xrightarrow{\gamma \rightarrow 0} \infty, \quad \|T_\gamma\|_{\mathcal{H}^2} = \frac{\gamma^{1/4}}{\sqrt{2}} \xrightarrow{\gamma \rightarrow 0} 0. \quad (84)$$

*Proof.* The constant self-energy  $-i\gamma$  satisfies  $\text{Im } \Sigma \leq 0$ , so Lemma 4.4 applies and (83) is admissible. Its modulus on the real axis is  $\gamma^{3/4}((E - \varepsilon_0)^2 + \gamma^2)^{-1/2}$ , maximal at  $E = \varepsilon_0$  with value  $\gamma^{3/4}/\gamma = \gamma^{-1/4}$ ; by the maximum principle this is  $\|T_\gamma\|_{\mathcal{H}^\infty}$ . For the mean-square norm,

$$\|T_\gamma\|_{\mathcal{H}^2}^2 = \frac{\gamma^{3/2}}{2\pi} \int_{\mathbb{R}} \frac{dE}{(E - \varepsilon_0)^2 + \gamma^2} = \frac{\gamma^{3/2}}{2\pi} \cdot \frac{\pi}{\gamma} = \frac{\gamma^{1/2}}{2}, \quad (85)$$

which is (84).  $\square$

**Corollary 7.5** (No mean-square certificate of worst-case smallness). [D] *There is no function  $\varphi : (0, \infty) \rightarrow (0, \infty)$  with  $\varphi(t) \rightarrow 0$  as  $t \rightarrow 0$  such that  $\|G\|_{\mathcal{H}^\infty} \leq \varphi(\|G\|_{\mathcal{H}^2})$  holds for all composite realizations. Indeed no bound*

*$\|G\|_{\mathcal{H}^\infty} \leq \varphi(\|G\|_{\mathcal{H}^2})$  with  $\varphi$  finite and nondecreasing can hold at all.*

*Proof.* Suppose such a  $\varphi$  existed. Applying it along the family (83) gives  $\gamma^{-1/4} \leq \varphi(\gamma^{1/4}/\sqrt{2})$  for all  $\gamma \in (0, 1)$ . Letting  $\gamma \rightarrow 0$ , the left side tends to  $\infty$  while the argument of  $\varphi$  tends to 0, so  $\varphi$  is unbounded near 0 and in particular not nondecreasing with finite values there.  $\square$

**Remark 7.6** (Physical reading). The family (83) is a single resonance of width  $\gamma$  and height  $\gamma^{-1/4}$ . Its mean-square norm is small because the resonance is narrow, and its worst-case norm is large because the resonance is tall. A criterion based on (82) certifies such a form as benign; the network built from it amplifies any disturbance whose spectrum happens to overlap the narrow band around  $\varepsilon_0$  by the factor  $\gamma^{-1/4}$ . Since the disturbance classes of Definition 2.5 carry no model of where the disturbance's spectral content lies — that is the content of Remark 2.6 — such an overlap cannot be excluded.

## 7.3 Small gain, its converse, and the absence of an analogue

**Theorem 7.7** (Small gain and stability radius; cited). [D] *Let  $G$  be stable. The feedback interconnection of  $G$  with a causal linear time-invariant  $\Delta$  is internally stable for every  $\Delta$  with  $\|\Delta\|_{\mathcal{H}^\infty} < \|G\|_{\mathcal{H}^\infty}^{-1}$ , and there exists such a  $\Delta$  with  $\|\Delta\|_{\mathcal{H}^\infty} = \|G\|_{\mathcal{H}^\infty}^{-1}$  for which it is not. Thus*

$$\sup\{r : \text{stable for all } \|\Delta\|_{\mathcal{H}^\infty} < r\} = \frac{1}{\|G\|_{\mathcal{H}^\infty}}. \quad (86)$$

*Proof sketch with citation.* Sufficiency is the small-gain theorem [ZDG96, Thm. 9.1]. For necessity, let  $E_*$  be a point at which  $\sigma_{\max}(G(E_*))$  is within  $\epsilon$  of the supremum, and let  $u, y$  be corresponding unit singular vectors with  $G(E_*)u = \sigma y$ . A constant  $\Delta_0$  of norm  $\sigma^{-1}$  carrying  $y$  to  $u$  makes  $I - G(E_*)\Delta_0$  singular; interpolating  $\Delta_0$  by a stable rational function of norm  $\sigma^{-1}$ , which is possible by the standard Nevanlinna–Pick construction, produces a destabilizing perturbation. The complete argument is [ZDG96, Thm. 9.1 and Ch. 9.3].  $\square$

**Corollary 7.8** (No mean-square small-gain theorem). [D] *There is no function  $\psi : (0, \infty) \rightarrow (0, \infty)$  with  $\psi(t) \rightarrow \infty$  as  $t \rightarrow 0$  such that every stable  $G$  is robustly stable against all  $\Delta$  with  $\|\Delta\|_{\mathcal{H}^\infty} < \psi(\|G\|_{\mathcal{H}^2})$ .*

*Proof.* By Theorem 7.7 the exact stability radius of  $T_\gamma$  in (83) is  $\|T_\gamma\|_{\mathcal{H}^\infty}^{-1} = \gamma^{1/4}$ , while  $\|T_\gamma\|_{\mathcal{H}^2} = \gamma^{1/4}/\sqrt{2}$ . A function  $\psi$  as

in the statement would give  $\gamma^{1/4} \geq \psi(\gamma^{1/4}/\sqrt{2})$ , whose right side tends to  $\infty$  and left side to 0 as  $\gamma \rightarrow 0$ .  $\square$

**Remark 7.9** (Why this is the decisive argument). Corollary 7.5 shows that the mean-square norm does not bound the worst-case norm. Corollary 7.8 shows more: even granting an arbitrarily small mean-square norm, no robustness margin whatever follows. Since Theorem 5.9 and Corollary 6.16 — the two survival statements of this paper — are both consequences of the small-gain theorem, and since (86) shows  $\|\cdot\|_{\mathcal{H}^\infty}$  to be not merely sufficient but exactly the reciprocal of the stability radius, the choice of norm is forced rather than preferred. This is the formal content of the requirement that the network survive under worst-case perturbation.

## 7.4 What each norm is the induced norm of

**Proposition 7.10** (Induced-norm characterizations). [D] *Let  $G$  be stable with  $J = 0$  and scalar channels. Then*

- (i)  $\|G\|_{\mathcal{H}^\infty}$  is the induced norm  $\mathcal{W}_2 \rightarrow \mathcal{W}_2$  (Lemma 3.5);
- (ii)  $\|G\|_{\mathcal{H}^\infty}$  is the induced power gain on  $\mathcal{W}_{\text{pow}}$ , that is  $\sup\{\|y\|_{\text{pow}}/\|w\|_{\text{pow}}\}$  over nonzero  $w \in \mathcal{W}_{\text{pow}}$ ;
- (iii)  $\|G\|_{\mathcal{H}^2}$  is the induced norm  $\mathcal{W}_2 \rightarrow L^\infty$ , that is  $\sup\{\|y\|_\infty/\|w\|_2\}$ ;
- (iv)  $\|G\|_{\mathcal{H}^2}$  is the root-mean-square amplitude of the output when the input is unit-intensity white noise.

*Proof.* (i) is Lemma 3.5. (iii): with impulse response  $g$ ,  $|y(t)| = |\int g(t-\tau)w(\tau)d\tau| \leq \|g\|_{L^2}\|w\|_{L^2}$  by Cauchy–Schwarz, and  $\|g\|_{L^2} = \|G\|_{\mathcal{H}^2}$  by Plancherel; equality is attained by the time-reversed shifted impulse response, so the bound is the induced norm. (ii) and (iv) are standard; see [ZDG96, Ch. 4].  $\square$

**Remark 7.11** (The two norms answer different questions). By Proposition 7.10 the mean-square norm is the correct quantity for two questions: the largest output excursion produced by a disturbance of bounded energy, and the average response to noise of known statistics. Neither is the question of this paper. The disturbance classes that Remark 2.6 identifies as the operative ones are  $\mathcal{W}_{\text{pow}}$  and  $\mathcal{W}_\infty$ , which are persistent and carry no statistical model; on the first of these  $\|\cdot\|_{\mathcal{H}^\infty}$  is the exact induced gain by Proposition 7.10(ii), while a finite-energy disturbance is by definition one that ends.

## 7.5 The one case where $\|\cdot\|_{\mathcal{H}^\infty}$ is the wrong measure

The uncertainty class of Theorem 7.7 is the unstructured causal class. Part of the uncertainty in the present setting is not of that kind: a perturbation of the active region that is itself a Hamiltonian is self-adjoint, and self-adjoint perturbations cannot destabilize a dissipative realization at all.

**Proposition 7.12** (Self-adjoint perturbations never destabilize). [A] *Let  $\delta H = \delta H^* \in \mathcal{B}(\mathcal{H}_{\text{pri}})$  and set  $M_\delta(z) := z - H_p - \delta H - \Sigma(z)$ . Then for every  $z \in \mathbb{C}_+$ ,  $M_\delta(z)$  is invertible with  $\|M_\delta(z)^{-1}\| \leq (\text{Im } z)^{-1}$ . In particular the perturbed realization has no pole in  $\mathbb{C}_+$ , for any  $\|\delta H\|$  whatever.*

*Proof.*  $\text{Im}(\delta H) = 0$ , so  $\text{Im } M_\delta(z) = \text{Im } M(z) \geq (\text{Im } z)I$  by Lemma 4.4, and the argument of that lemma applies unchanged.  $\square$

**Remark 7.13** (Consequence). Proposition 7.12 shows that the stability radius (86) is strictly conservative for the self-adjoint class: it reports a finite margin  $\|G\|_{\mathcal{H}^\infty}^{-1}$  where the true margin against self-adjoint perturbations is infinite in the sense of pole location. What a large self-adjoint perturbation can do is drive a pole toward the real axis, that is, create an embedded mode in the sense of Remark 4.18, making  $\|G\|_{\mathcal{H}^\infty}$  infinite without ever making the realization unstable. The correct margin for this class is therefore a distance to the embedded-mode set, not a gain.

**Definition 7.14** (Embedded-mode margin). For a composite realization with  $\Lambda, \Gamma$  as in Proposition 4.13, extended by  $\Gamma := 0$  and  $\Lambda := \Sigma$  off the bands, set

$$r_{\text{emb}} := \inf_{\lambda \in \mathbb{R}} \inf_{\substack{\psi \in \ker \Gamma(\lambda) \\ \|\psi\|=1}} \|(\lambda - H_p - \Lambda(\lambda))\psi\|, \quad (87)$$

with  $\inf \emptyset := +\infty$ .

**Theorem 7.15** (The margin is exact). [D]  *$r_{\text{emb}}$  is precisely the smallest norm of a self-adjoint  $\delta H \in \mathcal{B}(\mathcal{H}_{\text{pri}})$  for which the perturbed realization possesses an embedded mode, that is, for which there are  $\lambda \in \mathbb{R}$  and  $\psi \neq 0$  with*

$$(\lambda - H_p - \delta H - \Lambda(\lambda))\psi = 0 \quad \text{and} \quad \Gamma(\lambda)\psi = 0. \quad (88)$$

*Consequently, by Theorem 4.17, every self-adjoint  $\delta H$  with  $\|\delta H\| < r_{\text{emb}}$  leaves  $\|G\|_{\mathcal{H}^\infty}$  finite, and some self-adjoint  $\delta H$  with  $\|\delta H\| = r_{\text{emb}}$  makes it infinite.*

*Proof. Lower bound.* Suppose  $\delta H = \delta H^*$  and  $(\lambda, \psi)$  satisfy (88) with  $\|\psi\| = 1$ . Put  $\phi := (\lambda - H_p - \Lambda(\lambda))\psi$ . Then  $\delta H\psi = \phi$  and  $\psi \in \ker \Gamma(\lambda)$ , so

$$\|\delta H\| \geq \|\delta H\psi\| = \|\phi\| \geq r_{\text{emb}} \quad (89)$$

by (87).

*Attainment.* Conversely, fix  $\lambda$  and a unit  $\psi \in \ker \Gamma(\lambda)$ , and let  $\phi$  be as above. Since  $\lambda \in \mathbb{R}$  and  $H_p, \Lambda(\lambda)$  are self-adjoint, the scalar  $c := \langle \psi, \phi \rangle$  is real. Write  $\phi = c\psi + \phi_\perp$  with  $\phi_\perp \perp \psi$ , put  $b := \|\phi_\perp\|$  and, if  $b > 0$ ,  $e := \phi_\perp/b$ . Define  $\delta H$  to act as 0 on  $\{\psi, e\}^\perp$  and, in the ordered orthonormal basis  $(\psi, e)$  of their span, as

$$\delta H|_{\text{span}\{\psi, e\}} = \begin{pmatrix} c & b \\ b & -c \end{pmatrix}. \quad (90)$$

This is self-adjoint and satisfies  $\delta H\psi = c\psi + be = \phi$ . Its eigenvalues are  $\pm\sqrt{c^2 + b^2}$ , so  $\|\delta H\| = \sqrt{c^2 + b^2} = \|\phi\|$ . If  $b = 0$  take  $\delta H = c|\psi\rangle\langle\psi|$ , of norm  $|c| = \|\phi\|$ . Taking the infimum over  $\lambda$  and  $\psi$  shows the value  $r_{\text{emb}}$  is attained whenever the infimum in (87) is, and approached otherwise.  $\square$

**Example 7.16** (The two margins on the witness). [D] For the witness realization of Definition A.6 with  $\varepsilon = 0$  and  $0 < v^2 < 2$ : on the band  $\Gamma(\lambda) = v^2\sqrt{4 - \lambda^2}/2 > 0$ , so  $\ker \Gamma(\lambda) = \{0\}$  and the band contributes nothing to (87); off the band  $\Gamma = 0$ ,  $\ker \Gamma = \mathbb{C}$ , and

$$r_{\text{emb}} = \inf_{|\lambda| > 2} |\lambda - v^2 g_s(\lambda)| = 2 - v^2, \quad (91)$$

the infimum being approached at the band edges, in agreement with the threshold  $|\varepsilon| < 2 - v^2$  of Proposition A.9. By contrast Proposition A.8 gives

$$\frac{1}{\|G\|_{\mathcal{H}^\infty}} = \begin{cases} v^2, & v^2 \leq 1, \\ 2 - v^2, & v^2 \geq 1. \end{cases} \quad (92)$$

The two agree for  $v^2 \geq 1$  and differ sharply for small  $v$ : at  $v^2 = \frac{1}{4}$  the gain margin is  $\frac{1}{4}$  while the self-adjoint margin is  $\frac{7}{4}$ , a factor of seven.

*Remark 7.17* (Reading of Example 7.16). A weakly coupled form is fragile in the unstructured sense — a small non-self-adjoint perturbation of the feedback path destabilizes it — and simultaneously very robust in the self-adjoint sense, since a large Hamiltonian perturbation is needed before a mode is pushed out of the band and ceases to hand off. The two numbers measure different failures, and a design that is required to survive both must satisfy both. Nothing in this paper combines them into a single quantity; doing so is Target 7.19.

## 7.6 What the worst-case norm does not certify

*Remark 7.18* (Gain is not admissibility). By Proposition 7.10 the quantity  $\|G\|_{\mathcal{H}^\infty}$  bounds a ratio of norms of signals. The admissible domain of Axiom A12 is defined by the pointwise strict inequalities  $C_j(x) < \tau_j$ , which are conditions on a state and not on a signal norm. Without Hypothesis 2.22 the two are not related at all; with it, the relevant induced norm is  $\mathcal{W}_\infty \rightarrow L^\infty$  or  $\mathcal{W}_2 \rightarrow L^\infty$ , that is a peak gain, and by Proposition 7.10(iii) the latter is the mean-square norm. Thus the mean-square quantity, dismissed above as a robustness certificate, is the correct quantity for the other half of the survival claim; the error would be to use either for the other's purpose. This is developed in Section 8.

## 7.7 Targets

**Theorem 7.19** (Target: a combined margin). [T] *Target: construct a single quantity  $r(x)$ , for a form  $x$ , that lower bounds simultaneously the unstructured stability radius  $\|G_x\|_{\mathcal{H}^\infty}^{-1}$  of Theorem 7.7 and the self-adjoint embedded-mode margin  $r_{\text{emb}}(x)$  of Theorem 7.15, and that is computable from the composite datum of Definition 4.1. The natural candidate is a structured singular value with a mixed real and complex uncertainty block structure, real blocks encoding the self-adjoint part; its computation is NP-hard in general, so a tractable bound rather than an exact value is the realistic target.*

**Theorem 7.20** (Target: the margin along a transformation route). [T] *Target: bound  $r_{\text{emb}}(y)$  in terms of  $r_{\text{emb}}(x)$  and the transformation distance  $d_{\mathcal{H}^\infty}(x, y)$  of Definition 5.5, in analogy with Proposition 5.3 and Theorem 5.9. Such a bound would give the self-adjoint class its own survival radius. It is not available from the results of this paper, since  $\delta_u$  compares transfer functions while  $r_{\text{emb}}$  is computed from  $H_p$  and  $\Lambda$ , which (28) does not determine uniquely.*

## 8 Environmental Compatibility, Pointwise Admissibility, and a Taxonomy of Distances

This section completes the survival claim and situates the quantities that the corpus uses alongside  $d_{\mathcal{H}^\infty}$ . Section 8.1 treats the second survival mode, compatibility of a form with an environment that the form itself may shape. Section 8.2 records the structured version of the robustness question. Section 8.3 supplies the half of the survival claim that Remark 7.18 identified as outstanding: invariance of the admissible domain is a peak-gain statement, not an  $\mathcal{H}^\infty$  statement. Sections 8.4–8.5

situate the graph-theoretic and optimal-transport quantities, and Section 8.6 collects the result.

## 8.1 Environmental compatibility and nucleation

**Definition 8.1** (Compatibility margin). Assume Hypotheses 3.4 and 2.19. The form  $x$  and the environment state  $e$  are *compatible* if the feedback interconnection of  $G_x$  with  $E_e$  is well posed and internally stable. The *compatibility margin* of  $x$  is

$$\text{cm}(x) := \sup\{r > 0 : x \text{ is compatible with every } e \text{ with } \|E_e\|_{\mathcal{H}^\infty} \leq r\} \quad (93)$$

**Proposition 8.2** (Compatibility margin equals the reciprocal gain). [D]  $\text{cm}(x) = \|G_x\|_{\mathcal{H}^\infty}^{-1}$ , and for  $y \in \Omega_{\text{adm}}$ ,

$$\text{cm}(y) \geq (\|G_x\|_{\mathcal{H}^\infty} + d_{\mathcal{H}^\infty}(x, y))^{-1}. \quad (94)$$

*Proof.* The equality is Theorem 7.7 applied to  $G = G_x$ , the environment operators ranging over the unstructured causal class of that theorem. Inequality (94) follows from Proposition 5.3 and (51), exactly as in the proof of Theorem 5.9.  $\square$

**Definition 8.3** (Nucleation and persistence). Assume Hypothesis 2.20 with nucleation data  $\mathcal{N}$  and partial map  $\nu : (x, e) \mapsto e'$ . A form  $x$  *nucleates*  $e'$  from  $e$  if  $\nu(x, e) = e'$  is defined. The nucleated state *persists* if  $e'$  remains the environment state along subsequent transformations of the form coordinate; compatibility of a later form  $y$  with a persistent  $e'$  is compatibility in the sense of Definition 8.1. No chemical structure is asserted of  $e'$  by this definition.

**Proposition 8.4** (The nucleated complex is again a composite realization). [D] *Let  $x$  be a defect form with composite realization  $(\mathcal{H}_{\text{pri}}, \mathcal{H}_{\text{sec}}, H_p, H_s, W, B, C, J)$  and let  $e'$  be a nucleated environment state whose operator  $E_{e'}$  admits a finite-dimensional self-adjoint realization on a space  $\mathcal{H}_{e'}$  coupled to  $\mathcal{H}_{\text{pri}}$  by a bounded  $W_{e'}$ . Then the interconnected system has a composite realization in the sense of Definition 4.1, with active region  $\mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{e'}$  and the same second fibration  $\mathcal{H}_{\text{sec}}$ . Consequently Theorem 4.17 applies to the nucleated complex, and its gain is finite if and only if the enlarged active region carries no embedded mode.*

*Proof.* Set  $\mathcal{H}'_{\text{pri}} := \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{e'}$ ,  $H'_p := \begin{pmatrix} H_p & W_{e'} \\ W_{e'}^* & H_{e'} \end{pmatrix}$ ,  $W' := \begin{pmatrix} W \\ 0 \end{pmatrix}$ ,  $B' := \begin{pmatrix} B \\ 0 \end{pmatrix}$ ,  $C' := \begin{pmatrix} C & 0 \end{pmatrix}$ . Then  $\dim \mathcal{H}'_{\text{pri}} < \infty$ ,  $H'_p$  is self-adjoint,  $W'$  has finite rank, and the composite Hamiltonian (26) formed from these data is bounded and self-adjoint. All

requirements of Definition 4.1 are met, and  $\mathcal{H}_{\text{sec}}$ ,  $H_s$  are unchanged.  $\square$

*Remark 8.5* (Coupling of the first and second survival modes). By Proposition 3.12 a rearranged unit is a defect form; by Hypothesis 2.20 a defect may carry a nucleation datum; by Proposition 8.4 the resulting complex is again subject to the analysis of Section 4. The first survival mode, transformation of a node, can therefore *produce* the second, compatibility with a nucleated environment, and the two are not independent alternatives. What Proposition 8.4 does not assert is that nucleation improves the margin: enlarging the active region adds modes, and by Theorem 4.17 each added mode is a fresh opportunity for an embedded mode orthogonal to  $W'$ . Whether nucleation raises or lowers  $\text{cm}$  is a property of the specific data.

*Remark 8.6* (Chemical reading). [E] Under Hypothesis 2.23, the nucleation datum of Remark 2.35 is the five-membered ring of (9) with its exocyclic phosphate; the alternatives admitted by Hypothesis 2.20 are a coordinated metal ion and a defect of the network of Axiom A6. The assertion that such a datum nucleates a specified catalytic pocket from environmental constituents is of type [E] and is in addition conditional on the Identification Hypothesis  $\Phi$  of [Sch26b]. Nothing in Sections 8.1–8.6 depends on it.

## 8.2 Structured uncertainty

**Definition 8.7** (Admissible block structure). Let  $\mathcal{K} = \bigoplus_{i=1}^r \mathcal{K}_i$  be an orthogonal decomposition of the channel space and let

$$\Delta := \left\{ \Delta = \bigoplus_{i=1}^r \Delta_i : \Delta_i \in \mathcal{B}(\mathcal{K}_i) \right\}. \quad (95)$$

The *structured singular value* of  $M \in \mathcal{B}(\mathcal{K})$  with respect to  $\Delta$  is

$$\mu_\Delta(M) := \left( \min \{ \sigma_{\max}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0 \} \right)^{-1}, \quad (96)$$

with  $\mu_\Delta(M) := 0$  if no such  $\Delta$  exists.

**Proposition 8.8** (Elementary bounds and robust stability). [D]  $\mu_\Delta(M) \leq \sigma_{\max}(M)$ , with equality when  $r = 1$ ; and  $\mu_\Delta(M) \geq \rho(M)$ , the spectral radius. The interconnection of  $G$  with every  $\Delta \in \Delta$  satisfying  $\|\Delta\|_{\mathcal{H}^\infty} < \beta^{-1}$  is stable if and only if  $\sup_{E \in \mathbb{R}} \mu_\Delta(G(E)) \leq \beta$  [Doy82, PD93]. Moreover  $\mu_\Delta(M) \leq \inf \{ \sigma_{\max}(DMD^{-1}) : D \in \Delta' \}$ , where  $\Delta'$  is the commutant of  $\Delta$ .

*Proof sketch with citation.* The bounds follow from (96) by restricting, respectively enlarging, the class of admissible  $\Delta$ : a scalar multiple of the identity lies in  $\mathbf{\Delta}$ , giving  $\mu_{\Delta} \geq \rho$ , while  $\Delta$  is contained in the full block, giving  $\mu_{\Delta} \leq \sigma_{\max}$ . For  $D \in \mathbf{\Delta}'$ ,  $\det(I - M\Delta) = \det(I - DMD^{-1}\Delta)$ , so the scaled matrix has the same  $\mu_{\Delta}$  and the upper bound follows from the first inequality applied to  $DMD^{-1}$ . The robust-stability equivalence is [Doy82, Thm. 2]; see [PD93] for the general treatment.  $\square$

**Remark 8.9** (Where the block structure comes from). The decomposition (95) must be a *direct sum*. Three such decompositions are available in the formalism: the terminal decomposition  $\mathcal{H}_L \oplus \mathcal{H}_B \oplus \mathcal{H}_R$  of Axiom A10, the decomposition of the channel space by lattice direction, and its decomposition by site. The tensor factorization of the local structural tensor is *not* such a decomposition: uncertainty of Kronecker type is not of the form (95) and is not treated by  $\mu_{\Delta}$ . Confusing the two would attribute to  $\mu_{\Delta}$  a scope it does not have.

**Remark 8.10** (Computational status). Exact computation of  $\mu_{\Delta}$  is NP-hard in general, so the scaled upper bound of Proposition 8.8 is the operative quantity in practice. No claim of exact computability is made anywhere in this paper.

### 8.3 Pointwise admissibility is a peak-gain statement

**Definition 8.11** (Peak-to-peak gain). For a stable form realization with impulse response  $g$ , set  $\|G\|_{\text{pk}} := \|g\|_{L^1}$ , the induced norm  $\mathcal{W}_{\infty} \rightarrow L^{\infty}$ .

**Hypothesis 8.12** ( $\mathcal{H}'_{\text{out}}$ , Lipschitz constraint readout). Hypothesis 2.22 holds with constants  $\kappa_j > 0$  such that  $C_j$  evaluated along a trajectory satisfies  $C_j(t) \leq C_j(0) + \kappa_j \|y(t)\|$  for every  $j$ .

**Theorem 8.13** (Invariance of the admissible domain). [D] Assume Hypothesis 8.12, let  $x \in \Omega_{\text{adm}}$  and let  $\theta_j := \tau_j - C_j(x) > 0$  be its constraint margins, which are strictly positive because  $\Omega_{\text{adm}}$  is defined by strict inequalities. If the disturbance satisfies

$$\|w\|_{\infty} < \min_{1 \leq j \leq m} \frac{\theta_j}{\kappa_j \|G_x\|_{\text{pk}}}, \quad (97)$$

then  $C_j(t) < \tau_j$  for every  $j$  and every  $t \geq 0$ ; that is, the trajectory remains in  $\Omega_{\text{adm}}$ .

*Proof.* By Young's convolution inequality,  $\|y(t)\| \leq \|g\|_{L^1} \|w\|_{\infty} = \|G_x\|_{\text{pk}} \|w\|_{\infty}$  for every  $t$ . Hypothesis 8.12 then gives  $C_j(t) \leq C_j(x) + \kappa_j \|G_x\|_{\text{pk}} \|w\|_{\infty}$ , which by (97) is strictly less than  $C_j(x) + \theta_j = \tau_j$ .  $\square$

**Corollary 8.14** (Finite-energy version). [D] Under the same hypotheses, if  $w \in \mathcal{W}_2$  with  $\|w\|_2 < \min_j \theta_j / (\kappa_j \|G_x\|_{\mathcal{H}^2})$  then the trajectory remains in  $\Omega_{\text{adm}}$ .

*Proof.* Proposition 7.10(iii) gives  $\|y\|_{\infty} \leq \|G_x\|_{\mathcal{H}^2} \|w\|_2$ ; the argument of Theorem 8.13 applies unchanged.  $\square$

**Proposition 8.15** (No  $\mathcal{H}^{\infty}$  certificate of admissibility). [D] One has  $\|G\|_{\mathcal{H}^{\infty}} \leq \|G\|_{\text{pk}}$  always, but there is no converse bound: there is a family  $(G_n)$  of stable realizations with  $\|G_n\|_{\mathcal{H}^{\infty}} = 1$  for every  $n$  and  $\|G_n\|_{\text{pk}} \rightarrow \infty$ .

*Proof.* The inequality is immediate from  $\sigma_{\max}(G(E)) = \|\int g(t)e^{iEt} dt\| \leq \|g\|_{L^1}$ . For the family, take the all-pass functions

$$\widehat{G}_n(s) = \left( \frac{1-s}{1+s} \right)^n, \quad (98)$$

each of which satisfies  $|\widehat{G}_n(i\omega)| = 1$  identically, hence  $\|G_n\|_{\mathcal{H}^{\infty}} = 1$ . Their impulse responses are, up to the delta at the origin, Laguerre functions, whose  $L^1$  norms diverge; numerically  $\|G_n\|_{\text{pk}} = 3.000, 3.943, 5.273, 7.148, 9.796, 13.538, 18.828$  for  $n = 1, 2, 4, 8, 16, 32, 64$ , growing like  $2.4\sqrt{n}$  (Remark 8.28).  $\square$

**Remark 8.16** (The two halves of the survival claim). Theorem 5.9 and Corollary 6.16 certify robust stability, and by Theorem 7.7 the quantity that controls them is exactly  $\|\cdot\|_{\mathcal{H}^{\infty}}$ . Theorem 8.13 certifies that the state does not leave  $\Omega_{\text{adm}}$ , and by Proposition 8.15 the quantity that controls it is a peak gain, which  $\|\cdot\|_{\mathcal{H}^{\infty}}$  does not bound. Neither statement implies the other. The survival claim of this paper is the conjunction, and a form may satisfy either half while failing the other: an all-pass form (98) is maximally robust in the sense of Definition 8.1 while violating (97) at arbitrarily small disturbance amplitude.

### 8.4 Connectivity of the transformation graph

**Definition 8.17** (Weighted Laplacian, Fiedler value, effective resistance). Let  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$  be finite and let  $L$  denote the Laplacian of the weighted graph with conductances  $\kappa(e) := w(e)^{-1}$ ,  $w$  the weight function of Axiom A12. The Fiedler value  $\lambda_2(L)$  is its second smallest eigenvalue, and  $R_{\text{eff}}(x, y)$  the effective resistance between  $x$  and  $y$  when each edge  $e$  is a resistor of resistance  $w(e)$ .

**Proposition 8.18** (Connectivity trichotomy and the resistance bound). [D] For a finite  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$  the following are equivalent:  $\lambda_2(L) > 0$ ;  $d_C^*(x, y) < \infty$  for all  $x, y$ ; and  $d_{\mathcal{H}^\infty}(x, y) < \infty$  for all  $x, y$ . Moreover

$$R_{\text{eff}}(x, y) \leq d_C^*(x, y) \quad \text{for all } x, y \in \Omega_{\text{adm}}. \quad (99)$$

*Proof.* The first equivalence is the standard characterization of  $\lambda_2 > 0$  as connectedness of a finite weighted graph [Fie73]; the second is the last assertion of Proposition 5.6 together with the corresponding statement for  $d_C^*$ , both being infinite exactly across components. For (99), let  $\gamma$  be any  $\mathcal{E}_{\text{adm}}$ -path from  $x$  to  $y$ . Deleting every edge not on  $\gamma$  cannot decrease the effective resistance, by Rayleigh’s monotonicity law, and the resulting series network has resistance  $\sum_{e \in \gamma} w(e) = L_w(\gamma)$ . Taking the infimum over  $\gamma$  gives (99); see [KR93].  $\square$

**Remark 8.19** (What  $\lambda_2$  is and is not).  $\lambda_2(L)$  is a scalar invariant of the transformation graph, not a distance, and it cannot be substituted for one: it reports how robustly connected the graph of admissible transformations is, and hence how much redundancy of survival routes exists, but assigns no value to a pair of forms. The metric relative of  $\lambda_2$  is  $R_{\text{eff}}$ , which by (99) is dominated by  $d_C^*$  and which, unlike a shortest-path distance, decreases when alternative routes are added. Neither quantity refers to the realizations  $G_x$ , so neither bounds nor is bounded by  $d_{\mathcal{H}^\infty}$ .

## 8.5 Optimal transport

**Definition 8.20** (Transport distance over the admissible domain). For probability measures  $\pi_1, \pi_2$  on  $\Omega_{\text{adm}}$  with finite first moment, let

$$W_1(\pi_1, \pi_2) := \inf_{\Pi} \int_{\Omega_{\text{adm}} \times \Omega_{\text{adm}}} d_{\mathcal{H}^\infty}(x, y) d\Pi(x, y), \quad (100)$$

the infimum over couplings  $\Pi$  with marginals  $\pi_1, \pi_2$ .

**Proposition 8.21** ( $W_1$  inherits the pseudometric structure). [D]  $W_1$  is an extended pseudometric on the set of such measures, and  $W_1(\delta_x, \delta_y) = d_{\mathcal{H}^\infty}(x, y)$ . It is a metric exactly on the quotient by the relation of Proposition 5.7.

*Proof.* The ground cost  $d_{\mathcal{H}^\infty}$  is an extended pseudometric by Proposition 5.6; the standard theory of optimal transport with a pseudometric cost gives the assertions, the triangle inequality by the gluing lemma [Vil09, Ch. 6]. The value on Dirac measures is immediate since the only coupling is  $\delta_x \otimes \delta_y$ .  $\square$

**Remark 8.22** (Transport adds nothing for a single form). By Proposition 8.21,  $W_1$  agrees with  $d_{\mathcal{H}^\infty}$  on Dirac measures, and the same holds for every  $W_p$ ,  $p \geq 1$ , and for the  $\infty$ -transport distance. A transport distance is therefore informative only for *ensembles* of forms — a population of network sites in differing structural states — and carries no information beyond  $d_{\mathcal{H}^\infty}$  about an individual form. Where the corpus applies a transport distance to a single deterministic configuration, the content is that of the underlying ground metric.

**Remark 8.23** (The spectral distance is already a transport distance). The Connes distance associated with the spectral triple of Axiom A9 is, by the Kantorovich–Rubinstein duality, the noncommutative  $W_1$  distance for the Lipschitz seminorm  $\|[D_{\mathcal{C}}, \cdot]\|$  [Rie99, DM10]. Optimal transport is therefore not a further candidate metric to be added to the formalism: it is the dual description of a structure Axiom A9 already supplies. A  $W_2$ -type distance, by contrast, requires the additional datum of a quantum Markov semigroup with respect to which the gradient flow is taken [CM17]; where the monographs invoke a  $W_2$  action functional, that semigroup must be specified, and is not supplied by Axioms A1–A12.

## 8.6 Summary

**Remark 8.24** (Reading of Table 1). Table 1 classifies the quantities discussed in this paper by the object they measure, the axioms they satisfy, the question they answer, and their logical status here. The classification is intended to prevent substitution: the quantities in the table are not interchangeable, and the failure modes documented above (Remark 8.16, Proposition 8.15, Remark 8.19, Remark 8.22) are each a case of one being used for another’s purpose.

## 8.7 Targets

**Theorem 8.25** (Target: the dual-cipher commutant as a scaling class). [T] Target: identify the scaling class  $\Delta'$  of Proposition 8.8 with the dual-cipher commutant  $\mathfrak{M}^{\text{dual}} = (\mathfrak{M}^{\text{pri}})'$  of Axiom A8, for a block structure  $\Delta$  whose uncertainty is constrained to  $\mathfrak{M}^{\text{pri}}$ . If established, the  $D$ -scaled upper bound would be computed over an algebra that the formalism supplies rather than over an ad hoc class. The obstruction is that  $\mathfrak{M}^{\text{pri}}$  and  $\mathfrak{M}^{\text{dual}}$  act on  $\mathcal{H}_\omega$ , while  $\Delta$  acts on the channel space  $\mathcal{K}$  of Axiom A10; the identification requires the intertwiner of [Sch26c, Rem. 33], which is not available here.

| Quantity                         | Measures              | Axioms satisfied            | Answers                                  | Status |
|----------------------------------|-----------------------|-----------------------------|------------------------------------------|--------|
| $\delta_u$                       | pairs of realizations | pseudometric                | how far apart two forms transfer         | [D]    |
| $d_{\mathcal{H}^\infty}$         | pairs of forms        | extended pseudometric       | survival radius under transformation     | [D]    |
| $d_{\mathcal{H}^\infty}^\vee$    | pairs of forms        | extended pseudo-ultrametric | worst single transformation              | [D]    |
| $d_{\text{att}}$                 | pairs of ports        | quasi-pseudometric          | directional attenuation                  | [D]    |
| $d_C^*$                          | pairs of forms        | extended pseudometric       | admissible path length                   | A12    |
| $\ \cdot\ _{\mathcal{H}^\infty}$ | one form              | norm                        | robust stability margin                  | [D]    |
| $r_{\text{emb}}$                 | one form              | margin                      | self-adjoint perturbation tolerance      | [D]    |
| $\ \cdot\ _{\text{pk}}$          | one form              | norm                        | invariance of $\Omega_{\text{adm}}$      | [D]    |
| $\ \cdot\ _{\mathcal{H}^2}$      | one form              | norm                        | bounded-energy excursion; noise response | [D]    |
| $\mu_\Delta$                     | one form              | not a metric                | structured robustness                    | [D]    |
| $\lambda_2(L)$                   | the graph             | not a metric                | redundancy of routes                     | [D]    |
| $R_{\text{eff}}$                 | pairs of forms        | metric                      | route redundancy, pairwise               | [D]    |
| $W_1$                            | ensembles of forms    | extended pseudometric       | displacement of a population             | [D]    |
| Connes $d_D$                     | pairs of states       | extended pseudometric       | spectral geometry                        | A9     |

**Table 1:** Taxonomy of the quantities of this paper. “Axioms satisfied” refers to the metric axioms established here, not to Axioms A1–A12; the last column gives the logical status of the corresponding result in the sense of Definition 2.7.

**Theorem 8.26** (Target: a joint certificate). [T] *Target: give a single computable condition on a composite realization implying both the robust-stability bound of Theorem 5.9 and the invariance bound of Theorem 8.13, and propagate it along a transformation route as in Theorem 6.13. By Proposition 8.15 no such condition can be a bound on  $\|\cdot\|_{\mathcal{H}^\infty}$  alone.*

**Theorem 8.27** (Target: nucleation as a well-posed dynamics). [T] *Target: replace the partial map  $v$  of Hypothesis 2.20 by a dynamics on  $\Omega_{\text{adm}} \times \Omega_{\text{env}}$  for which nucleation and persistence are derived rather than posited, and determine conditions under which nucleation increases the compatibility margin  $\text{cm}$  of Definition 8.1 rather than decreasing it, which Remark 8.5 shows is not automatic.*

**Remark 8.28** (Independent numerical verification). The values quoted in Proposition 8.15 were obtained by computing the impulse response of (98) from a state-space realization and integrating its modulus; the all-pass property  $|\widehat{G}_n(i\omega)| = 1$  was confirmed to within  $2.2 \times 10^{-16}$  at sample frequencies, and the ratio  $\|G_n\|_{\text{pk}}/\sqrt{n}$  was found to decrease slowly from 3.00 to 2.35 over  $n = 1, \dots, 64$ , consistent with growth of order  $\sqrt{n}$ .

## 9 Non-Implications and the Deferred Program

The formalism’s governing restraint is that restoring a foundational object never licenses restoring an unproved implication of that object [Sch26b, §3.1]. This section states, in the style of the

non-implications [Sch26b, Eqs. (2)–(3)], what the constructions of this paper do *not* establish. Each item is refuted by a witness already exhibited above, so the section adds no new argument; its purpose is to make the list checkable in one place.

### 9.1 Non-implications

**Proposition 9.1** (Non-implications). [D] *None of the following implications holds in general; each is refuted by the witness named in its proof.*

- (i)  $d_{\mathcal{H}^\infty}(x, y) = +\infty \Rightarrow$  autonomous physical scission of the network.
- (ii)  $\Gamma \neq 0$ , or decay of the active-region occupation,  $\Rightarrow$  loss or annihilation of the decaying component.
- (iii)  $\|G_x\|_{\mathcal{H}^\infty} < \infty \Rightarrow$  the trajectory remains in  $\Omega_{\text{adm}}$ .
- (iv) two forms have isomorphic molecular graphs  $\Rightarrow$  they are transfer isomorphic; and conversely.
- (v) every frozen form is robustly stable  $\Rightarrow$  the network survives transformation among them.
- (vi)  $\delta_u(G_x, G_y)$  is small  $\Rightarrow d_{\mathcal{H}^\infty}(x, y)$  is small.
- (vii) the admissible sector is exactly preserved  $\Rightarrow$  the form is robustly stable.
- (viii)  $\|G\|_{\mathcal{H}^2}$  is small  $\Rightarrow \|G\|_{\mathcal{H}^\infty}$  is small, or  $\Rightarrow$  any robustness margin whatever.
- (ix) the coupling to the second fibration is nonzero  $\Rightarrow$  the gain is finite.
- (x) the scatterer is lossless  $\Rightarrow$  the attenuation quantity is symmetric.

- (xi) *a symmetry of the carbon framework  $\Rightarrow$  the corresponding realizations are unitarily equivalent.*
- (xii)  *$\mu_\Delta$  becomes singular  $\Rightarrow$  topological scission.*
- (xiii)  *$\lambda_2(L) > 0 \Rightarrow$  any bound on  $d_{\mathcal{H}^\infty}$ .*
- (xiv)  *$d_{\mathcal{H}^\infty}$  is defined  $\Rightarrow$  any relation holds between  $d_{\mathcal{H}^\infty}$  and the spectral distance of Axiom A9.*

*Proof.* (i) is [Sch26b, Eq. (3)], preserved here;  $d_{\mathcal{H}^\infty} = +\infty$  means disconnection of  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$  by Proposition 5.6, which Axiom A12 does not interpret physically. (ii) Proposition 4.20: the composite evolution is unitary and  $\|P_{\text{pri}}\psi_t\|^2 + \|P_{\text{sec}}\psi_t\|^2$  is conserved. (iii) The all-pass family (98) has  $\|G_n\|_{\mathcal{H}^\infty} = 1$  and  $\|G_n\|_{\text{pk}} \rightarrow \infty$  (Proposition 8.15), so by Theorem 8.13 the admissible bound degrades without bound while the gain is constant. (iv) Both directions are refuted in Proposition 3.8. (v) Proposition 6.6. (vi) The witness of Proposition A.11 has  $\delta_u(G_1, G_4) = 1$  and  $d_{\mathcal{H}^\infty}(x_1, x_4) = 2$ ; the ratio is unbounded in general, since inserting further intermediate forms along the route increases  $d_{\mathcal{H}^\infty}$  while leaving  $\delta_u$  unchanged. (vii) Proposition 4.22: exact preservation forces  $\|G\|_{\mathcal{H}^\infty} = \infty$ , so the implication fails in the strong sense that its hypothesis excludes its conclusion. (viii) Proposition 7.4 and Corollary 7.8. (ix) Example A.15: the coupling is nonzero and  $\Gamma \neq 0$ , yet an antisymmetric mode is orthogonal to the range of the coupling and the gain is infinite. (x) Example 5.19: a lossless three-port with  $\sup_E |t_{21}| = \frac{12}{13}$  and  $\sup_E |t_{12}| = 3^{-1/2}$ . (xi) Remark 2.32, resting on [Sch26c, Rem. 31] and on Proposition 3.8(i). (xii) Proposition 8.8 yields loss of robust stability and nothing more; the inference to scission is the error recorded in Table 5. (xiii) Remark 8.19:  $\lambda_2$  is computed from  $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}}, w)$  and does not refer to the realizations  $G_x$  at all. (xiv) Remark 5.24, resting on [Sch26c, Prop. 27 and Rem. 29].  $\square$

*Remark 9.2* (The two that matter most). Items (iii) and (v) are the ones on which a reader is most likely to over-read this paper. The construction of Sections 5–6 certifies robust stability, under transformation and under switching; it does not certify that the network’s constraint functionals stay below their thresholds, which is Theorem 8.13 and requires a different quantity. A form may therefore be maximally robust in the sense of Theorem 5.9 and leave  $\Omega_{\text{adm}}$  at arbitrarily small disturbance amplitude. Item (vii) is the one most likely to surprise: it inverts the expected relation between confinement and survival, and its consequences for the corpus are recorded in Remark B.1.

| Deferred target of [Sch26c]                            | Relation to this paper                                                                                   |
|--------------------------------------------------------|----------------------------------------------------------------------------------------------------------|
| $N_y = N_x + 2$ directional closure law                | not touched                                                                                              |
| $GF(4)$ field multiplication on $\Sigma_{\mathcal{E}}$ | not touched; see Target 8.25 for a distinct use of A8                                                    |
| directional transfer and optical anisotropy            | companion only: $\delta_u$ quantifies the difference between two realizations without proving anisotropy |
| spectral–path metric reconstruction                    | not approached: $d_{\mathcal{H}^\infty}$ is not a candidate (Rem. 5.23)                                  |
| real spectral structure and spectral dimension         | not touched                                                                                              |
| higher-algebraic braiding                              | not touched                                                                                              |
| emergent assembly chirality                            | not touched                                                                                              |
| dynamical scission                                     | not discharged; explicitly not licensed (Prop. 9.1(i), (ii))                                             |

**Table 2:** The deferred program of [Sch26c] against the present paper. No entry is “discharged”.

## 9.2 Status of the deferred program of the twelve-axiom paper

*Remark 9.3* (Reading of Table 2). [Sch26c] states eight deferred targets, recalled in [Sch26b, §3.6]. Table 2 records this paper’s relation to each. No target is discharged. Two are touched: the directional transfer target acquires a quantitative companion that does not prove it, and the spectral–path target is shown *not* to be approached by the present construction, which is a negative result about scope rather than progress.

## 9.3 Targets introduced here

*Remark 9.4* (Inventory). This paper introduces twelve targets of its own, all stated in the form of [Sch26c, Def. 3] with their hypotheses named: Targets 4.24, 4.25 and 4.26 in Section 4; Targets 6.24, 6.25 and 6.26 in Section 6; Targets 7.19 and 7.20 in Section 7; Targets 8.25, 8.26 and 8.27 in Section 8; and Target 10.9 in Section 10. Their number is not offered as evidence of anything. Three of them — 6.26, 6.24 and 8.26 — are load-bearing in the sense that the paper’s own results are incomplete without them, and are singled out below.

*Remark 9.5* (The three load-bearing targets). Target 6.26 is the gap recorded as item 7 of Table 6: Section 6 assumes Hypothesis 6.2, which the witness of Appendix A does not

satisfy, so the switching results have no exhibited instance. Target 6.24 is what would make  $d_{\mathcal{H}^\infty}$  the controlling quantity for survival across transformations as well as within them; without it, Section 5 governs the forms and Section 6 governs the passage between them, with no established relation between the two. Target 8.26 is the single certificate for both halves of the survival claim, which Proposition 8.15 shows cannot be a bound on  $\|\cdot\|_{\mathcal{H}^\infty}$  alone. Until these three are discharged, the survival claim of this paper is a conjunction of separately established statements and not a single theorem.

*Remark 9.6 (Status of Target 8.26).* Target 8.26 asks for a single certificate controlling both halves of the survival claim. Since Proposition 8.15 gives  $\|G\|_{\mathcal{H}^\infty} \leq \|G\|_{\text{pk}}$  with no converse, a bound on the peak gain controls both halves for a single form, and the orbit construction of Definition 5.1 applied to the impulse response rather than the transfer function yields a quantity dominating  $\delta_u$  and hence a route-propagating version. That development is not carried out here: it is new mathematics, and it is deferred to a companion note to be reviewed on its own. The survival claim of this paper remains the conjunction of two separately certified statements, as Remark 9.5 records.

## 10 Transfer to Other Fibrated Networks

The proofs of Sections 4–8 use the molecular content of Axioms A1–A3 nowhere. This section makes that observation precise, states what must be exhibited for another network to inherit the results, and states equally precisely what is *not* thereby established.

### 10.1 The induced network

**Proposition 10.1** (Every composite realization induces a fibrated transfer network). [D] *Let  $x \in \Omega_{\text{adm}}$  carry a composite realization  $\mathfrak{G}_x$  (Definition 4.1) under Hypotheses 2.12, 2.13, 2.14, 3.4, 2.15 and 4.9. Then the data*

$$\begin{aligned} \Lambda &:= \mathbb{Z}^3, \quad \Sigma_{\text{sp}} := \{\mathbf{p}, \mathbf{d}\}, \\ \mathcal{S} &:= \Omega_{\text{adm}}, \quad \mathcal{T} := \mathcal{E}_{\text{adm}}, \quad \mathcal{H}_s := \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}, \\ G_s &:= \text{the transfer function (28)}, \quad \mathcal{D} := (\mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}, W) \end{aligned} \quad (101)$$

*constitute an admissible fibrated transfer network of rank 3 in the sense of Definition 2.10, with the constraint functionals of Axiom A12 supplying admissibility.*

*Proof.* (N1) and (N2) are Hypotheses 4.9 and 2.16; (N3) is Axiom A12; (N4) is Hypothesis 2.14 together with Definition 3.2, the analyticity and continuity requirements holding by

Lemma 4.4 and Proposition 4.13; (N5) is the decomposition (26) with its coupling. Admissibility is Definition A.1 in the general case, that is the restriction of  $\mathcal{T}$  to pairs satisfying  $C_j < \tau_j$ .  $\square$

**Theorem 10.2** (Portability). [D] *Let  $\mathfrak{R}$  be any admissible fibrated transfer network of rank  $d$  (Definition 2.10) whose data satisfy the hypotheses named in the statement of a given result below. Then that result holds for  $\mathfrak{R}$ , with the same proof,  $\mathbb{Z}^3$  replaced by  $\mathbb{Z}^d$  throughout. This applies to: Lemma 4.4; Proposition 4.5; Corollary 4.7; Proposition 4.13; Theorem 4.17; Proposition 4.20; Proposition 4.22; Lemma 3.5; Propositions 3.7 and 3.8; Proposition 3.12 and Lemma 3.14; Lemma 3.16; all of Section 5; all of Section 6; all of Section 7; and Sections 8.1–8.5.*

*Proof.* Inspection of the proofs. Each cited proof uses only: the direct-sum decomposition (N5) and self-adjointness of the two diagonal blocks; the translation covariance and finite range of (N1)–(N2); the analyticity and continuity of the realizations (N4); the admissibility structure of (N3); and the named hypotheses of the hypothesis register (Section 2.6). No proof invokes Axioms A1–A3, the comparison class  $\mathcal{B}_4$ , the locant structure, or any statement about the reference molecule. The replacement of  $\mathbb{Z}^3$  by  $\mathbb{Z}^d$  affects only the dimension of the torus in Definition 2.2 and the Hausdorff measure in (35), which becomes  $(d - 1)$ -dimensional.  $\square$

*Remark 10.3 (What is not portable).* Three classes of statement do not transfer. First, everything of type [E]: Remarks 4.19, 8.6, 2.34, 2.35, B.4 and B.6 concern the reference molecule and have no analogue elsewhere. Second, the combinatorial statements of Section 2.7, including Proposition 2.31, which are about the chrysene carbon framework. Third, the closed forms of Appendix A, which are properties of the particular half-line fibration chosen there, not of fibrated transfer networks in general.

### 10.2 What an instance must exhibit

**Definition 10.4** (Instance schema). To assert that a given system is an admissible fibrated transfer network, and thereby to invoke Theorem 10.2, one must exhibit: a site lattice and its translation action (N1); a finite species alphabet and a labelling (N2); a set of forms with an admissibility structure, that is constraint functionals and thresholds, and a transition relation restricted accordingly (N3); a common channel pair  $(\mathcal{K}, \mathcal{Y})$  and, for each form, a transfer realization with the required analyticity (N4); and a dissipation datum, that is a distinguished sector into which the active region hands off, together with its coupling (N5). One must then verify, for

each result invoked, the hypotheses named in its statement.

**Remark 10.5** (The verification burden is not nominal). Definition 10.4 is not satisfied by the observation that a system is a network with dynamics. Three of its clauses are substantive and are the ones most likely to fail. Clause (N5) requires an identified sector playing the role of the second fibration, with an absolutely continuous band structure in the sense of Hypothesis 4.12; without it, Proposition 4.5 gives  $\Gamma \equiv 0$  and every result of Section 4 is vacuous. Clause (N3) requires the constraint functionals to be stated explicitly, since by Hypothesis 8.12 and Theorem 8.13 the admissibility half of the survival claim is otherwise unavailable. Clause (N4) requires a common channel pair across forms, failing which  $\delta_u$  of Definition 5.1 is undefined and there is no transformation geometry at all.

**Remark 10.6** (Candidate domains). [E] The applied clusters of [Sch26b, Tab. 1] suggest four domains in which Definition 10.4 might be discharged. In a clinical or physiological network, the forms would be therapeutic or physiological configurations, the constraint functionals the clinical limits, and the dissipation datum a reservoir into which perturbation is absorbed. In a cryptographic or containment architecture, the forms would be system configurations and the constraint functionals the containment predicates, the admissible region playing the role Definition 2 of [Sch26b] assigns to  $F_{\text{adm}}$ . In a financial network, the forms would be market configurations, the constraint functionals liquidity or solvency bounds, and the second fibration the external funding channel. In each case the assertion that the system *is* such a network is of type [E], to be discharged by exhibiting the data of Definition 10.4 and not by analogy.

### 10.3 What this section does not establish

**Remark 10.7** (No retroactive certification). Theorem 10.2 makes results available to a network that satisfies Definition 10.4. It certifies no existing paper. In particular, the papers of Volume 1, Number 1 that apply  $\mathcal{H}^\infty$  methods to clinical, cryptographic and financial settings [Sch26b, §6] were reviewed under the Level 1 protocol, were not evaluated against  $F_{\text{adm}} = M \cap C \cap H$ , and have not been checked here against Definition 10.4 or against the corrections recorded in Table 5. Two of those corrections — the type mismatch of Remark B.2 and the confinement-robustness incompatibility of Remark B.1 — bear directly on such applications. A healthcare-directed instantiation falls within the scope condition of [Sch26b, §4.5] and would itself require orthogonal-constraint review.

**Remark 10.8** (Substrate independence, restated). The claim that the formalism's structure is substrate-independent, advanced in [Sch26f], is here restated in the only form the results support: the structure that transfers is that of Axioms A4–A12 — fibration, translation covariance, sequence structure, admissibility, Hamiltonian and response — and not that of Axioms A1–A3, which fix the molecular reduct. Consequently the reference molecule plays, for the mathematics of this paper, the role of an exogenous witness: it supplies an instance (Remark A.7) and, through [Sch26c, Thm. 61], a non-vacuity guarantee for the axioms the instance satisfies. It does not supply any step in any proof of Sections 4–8. Whether the chrysene instance is distinguished among instances — whether A1–A3 contribute anything beyond fixing parameter values in A4–A12 — is not settled here and is recorded as Target 10.9.

**Theorem 10.9** (Target: is the reference instance distinguished?). [T] *Target: exhibit a statement provable for the fibrated transfer network induced by  $\mathfrak{R} \in \text{Ext}_{12}(\mathbb{C}^{\text{chr}})$  via Proposition 10.1 that is false for some other admissible fibrated transfer network of rank 3 satisfying the same named hypotheses. A positive answer would show that Axioms A1–A3 constrain the transfer geometry and not merely the interpretation; a negative answer would show that the results of this paper are, in their entirety, results about A4–A12, and that the reference molecule's role in them is exhausted by Remark 10.8.*

## 11 Conclusion

The question this paper set out to answer was whether the worst-case operator norm, used pervasively and without derivation throughout the applied corpus of the Chrysene Formalism, can be made into a mathematical structure on the formalism's admissible domain. The answer is affirmative but conditional, and the conditions are the substance of the result.

The norm is not finite by default. It is finite exactly when the active region of a form carries no mode that is energetically embedded in the continuum of the second fibration yet invisible to every channel into it (Theorem 4.17). The formalism's own explicit model fails this criterion (Remark 4.6), so a separate witness had to be constructed (Appendix A). The dissipation that makes finiteness possible is not an added postulate: it is the coupling-weighted density of states of the second fibration (Proposition 4.13), and the norm it removes from the admissible sector is transferred there, not destroyed (Proposition 4.20).

Granted finiteness, the geometry is well behaved. The unitary-orbit distance is a pseudometric with no logarithm required, its path closure over admissible transformations is an extended pseudometric, and its zero set is transfer isomorphism — a

relation logically independent of isomorphism of the molecular graphs (Proposition 3.8), which is what permits a structural rearrangement to leave the network's transfer behaviour intact. A ball in this metric is a set of forms guaranteed to remain robustly stable against a disturbance of given size (Theorem 5.9).

Three findings run against the reading the corpus has so far taken. First, exact confinement of the admissible sector is incompatible with finite gain (Proposition 4.22): survival is purchased by opening a handoff channel, not by closing one. Second, stability of every form separately does not give survival under transformation among them (Proposition 6.6); what does is a common storage functional, whose structural sufficient condition is uniform absorption along the route (Corollary 6.16) — the same channel again. Third, the worst-case norm does not control invariance of the admissible domain, which is a peak-gain statement (Theorem 8.13, Proposition 8.15); the survival claim is therefore a conjunction of two separately certified halves, and a single certificate for both is not supplied here (Target 8.26).

What the paper does not do is stated as plainly as what it does. No deferred target of [Sch26c] is discharged. Fourteen non-implications are recorded (Proposition 9.1), twelve new targets are opened, of which three are load-bearing (Remark 9.5), and the concordance appendix records five corrections to the prior corpus, including a type mismatch present in the monographs from which several applied papers derive (Remark B.2). One error in the proof of this paper's own central theorem was found during drafting and is recorded with its repair (Remark A.10). The results transfer to any network exhibiting the data of Definition 10.4 (Theorem 10.2), and they certify no existing application (Remark 10.7).

## A An Explicit Witness Realization

By Remark 4.6 the explicit twelve-axiom model  $\mathfrak{R}_{\text{ex}}$  of [Sch26c, App. B] has infinite gain, so it does not witness any construction of Sections 4–5. This appendix supplies a witness. It agrees with  $\mathfrak{R}_{\text{ex}}$  on Axioms A1–A9 and A11, replaces the A10 and A12 data, and exhibits closed-form values for every quantity defined in this paper, together with explicit witnesses for each of the paper's negative results.

### A.1 The admissibility datum

**Definition A.1** (The witness A12 datum). Let  $\omega$  be the state of Axiom A7 carried by  $\mathfrak{R}_{\text{ex}}$  and let  $\varphi$  be any state of  $\mathfrak{A}_{\mathbb{C}}$  with  $\varphi(P_0) \neq \omega(P_0)$ , where  $P_0$  is the localized projection of [Sch26c, Thm. 8]. Put

$$X := \{x_1, \dots, x_5\}, \quad s_1 = 0, s_2 = \frac{1}{5}, s_3 = \frac{2}{5}, s_4 = \frac{3}{5}, s_5 = 1, \quad (102)$$

and define

$$J_{\text{str}}(x_i) := (1 - s_i) \omega + s_i \varphi \in S(\mathfrak{A}_{\mathbb{C}}), \quad (103)$$

one constraint functional  $C_1(x_i) := s_i$  with threshold  $\tau_1 := 1$ , the symmetric transition relation generated by

$$\mathcal{E}_0 \supseteq \{\{x_1, x_3\}, \{x_1, x_2\}, \{x_2, x_4\}, \{x_4, x_5\}\}, \quad (104)$$

and the symmetric positive weights

$$w(x_1 x_3) = 1, \quad w(x_1 x_2) = \frac{1}{10}, \quad w(x_2 x_4) = 1, \quad w(x_4 x_5) = 1. \quad (105)$$

**Proposition A.2** (The datum satisfies Axiom A12). [D] *Given the constructed datum of Definition A.1 — the five-point set  $X$ , the mixing parameters  $s_i$ , the auxiliary state  $\varphi$ , the edge set and the weights, none of which is fixed by A1–A12 —  $J_{\text{str}}$  is injective,  $\Omega_{\text{adm}} = \{x_1, x_2, x_3, x_4\}$ ,  $x_5 \in \partial_C \Omega_{\text{adm}}$ , and  $\mathcal{E}_{\text{adm}} = \{\{x_1, x_3\}, \{x_1, x_2\}, \{x_2, x_4\}\}$ . The induced path distance takes the values*

$$d_C^*(x_1, x_3) = 1, \quad d_C^*(x_1, x_2) = \frac{1}{10}, \quad d_C^*(x_1, x_4) = \frac{11}{10}. \quad (106)$$

*Proof.* Each  $J_{\text{str}}(x_i)$  is a convex combination of states, hence a state. If  $i \neq j$  then  $s_i \neq s_j$  and

$$(J_{\text{str}}(x_i) - J_{\text{str}}(x_j))(P_0) = (s_i - s_j)(\varphi(P_0) - \omega(P_0)) \neq 0, \quad (107)$$

so  $J_{\text{str}}$  is injective. Since  $C_1(x_i) = s_i$  and  $\tau_1 = 1$ , the strict inequality  $C_1 < \tau_1$  holds exactly for  $i \leq 4$ , giving  $\Omega_{\text{adm}}$  as stated and  $C_1(x_5) = \tau_1$ , that is  $x_5 \in \partial_C \Omega_{\text{adm}}$ ; in particular  $\partial_C \Omega_{\text{adm}}$  is disjoint from  $\Omega_{\text{adm}}$ , as required.  $\mathcal{E}_{\text{adm}}$  is obtained by discarding from (104) the pairs meeting  $x_5$ . The values (106) are the shortest  $w$ -weighted path lengths in the resulting three-edge graph.  $\square$

### A.2 The second fibration and its self-energy

**Definition A.3** (Witness second fibration). Let  $\mathcal{H}_{\text{sec}} := \ell^2(\mathbb{N})$  with the nearest-neighbour operator  $H_s u(k) = u(k-1) + u(k+1)$ ,  $u(-1) := 0$ ; equivalently,  $H_s$  is the bi-infinite translation-covariant chain of Hypothesis 4.9 with  $m = 1$ ,  $F = \{\pm 1\}$ ,  $t_{\pm 1} = 1$ , modified by deletion of the bond joining sites  $-1$  and  $0$ , a modification of rank two.

**Proposition A.4** (Closed-form self-energy). [D] *Let  $g_s(z) := \frac{1}{2}(z - \sqrt{z-2}\sqrt{z+2})$  with principal branches, so that  $g_s(z) = z^{-1} + O(z^{-3})$  as  $|z| \rightarrow \infty$ . Then  $g_s(z) = \langle \delta_0, (z - H_s)^{-1} \delta_0 \rangle$  for  $z \in \mathbb{C}_+$ , and for a coupling  $W = v|\cdot\rangle\langle\delta_0|$  of strength  $v \in \mathbb{R}$  one has  $\Sigma(z) = v^2 g_s(z)$*

with

$$\Lambda(E) = \frac{v^2 E}{2}, \quad \Gamma(E) = \frac{v^2 \sqrt{4 - E^2}}{2} \quad (|E| < 2), \quad (108)$$

and  $\Gamma(E) = 0$ ,  $\Lambda(E) = v^2 g_s(E) \in \mathbb{R}$  for  $|E| > 2$ . The maps  $\Lambda, \Gamma$  are continuous on all of  $\mathbb{R}$ , and  $g_s(\pm 2) = \pm 1$ .

*Proof.* The band function of the bi-infinite chain is  $E(\theta) = 2 \cos \theta$  on  $\mathbb{T}^1$ , with  $\nabla E = -2 \sin \theta \neq 0$  for  $\theta \notin \{0, \pi\}$ , so Hypothesis 4.12 holds on every compact  $I \subset (-2, 2)$  and Proposition 4.13 applies to the unmodified chain; Definition A.3 modifies it by an operator of rank two, so Lemma 3.14 applies. The stated  $g_s$  is the classical half-line resolvent diagonal element: it is the root of  $g_s^2 - z g_s + 1 = 0$  selected by decay at infinity, which is the defining recursion of the half-line chain. Substituting  $z = E + i0$  with  $|E| < 2$  gives  $\sqrt{z - 2} \sqrt{z + 2} = i \sqrt{4 - E^2}$ , hence  $g_s(E + i0) = (E - i \sqrt{4 - E^2})/2$  and (108); for  $|E| > 2$  the square root is real and  $g_s$  is real. Continuity at  $E = \pm 2$  follows from  $\sqrt{4 - E^2} \rightarrow 0$  and  $g_s(\pm 2) = \pm 1$ .  $\square$

**Remark A.5** (Why a half-line and not a bulk chain). The bulk one-dimensional chain has resolvent diagonal  $(z^2 - 4)^{-1/2}$ , which diverges at the band edges; coupling a single level to it produces a real zero of  $M$  outside the band for every  $v \neq 0$ , hence infinite gain. The half-line chain has  $g_s(\pm 2) = \pm 1$  finite, which is the feature that a three-dimensional lattice possesses and that Hypothesis 4.12 is designed to capture. The witness therefore uses the half-line as the smallest closed-form model exhibiting the correct band-edge behaviour, not as a claim that the second fibration is one-dimensional.

### A.3 The forms

**Definition A.6** (Witness forms). Take  $\mathcal{K} = \mathcal{Y} = \mathbb{C}$ , so Hypothesis 3.4 holds with  $k = q = 1$ ; take  $\mathcal{H}_{\text{pri}} = \mathbb{C}$ ,  $H_p = \varepsilon$ ,  $J = 0$ ,  $C = 1$ , and assign

$$\begin{aligned} x_1 : (\varepsilon, v^2, B) &= (0, 1, 1), & x_2 : (0, 1, \rho), \\ x_3 : (0, 1, e^{i\psi}), & & x_4 : (0, \tfrac{1}{2}, 1), \end{aligned} \quad (109)$$

with  $\rho := \frac{1}{2}$  and  $\psi := \pi/3$ . The resulting transfer functions are  $G_j(z) = B_j(z - \varepsilon_j - v_j^2 g_s(z))^{-1}$ .

**Remark A.7** (These are Axiom A10 data). [D] Given the constructed motif datum of Definitions A.3 and A.6, which instantiate Axiom A10 under choices the axiom does not determine, put  $\mathcal{H}_L = \mathcal{Y}$ ,  $\mathcal{H}_R = \mathcal{K}$ ,  $\mathcal{H}_B = \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}$  and let  $H_B$  be the composite Hamiltonian (26), which is bounded

and self-adjoint with  $\sigma(H_B) \subseteq [-2 - |v| - |\varepsilon|, 2 + |v| + |\varepsilon|]$ . With  $W_L = C^*$  and  $W_R = B$ , both extended by zero on  $\mathcal{H}_{\text{sec}}$ , the Schur complement identity gives  $W_L^\dagger(z - H_B)^{-1} W_R = CM(z)^{-1} B$ , so the transfer operator of Axiom A10 for this motif is exactly  $G_j$  and [Sch26c, Prop. 26] applies verbatim.

**Proposition A.8** (Closed-form gain). [D] Let  $\varepsilon = 0$ ,  $B = 1$  and  $0 < v^2 < 2$ . Then

$$\|G\|_{\mathcal{H}^\infty} = \begin{cases} \frac{1}{v^2}, & 0 < v^2 \leq 1, \\ \frac{1}{2 - v^2}, & 1 \leq v^2 < 2, \end{cases} \quad (110)$$

the supremum being attained at  $E = 0$  in the first case and at  $E = \pm 2$  in the second. For  $v^2 = 1$  one has  $|M(E)| = 1$  for every  $|E| \leq 2$ , so the supremum is attained on the whole band and  $\|G\|_{\mathcal{H}^\infty} = 1$ .

*Proof.* By Proposition A.4, for  $|E| < 2$ ,

$$M(E) = E \left( 1 - \frac{v^2}{2} \right) + \frac{i v^2 \sqrt{4 - E^2}}{2}, \quad (111)$$

whence

$$|M(E)|^2 = E^2 \left[ \left( 1 - \frac{v^2}{2} \right)^2 - \frac{v^4}{4} \right] + v^4 = E^2(1 - v^2) + v^4. \quad (112)$$

For  $v^2 < 1$  the coefficient of  $E^2$  is positive and the minimum over the band is  $v^4$  at  $E = 0$ ; for  $v^2 > 1$  it is negative and the minimum is  $4(1 - v^2) + v^4 = (2 - v^2)^2$  at  $E^2 = 4$ ; for  $v^2 = 1$ , (112) gives  $|M| \equiv 1$ . For  $|E| > 2$ ,  $g_s$  is real with  $|g_s(E)| \leq 1$  and  $g_s(E)E > 0$ , so  $|M(E)| = |E - v^2 g_s(E)|$  is increasing in  $|E|$  with value  $2 - v^2 > 0$  at  $|E| = 2$ ; hence no smaller value occurs outside the band. Since  $\|G\|_{\mathcal{H}^\infty} = \sup_E |M(E)|^{-1}$  by Lemma 3.5 and the maximum principle, (110) follows.  $\square$

**Proposition A.9** (Exclusion of modes outside the band). [D] For  $\varepsilon \in \mathbb{R}$  and  $0 < v^2 < 2$ , the function  $M$  has no real zero with  $|E| > 2$  if and only if

$$|\varepsilon| < 2 - v^2. \quad (113)$$

*Proof.* For  $E > 2$  put  $F(E) := E - \varepsilon - v^2 g_s(E)$ . Then  $F$  is continuous,  $F(E) \rightarrow +\infty$  as  $E \rightarrow \infty$  because  $g_s(E) \rightarrow 0$ , and  $F(2^+) = 2 - \varepsilon - v^2$ . Moreover  $F$  is strictly increasing on  $(2, \infty)$ , since  $g_s$  is strictly decreasing there. Hence  $F$  has a zero in  $(2, \infty)$  iff  $F(2^+) < 0$ , that is iff  $\varepsilon > 2 - v^2$ . The same argument on  $(-\infty, -2)$ , where  $g_s(-2^-) = -1$  and  $F(-2^-) = -2 - \varepsilon + v^2$ , gives a zero iff  $\varepsilon < v^2 - 2$ . Combining the two gives (113).  $\square$

**Remark A.10** (The band-edge condition, and how it was found). Proposition A.9 is the witness that fixed the form of Hypothesis 4.12. An earlier statement of Theorem 4.17 quantified only over an interval interior to the bands and disposed of the remaining real axis by the smallness of  $\|\Sigma\|$ ; that argument is valid for  $|\lambda|$  large but not in the spectral gap adjacent to a band edge, where by Proposition A.9 a real zero of  $M$  appears as soon as  $|\varepsilon| \geq 2 - v^2$ . The interval  $I$  of (33) is chosen to contain the gaps for this reason, and the continuity clause of Hypothesis 4.12 is what the repaired Step 1 requires. The witness satisfies that clause: by Proposition A.4,  $g_s$  is continuous on all of  $\mathbb{R}$  with  $g_s(\pm 2) = \pm 1$ . The bulk chain of Remark A.5 does not, which is why it is not used.

## A.4 Metric values

**Proposition A.11** (Closed-form distances). [D] For the forms of Definition A.6 with  $\rho = \frac{1}{2}$ ,  $\psi = \pi/3$ :

$$\begin{aligned} \|G_1\|_{\mathcal{H}^\infty} &= 1, & \|G_2\|_{\mathcal{H}^\infty} &= \frac{1}{2}, \\ \|G_3\|_{\mathcal{H}^\infty} &= 1, & \|G_4\|_{\mathcal{H}^\infty} &= 2, \end{aligned} \quad (114)$$

and

$$\begin{aligned} c(x_1, x_3) &= \delta_u(G_1, G_3) = 0, \\ c(x_1, x_2) &= \delta_u(G_1, G_2) = 1 - \rho = \frac{1}{2}, \\ c(x_2, x_4) &= \delta_u(G_2, G_4) = \frac{3}{2}, \\ \delta_u(G_1, G_4) &= 1. \end{aligned} \quad (115)$$

Consequently

$$\begin{aligned} d_{\mathcal{H}^\infty}(x_1, x_3) &= 0, & d_{\mathcal{H}^\infty}(x_1, x_2) &= \frac{1}{2}, & d_{\mathcal{H}^\infty}(x_1, x_4) &= 2, \\ \text{and } d_{\mathcal{H}^\infty}^\vee(x_1, x_4) &= \frac{3}{2}. \end{aligned} \quad (116)$$

*Proof.* The gains (114) follow from Proposition A.8 and  $\|B_j G\| = |B_j| \|G\|$ . For (115), scalar channels make the unitary groups the circle, so for  $G_j = B_j G$  with a common  $G$ ,

$$\delta_u(G_i, G_j) = \min_{|c|=1} |cB_i - B_j| \cdot \|G\|_{\mathcal{H}^\infty} = ||B_i| - |B_j|| \cdot \|G\|_{\mathcal{H}^\infty}, \quad (117)$$

which gives the first three lines with  $\|G\|_{\mathcal{H}^\infty} = 1$ . For  $\delta_u(G_1, G_4)$  the two transfer functions have different  $v$ , so (117) does not apply; however  $G_1(0) = -i$  and  $G_4(0) = -2i$  by Proposition A.4, so for every  $|c| = 1$ ,  $\sup_E |cG_1 - G_4| \geq |-ic + 2i| \geq 1$ , while Proposition 5.3 gives  $\delta_u(G_1, G_4) \geq |1 - 2| = 1$ ; the value 1 is attained at  $c = 1$ , as verified in Remark A.19. The path values (116) and  $d_{\mathcal{H}^\infty}^\vee$  follow from Definitions 5.5 and 5.13 applied to the three-edge graph of Proposition A.2.  $\square$

**Remark A.12** (What the numbers witness). The values (116) establish, by exhibition:

- (i)  $d_{\mathcal{H}^\infty}$  is a pseudometric and not a metric:  $x_1 \neq x_3$  while  $d_{\mathcal{H}^\infty}(x_1, x_3) = 0$ , consistent with Proposition 5.7 since  $G_3 = e^{i\psi} G_1$  is transfer isomorphic to  $G_1$ ;
- (ii) the inequality (51) is strict in general:  $\delta_u(G_1, G_4) = 1 < 2 = d_{\mathcal{H}^\infty}(x_1, x_4)$ , the excess being the cost of routing through  $x_2$ , which is what  $d_{\mathcal{H}^\infty}$  records and  $\delta_u$  does not;
- (iii)  $d_{\mathcal{H}^\infty}^\vee < d_{\mathcal{H}^\infty}$  strictly:  $\frac{3}{2} < 2$ , so the bottleneck and additive closures are genuinely different, as claimed in Remark 5.16;
- (iv)  $d_{\mathcal{H}^\infty}$  and  $d_C^*$  are not comparable: by (106) and (116),  $d_C^*(x_1, x_3) = 1 > 0 = d_{\mathcal{H}^\infty}(x_1, x_3)$  while  $d_C^*(x_1, x_2) = \frac{1}{10} < \frac{1}{2} = d_{\mathcal{H}^\infty}(x_1, x_2)$ , so no edge-wise comparison constant exists in either direction, as stated in Remark 5.23.

**Remark A.13** (The survival radius on the witness). Theorem 5.9 applied at  $x_1$ , where  $\gamma_{x_1} = 1$ , states that every form within  $d_{\mathcal{H}^\infty}$ -distance  $\delta^{-1} - 1$  survives all environment perturbations of size  $\delta$ . For  $\delta = \frac{1}{4}$  the radius is 3 and the ball contains all four forms, consistent with the direct check  $\|G_4\|_{\mathcal{H}^\infty} \delta = \frac{1}{2} < 1$ . For  $\delta = \frac{2}{5}$  the radius is  $\frac{3}{2}$  and the ball contains  $x_1, x_2, x_3$  but not  $x_4$ , whereas the direct check  $\|G_4\|_{\mathcal{H}^\infty} \delta = \frac{4}{5} < 1$  shows  $x_4$  is in fact stable. The theorem is therefore sufficient and not necessary, the conservatism arising from the triangle inequality in (53).

## A.5 Witnesses for the negative results

**Example A.14** (Exact confinement, and the explicit model). [D] Take  $v = 0$  and  $\varepsilon = 0$  in Definition A.6. Then  $W = 0$ ,  $\Sigma \equiv 0$ ,  $\Gamma \equiv 0$  and  $G(z) = 1/z$ , which is precisely the transfer operator of the explicit model  $\mathfrak{R}_{\text{ex}}$  recalled in Remark 4.6. By Proposition 4.22,  $\|G\|_{\mathcal{H}^\infty} = \infty$ . Thus  $\mathfrak{R}_{\text{ex}}$  occupies the degenerate point  $v = 0$  of the witness family, and the family interpolates continuously from it: by Proposition A.8 the gain is  $v^{-2}$ , which diverges as  $v \downarrow 0$  and again as  $v^2 \uparrow 2$ .

**Example A.15** (An embedded mode that the continuum does not see). [D] Take  $\mathcal{H}_{\text{pri}} = \mathbb{C}^2$ ,  $H_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $v^2 = 1$ , coupling  $W$  supported on the symmetric vector  $u = \frac{1}{\sqrt{2}}(1, 1)^\top$ , so that  $\Sigma(z) = g_s(z) |u\rangle\langle u|$ , and  $B = C^\top = (1, 0)^\top$ . In the eigenbasis  $\{u, a\}$  of  $H_p$ , with  $a = \frac{1}{\sqrt{2}}(1, -1)^\top$  and  $H_p u = u$ ,

| Hypothesis        | Status on $\mathfrak{R}_{\text{wit}}$                               |
|-------------------|---------------------------------------------------------------------|
| $H_{\text{fib}}$  | verified, $\mathcal{H}_{\text{pri}} = \mathbb{C}$ or $\mathbb{C}^2$ |
| $H_{\text{lab}}$  | verified, Remark A.7                                                |
| $H_{\text{form}}$ | verified, $\mathcal{K} = \mathcal{Y} = \mathbb{C}$                  |
| $H_{\text{ch}}$   | verified, $k = q = 1$                                               |
| $H_{\text{sec}}$  | verified, Def. A.3                                                  |
| $H_{\text{pop}}$  | verified, $m = 1$ , rank-two modification                           |
| $H_{\text{sp}}$   | verified, single species d                                          |
| $H_{\text{cov}}$  | verified on the bulk chain                                          |
| $H_{\text{reg}}$  | verified on the interval $I$ of (33), the continuity clause holding |
| $H_{\text{min}}$  | verified, $n = 1$ ; and in Ex. A.15                                 |
| $H_{\text{rec}}$  | verified, scalar channels                                           |
| $H_{\text{env}}$  | not instantiated                                                    |
| $H_{\text{nuc}}$  | not instantiated                                                    |
| $H_{\text{out}}$  | not instantiated                                                    |
| $H_{\text{chem}}$ | not instantiated                                                    |

**Table 3:** Status of the hypothesis register on the witness realization  $\mathfrak{R}_{\text{wit}}$  of Appendix A.

$$H_p a = -a,$$

$$M(z) = \text{diag}(z-1-g_s(z), z+1), \quad G(z) = \frac{1}{2} \left[ \frac{1}{z-1-g_s(z)} + \frac{1}{z} \right] \quad (118)$$

The pair  $(\lambda, \psi) = (-1, a)$  satisfies (40):  $\Gamma(\lambda) = \Gamma(\lambda)|u\rangle\langle u|$  annihilates  $a$ , and  $(\lambda - H_p - \Lambda(\lambda))a = (-1 + 1)a = 0$ . The energy  $\lambda = -1$  lies in the interior of the band  $(-2, 2)$ , so this is an embedded mode in the sense of Remark 4.18. Accordingly  $\|G\|_{\mathcal{H}^\infty} = \infty$ , with  $|G(-1 + i\eta)| = \frac{1}{2\eta} + O(1)$  as  $\eta \downarrow 0$ .

*Remark A.16* (Role of Example A.15). Example A.15 is the witness for the necessity direction of Theorem 4.17. It also shows that the criterion is not implied by any condition on the coupling strength alone: here  $v \neq 0$ , the second fibration is coupled, and  $\Gamma \neq 0$ , yet the gain is infinite because a single mode of the active region is orthogonal to the range of the coupling. By Proposition 4.20 no norm is lost; the mode simply never enters the second fibration.

## A.6 Which hypotheses the witness discharges

*Remark A.17* (Register status on the witness). Table 3 records, for each hypothesis of Section 2.6 and Section 4, whether the witness verifies it. The three hypotheses marked “not instantiated” are not used by any statement proved on the witness; they are required only by Sections 6 and 8.

*Remark A.18* (What the witness does not establish).  $\mathfrak{R}_{\text{wit}}$  is a witness of non-vacuity for the constructions of Sections 4–5 and for nothing further. In particular it does not instantiate Hypothesis 2.23: the forms of Definition A.6 are not the rearranged frameworks of Definition 2.28, no molecular datum enters (109), and no statement of this appendix is of type [E]. It does not re-establish the non-vacuity of the twelve axioms, which remains [Sch26c, Thm. 61];  $\mathfrak{R}_{\text{wit}}$  modifies only the A10 and A12 data of  $\mathfrak{R}_{\text{ex}}$  and inherits A1–A9 and A11 unchanged, a replacement of downstream data that [Sch26c, Rem. 62] explicitly licenses while retaining membership in  $\text{Ext}_{12}(\mathfrak{C}^{\text{chr}})$ . It does not exhibit a mixed population, and therefore says nothing about Target 4.24.

*Remark A.19* (Independent numerical verification). The closed forms of Propositions A.8, A.9 and A.11 were checked numerically against direct evaluation of  $G$  on a refined grid of the real axis together with the far field, and the unitary-orbit distances by minimization over the phase. The computed values agreed with the closed forms to six significant figures: gains 4, 2, 1,  $\frac{1}{2-v^2}$  at  $v^2 = \frac{1}{4}, \frac{1}{2}, 1, \frac{121}{100}$ ; the threshold (113) confirmed by the transition between finite and numerically divergent gain across  $|\varepsilon| = 2 - v^2$ ; and  $\delta_u(G_1, G_4) = 1.000000$ ,  $\delta_u(G_2, G_4) = 1.500000$ ,  $\delta_u(G_1, G_2) = 0.500000$ ,  $\delta_u(G_1, G_3) = 0$  to grid resolution. The divergence rate of Example A.15 was confirmed as  $|G(-1 + i\eta)|\eta \rightarrow \frac{1}{2}$  over  $\eta \in [10^{-6}, 10^{-2}]$ .

## B Concordance, Chemistry Items, and Defect Ledger

### B.1 Scope and method

This appendix records, for every prior statement of the Chrysene Formalism corpus that the present paper touches, whether the statement is *preserved* (used unchanged), *extended* (used together with a further hypothesis, without alteration of its content), *restated* (reformulated here in a manner the reviewer must check is equivalent), *corrected* (found to be false or type-incorrect as written, with the repair given), *not applicable* (outside the scope of the objects considered here), or *superseded* (replaced for the purposes of this paper by a construction given here).

The classification is made against the criterion  $C$  of [Sch26b, Def. 2]: a statement of this paper is concordant when it neither contradicts a prior statement of the formalism nor silently weakens a structure that a prior statement relies upon. Where a correction is recorded, the entry states what the prior statement asserts, why it fails, and what replaces it; a correction is not a claim that the surrounding work is unaffected, and the affected downstream results are named where known.

## B.2 Concordance with the twelve-axiom paper

*Remark B.1* (The one extension, stated plainly). The single entry of Table 4 that changes the reading of a prior result is Proposition 4.22. The preservation criterion of [Sch26c, Prop. 31] identifies when the admissible sector is invariant under the dynamics. Nothing in that proposition is contradicted here. What is added is that invariance in that exact sense forces  $\|G\|_{\mathcal{H}^\infty} = \infty$ , so that a form cannot be simultaneously perfectly confined and robustly stable. Papers of Volume 1, Number 1 that treat confinement as the desideratum and robustness as a separate desideratum are, on the present analysis, pursuing two aims that cannot both be achieved exactly; the quantitative trade-off is Corollary 6.16, where the uniform lower bound  $\Gamma_x \geq \gamma_0 I$  is exactly a uniform failure of confinement.

## B.3 Concordance with the monograph series

*Remark B.2* (The type mismatch, in full). Several statements of [Sch26f, Sch26e] infer a bound on a constraint functional from a bound on a gain: [Sch26e, Thm. 14.5] concludes  $\sup_t \|\Theta_{\text{strain}}\| < \tau_{\max}$  from  $\sup_\omega \mu_\Lambda(M) < 1$ , and [Sch26f, Def. 4.1.1] defines viability by  $\sup_{x_i} \|S(x_i)\|_\infty < \tau_{\max}$ . A gain is a dimensionless ratio of signal norms (Proposition 7.10); a threshold  $\tau_j$  of Axiom A12 bounds the value of a constraint functional on a state. The two are of different type and the inequality as written has no content. The correct statement relating them is Theorem 8.13, in which the comparison is  $\kappa_j \|G_x\|_{\text{pk}} \|w\|_\infty < \tau_j - C_j(x)$ : it requires a peak gain rather than an  $\mathcal{H}^\infty$  gain, the readout constants  $\kappa_j$  of Hypothesis 8.12, the disturbance amplitude, and the constraint margin at the operating point. None of these four ingredients appears in the original comparison.

*Remark B.3* (The  $\mathcal{H}^\infty$  papers of Volume 1, Number 1). Eight papers of Volume 1, Number 1 are recorded in [Sch26b, Tab. 2] as using  $\mathcal{H}^\infty$  robust control. Their definitions have not been examined here, and no entry of Table 5 should be read as applying to them. Two items make an examination advisable: the type mismatch of Remark B.2, which is present in the monographs from which those papers derive, and the confinement–robustness incompatibility of Remark B.1. This paper supplies the apparatus by which such an examination could be carried out; it does not carry it out, and asserts nothing about the validity of those papers.

## B.4 Chemistry and domain items

The following items are those on which the chemistry review of [Sch26b, §4.3] is invited to rule. All are of type [E] or are

statements about labelled graphs; none is used in a proof.

*Remark B.4* (Reciprocity of the conjugated framework). [E] Hypothesis 2.21 asserts reciprocity, under which Proposition 5.18(i) gives symmetry of the attenuation quantity. For a substituted chrysene network in the absence of an applied magnetic field and of magnetic constituents, the electronic Hamiltonian may be taken real in a suitable basis and reciprocity is expected to hold. Nonreciprocity of the kind exhibited in Example 5.19 requires broken time-reversal symmetry; within the corpus the relevant construction is the Aharonov–Bohm phase gating of [Sch26b, paper 15]. The chemistry review is asked to confirm that the field-free case is the default and that Hypothesis 2.21 is therefore the normal, not the exceptional, situation.

*Remark B.5* (Point-group symmetry is not channel symmetry). [A] Proposition 2.31 establishes that the automorphism  $\sigma$  carries  $\mathcal{R}^{(3)}$  to  $\mathcal{R}^{(9)}$  and preserves both attachment sets. It does not follow that the two rearrangements have equal transfer realizations (Remark 2.32), nor that the two terminals of a motif are related by a symmetry: left–right channel symmetry, which by Proposition 5.21 produces perfect transmission and hence a degenerate attenuation cost, is a property of the realization data of Hypothesis 2.13 and is not implied by the  $C_{2h}$  symmetry of the carbon framework recorded in Axiom A2.

*Remark B.6* (The conjugated  $\pi$ -framework hypothesis). [E] The hypothesis  $H_{\partial\pi}$  of [Sch26c, App.] records delocalized perimeter  $\pi$  conjugation without making conductivity an axiom. In the present construction the quantity into which such delocalization would enter is  $\Gamma$  of Proposition 4.13, the coupling-weighted density of states of the second fibration: a more delocalized perimeter increases the coupling  $\tilde{W}$  and hence, by (35), the absorption that Theorem 4.17 requires. This is an interpretive remark and not a derivation; no value of  $\hat{W}$  is computed from  $H_{\partial\pi}$  anywhere in this paper.

*Remark B.7* (Provenance of the rearrangement, and what each instrument covers). [E] Three instruments bear on the rearrangement of Definition 2.28, and they cover different things.

- (i) The granted composition-of-matter patent [Sch25] claims the substituted chrysene molecule, and its specification fixes the substituent class of Definition 2.26.
- (ii) Its granted divisional, US 12,473,247, claims systems of that composition, including the stacked assembly that Hypothesis 4.9 abstracts.

| Prior statement                             | Content relied upon                                                    | Disposition                   | Locus here                                                 |
|---------------------------------------------|------------------------------------------------------------------------|-------------------------------|------------------------------------------------------------|
| A1–A12                                      | the twelve axioms in full                                              | preserved                     | throughout; no axiom added (Rem. 2.8)                      |
| Def. 1, 3, 5, 6                             | logical status vocabulary                                              | preserved                     | Def. 2.7                                                   |
| [Sch26c, p. 2 and (444)]; [Sch26b, Eq. (2)] | geometric anisotropy $\Rightarrow$ transfer anisotropy                 | preserved                     | Rem. 3.9, Prop. 3.8                                        |
| [Sch26c, p. 2 and (444)]; [Sch26b, Eq. (3)] | $d_C^* = +\infty \Rightarrow$ scission                                 | preserved                     | Rem. 4.21                                                  |
| Rem. 14                                     | sequence order is not physical time                                    | preserved                     | Rem. 2.4                                                   |
| Thm. 8                                      | sequence-induced noncommutativity                                      | preserved, not used           | Rem. 4.10                                                  |
| Thm. 15, Cor. 3                             | $\mathfrak{M}^{\text{dual}} = (\mathfrak{M}^{\text{pri}})'$            | preserved                     | Rem. 2.17; Target 8.25                                     |
| Prop. 26                                    | resolvent bound on the transfer operator                               | preserved                     | Cor. 4.7, Rem. A.7                                         |
| Prop. 27, Rem. 29                           | $D_{\mathfrak{C}}$ and $H$ logically independent                       | preserved                     | Rem. 5.24                                                  |
| [Sch26c, Rem. 32 and (444)]                 | framework symmetry $\Rightarrow$ transfer symmetry                     | preserved                     | Rem. 2.32                                                  |
| Prop. 30                                    | $d_C^*$ is an extended pseudometric                                    | preserved; analogue proved    | Prop. 5.6                                                  |
| Prop. 31                                    | preservation criterion for the admissible sector                       | extended                      | Prop. 4.22: exact preservation $\Rightarrow$ infinite gain |
| Prop. 39                                    | acyclicity of the axiom ladder                                         | preserved                     | Rem. 2.8                                                   |
| Thm. 36                                     | identification of $\mathfrak{C}^{\text{chr}}$                          | not applicable                | Rem. 2.33: rearranged framework $\notin \mathcal{B}_4$     |
| Thm. 61; Rem. 62                            | non-vacuity of A1–A12; licensed replacement of downstream A10/A12 data | preserved, not re-established | Rem. A.18                                                  |
| Thm. 42 (Target)                            | directional transfer anisotropy                                        | not discharged                | $\delta_u$ gives a quantitative companion only             |
| Thm. 44 (Target)                            | spectral–path metric compatibility                                     | not addressed                 | Rem. 5.23: $d_{\mathcal{H}^\infty}$ is not a candidate     |
| Thm. 49 (Target)                            | dynamical scission                                                     | not discharged                | Rem. 4.21                                                  |
| App. B.7                                    | explicit model, $T(z) = 1/z$                                           | superseded as a witness       | Rem. 4.6, Ex. A.14, App. A                                 |
| App. A.5                                    | locant and adjacency certificate                                       | preserved                     | Def. 2.25, Prop. 2.31                                      |

**Table 4:** Concordance with [Sch26c]. Every locator in the first column was verified against the source at the concordance pass.

(iii) The transformation itself — oxidative cleavage of the C1–C12a bond followed by closure of the five-membered ring — is shown schematically in [Sch25, Fig. 6, sheet 19]. It is *not* claimed as a process by either granted patent; a divisional application directed to the process is contemplated and has not issued.

Accordingly this paper cites [Sch25] for the substituent class and for the schematic transformation, and makes no claim that the product of Definition 2.28 is a granted composition of matter or that the process is a granted process. The locants used here are those of Definition 2.25 and need not coincide with the numbering of the figure.

B.5 Self-identified defect ledger

The following records, in the form of [Sch26b, Def. 3], the criticisms raised against this manuscript during its own drafting and in disciplinary review, their disposition, and whether the repair has been independently examined.

*Remark B.8* (The item the ledger does not close). Item 7 of Table 6 remains open at submission. It is a genuine gap in coverage: the results of Section 6 hold under Hypothesis 6.2, which the only witness constructed here does not satisfy, so that section is currently supported by no explicit instance.

B.6 Bibliographic items

*Remark B.9* (Citation inconsistencies in the record). Four items in the surrounding record are noted for correction; none affects any result of this paper. The self-citation in [Sch26c] gives its own pagination as 405–476 and article number 38, while [Sch26b] lists it at 476–547; the two are inconsistent and one collides with entries assigned within Volume 1, Number 1, an inconsistency [Sch26b, §2] itself records as outstanding. The citation block of [Sch26b] describes the paper as a survey of Volume 1, Number 2 where Volume 1, Number 1 is meant, and states first publication in Volume 1, Number 1 where Number 2 is meant. The closing paragraph of [Sch26b, §10] terminates

| Prior statement           | Content as written                                                                                                        | Disposition                    | Locus here                                                                                                                                                                                                                                                                         |
|---------------------------|---------------------------------------------------------------------------------------------------------------------------|--------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| [Sch26d, Lem. 8.1]        | noncommutative bounded real lemma, block operator matrix inequality                                                       | extended                       | Def. 6.12, Thm. 6.13: one storage operator across forms                                                                                                                                                                                                                            |
| [Sch26d, Thm. 8.2]        | $\ S\ _\infty \leq 1$ unconditionally, proved by excision of expansive operators                                          | conclusion preserved, restated | pre-proof<br>contractivity follows from passivity of the coupling (Lem. 4.4, Prop. 5.18), not from excision                                                                                                                                                                        |
| [Sch26d, Rem. 8.1]        | coupling to an unbounded exterior tensor space yields $\mathcal{H}^\infty$ gain exceeding unity; discreteness prevents it | corrected                      | the quoted attribution is false, not merely superseded: a passive coupling to an exterior continuum is contractive, so continuity of the exterior spectrum does not produce gain exceeding unity and discreteness is not what secures $\ S\ _\infty \leq 1$ (Lem. 4.4, Prop. 4.20) |
| [Sch26f, Def. 1.3.3]      | discrete principal $\mathbb{Z}_2$ bundle over the longitudinal sequence                                                   | restated                       | not the registry label of A6, not the second fibration (Rem. 2.17)                                                                                                                                                                                                                 |
| [Sch26f, Def. 2.1.3]      | $\mathcal{H}^\infty$ evaluated on the discrete frequency domain; the continuous Fourier domain rejected                   | restated                       | Rem. 2.4: the discrete variable is $\theta_z$ ; the energy variable is separate and continuous                                                                                                                                                                                     |
| [Sch26f, Def. 4.1.1]      | $J(X) = \sup_{x_i} \ S(x_i)\ _\infty$ ; $\Omega_{\text{viable}} = \{J < \tau_{\max}\}$                                    | restated                       | bottleneck functional, hence ultrametric (Prop. 5.14, Rem. 5.16)                                                                                                                                                                                                                   |
| [Sch26f, Thm. 4.4.2]      | strict topological dissipation inequality with storage functional $V_T$                                                   | extended                       | Def. 6.12, Thm. 6.13                                                                                                                                                                                                                                                               |
| [Sch26f, Thm. 9.2.5]      | strict immunity to Arnold diffusion and ergodic chaos                                                                     | corrected                      | KAM gives large-measure persistence of invariant tori, not immunity; the source proof asserts only survival of a large-measure set of tori                                                                                                                                         |
| [Sch26f, Def. B.1.2]      | action-cost functional $\Delta F_{\text{sciss}}$                                                                          | restated                       | jump factor $\mu$ of (80) (Rem. 6.20)                                                                                                                                                                                                                                              |
| [Sch26f, Thm. B.1.3]      | conditional deletion: mapping to the null ideal when the action cost exceeds the available gradient                       | restated                       | transfer, not deletion (Prop. 4.20); the cost/dissipation exchange is Thm. 6.19                                                                                                                                                                                                    |
| [Sch26f, Thm. B.2.3]      | percolation of stability                                                                                                  | restated                       | widest-route margin (Prop. 5.15)                                                                                                                                                                                                                                                   |
| [Sch26e, Def. 6.2]        | operator Hardy space of $\mathfrak{A}_{\mathbb{C}}$ -valued analytic functions                                            | corrected                      | the transfer operator of A10 maps $\mathcal{K} \rightarrow \mathcal{Y}$ between terminal spaces and is not an element of $\mathfrak{A}_{\mathbb{C}}$ ([Sch26c, Prop. 27 and Rem. 29]; Def. 3.2)                                                                                    |
| [Sch26e, Thm. 6.2]        | a boundary-violating vector generates a pole, leaves $\mathcal{H}^\infty$ , and is aborted                                | corrected                      | pole location and boundary violation are independent: by Prop. 4.22 an exactly confined form has a pole precisely when it does <i>not</i> breach                                                                                                                                   |
| [Sch26e, Thm. 14.5]       | $\sup_\omega \mu_\Lambda(M) < 1$ implies $\sup_t \ \Theta_{\text{strain}}\  < \tau_{\max}$                                | corrected                      | $\mu_\Lambda < 1$ gives robust stability, not a bound on the peak of a state functional; the correct inference is Thm. 8.13 (Rem. B.2)                                                                                                                                             |
| [Sch26e, Ch. 2 and Ch. 6] | mixed mean-square and worst-case criteria; boundary breach as the limit of the $H^2$ framework                            | restated                       | Prop. 7.10, Rem. 8.16                                                                                                                                                                                                                                                              |
| [Sch26f, Sch26e] (passim) | $W_2$ action functionals                                                                                                  | extended                       | Rem. 8.23: a quantum Markov semi-group must be specified                                                                                                                                                                                                                           |

**Table 5:** Concordance with the monograph series. Five rows are disposed *corrected*. Locators into [Sch26d] and [Sch26f] were verified at the concordance pass; those into [Sch26e] were verified editorially, that volume not having been among the materials supplied to that pass. The final row is a *passim* entry whose full extent has not been enumerated.

mid-sentence. The document metadata of the [Sch26b] file carries the title of a different paper of the corpus.

## C Established Mathematical Literature

As in the literature appendices of [Sch26c, Sch26b], none of the subjects below is claimed to be novel. The purpose is to identify the established machinery each part of this paper draws upon, and to keep that machinery distinct from the specific organization the Chrysene Formalism proposes.

### C.1 Robust control and the worst-case norm

The state-space theory of the worst-case norm, the bounded real lemma used in Remark 6.14, and the small-gain theorem on which Theorems 5.9 and 7.7 rest are developed in [ZDG96]; the small-gain principle in its input-output form originates with Zames. The exactness of the stability radius (86) for unstructured uncertainty, and the interpolation construction in the proof of Theorem 7.7, are standard there. The structured singular value of Section 8.2 is due to [Doy82], with the general theory in [PD93]. The dissipativity and storage formalism of Definition 6.12 is [Wil72]. The peak-to-peak gain of Section 8.3 and its optimal control theory are developed in [DP87]. The gap and  $\nu$ -gap metrics mentioned in Table 1 are [ZES80, GS90] and [Vin01].

### C.2 Switched systems

Section 6 draws on the standard theory of switched linear systems as presented in [Lib03], including the common-Lyapunov and multiple-Lyapunov constructions and the classical observation, reproduced with explicit constants in Proposition 6.6, that switching between Hurwitz systems may be unstable. The average dwell-time condition of Theorem 6.19 is [HM99].

### C.3 Spatially invariant systems

The treatment of a lattice-translation-covariant system by carrying a spatial transform variable alongside the temporal one, which Definition 2.3 and Remark 2.4 follow, is developed in [BPD02].

### C.4 Scattering theory, resolvents, and embedded modes

The Feshbach projection giving the self-energy of Definition 4.2 is [Fes58], with the operator-theoretic treatment in [MW69]; the relation between transmission blocks and resolvent kernels used in Section 5.5 is [FL81]. Perturbation of spectra, the second resolvent identity of Lemma 3.14, and the Kato–Rosenblum theorem are in [Kat66]; scattering theory and the RAGE theorem are in [RS79]. The Lax–Phillips formulation referred to in

Target 4.26 is [LP67], and the identification of a scattering matrix with the characteristic function of a contraction is the theory of [SNF70]. Privalov’s theorem, used in the proof of Lemma 3.14, and the boundary behaviour of bounded analytic functions used throughout Section 4, are in [Gar07].

### C.5 Periodic operators and band structure

The direct-integral decomposition of Definition 4.11, the analyticity of band functions, and the absolute continuity of the spectrum on non-flat bands are standard Floquet–Bloch theory; see [RS78] and [Kuc93]. The co-area formula used in the proof of Proposition 4.13 is in [EG15].

### C.6 Operator algebras and noncommutative geometry

The quasi-local and crossed-product constructions recalled in Section 2.1 are standard; see [BR87]. The spectral triple of Axiom A9 belongs to the programme of [Con94]. The identification of the Connes distance with a Kantorovich–Rubinstein transport distance (Remark 8.23) is [Rie99], with the transport reading developed in [DM10].

### C.7 Graph theory

The algebraic connectivity of Definition 8.17 is [Fie73]. Resistance distance and the Rayleigh monotonicity argument in the proof of Proposition 8.18 are [KR93] and [DS84].

### C.8 Optimal transport

The transport distance of Definition 8.20, the gluing lemma, and the duality invoked in Remark 8.23 are in [Vil09]. The quantum  $W_2$  construction referred to there, which requires a specified Markov semigroup, is [CM17].

### C.9 What is drawn from the Chrysene Formalism itself

Axioms A1–A12, the status vocabulary, the explicit model, and the locant and adjacency certificate are [Sch26c]. The review architecture, the editorial classification, and the non-implication conventions are [Sch26b], with the case study in [Sch26a]. The monograph material restated or corrected in Appendix B is [Sch26d, Sch26f, Sch26e]. The composition-of-matter record referred to in Remark 2.34 is [Sch25]. None of these is external literature in the sense of this appendix; they are listed here so that the boundary between established machinery and the formalism’s own claims is visible in one place.

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| No. | crit( <i>R</i> )                                                                                                                                                       | rev( <i>R</i> )                                                                                                                                                                | ver( <i>R</i> )                                                                                                                                                    |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1   | Step 1 of the proof of Thm. 4.17 asserts invertibility of <i>M</i> outside <i>I</i> on the ground that $\ \Sigma\ $ is small; false in the gap adjacent to a band edge | fully accepted; repair stated in Rem. A.10, <i>I</i> chosen by (33) to contain the gaps                                                                                        | explicit: Prop. A.9 exhibits the failure and the threshold                                                                                                         |
| 2   | the transfer operator of A10 is indexed by motifs, not by pairs of admissible states, so no distance between states follows from A10 alone                             | fully accepted; Hypotheses 2.12, 2.13 introduced and named in every dependent statement                                                                                        | explicit: Rem. 3.3                                                                                                                                                 |
| 3   | symmetry of the attenuation quantity was first claimed to require loss together with nonreciprocity                                                                    | fully accepted; corrected to reciprocity, losslessness sufficing only for two ports                                                                                            | explicit: Ex. 5.19, a lossless nonreciprocal three-port                                                                                                            |
| 4   | order dependence of the band set was claimed for the second fibration                                                                                                  | fully accepted; withdrawn under Hyp. 4.9, retained for mixed populations only                                                                                                  | explicit: Rem. 4.10, Target 4.24                                                                                                                                   |
| 5   | the explicit model of [Sch26c, App. B] does not witness any construction of this paper                                                                                 | fully accepted; separate witness constructed                                                                                                                                   | explicit: App. A, Ex. A.14                                                                                                                                         |
| 6   | Thm. 5.9 certifies frozen forms only and was described as a survival theorem                                                                                           | fully accepted; Section 6 added and the radius renamed                                                                                                                         | explicit: Prop. 6.6                                                                                                                                                |
| 7   | the witness of App. A does not satisfy Hyp. 6.2, so Section 6 has no instantiated example                                                                              | accepted, not repaired                                                                                                                                                         | none; recorded as Target 6.26                                                                                                                                      |
| 8   | $\mathcal{H}^\infty$ was used for the admissibility half of the survival claim                                                                                         | fully accepted; peak-gain treatment added                                                                                                                                      | explicit: Thm. 8.13, Prop. 8.15                                                                                                                                    |
| 9   | $\mathcal{H}_{\text{sec}}$ of Def. 4.1 was read as the dual-cipher fibration alone, omitting the complement of the defect region                                       | fully accepted; reading fixed                                                                                                                                                  | explicit: Rem. 3.13, Lem. 3.14                                                                                                                                     |
| 10  | the word <i>isomorphic</i> was used without distinguishing graph isomorphism from equality of transfer                                                                 | fully accepted                                                                                                                                                                 | explicit: Prop. 3.8, Rem. 2.34                                                                                                                                     |
| 11  | monograph theorem numbers had not been checked against the sources                                                                                                     | fully accepted; every locator in Table 5 verified, one entry ([Sch26e, Thm. 14.5]) found to have been mischaracterized and rewritten, one unsupported entry removed            | explicit for [Sch26d], [Sch26f]; editorial for [Sch26e]                                                                                                            |
| 12  | the valence ledger of the rearrangement did not close: the ring was called saturated, $C\alpha$ would be pentavalent, and C12a was left uncapped                       | fully accepted; oxidation record added to Def. 2.28 and the accounting stated                                                                                                  | explicit: Rem. 2.29, Rem. B.7                                                                                                                                      |
| 13  | the substituent was named <i>pegylating</i> , which imports a claim the two-carbon chain does not earn                                                                 | fully accepted; renamed 2-phosphoethoxy throughout                                                                                                                             | explicit: Def. 2.26, Rem. 2.27                                                                                                                                     |
| 14  | naming the handoff sector the <i>dual-cipher fibration</i> introduces a third use of a term already carrying two senses in the corpus                                  | fully accepted; the sector is named the <i>second fibration</i> throughout, and the dual-cipher species is named only in Hyp. 4.9, where the population is the object at issue | explicit: Rem. 2.17; the adjective occurs nowhere outside statements about $A_{\mathbb{C}}^{\text{dual}}$ , the species letter d, and $\mathfrak{M}^{\text{dual}}$ |

**Table 6:** Self-identified defect ledger for this manuscript, in the crit/rev/ver form of [Sch26b, Def. 3]. Item 7 is recorded as accepted and unrepaired; it is not resolved by this paper.