

The Encoded Stack: A Frame-Indexed Base-Four Structure on the $\mathbb{Z}^3$ -Fibrated Chrysene Network, its Sequence Algebra, and the Set-Up for Survival

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Abstract

We construct, from Axioms A1, A2, A6, A7, A8 and A12 of the Chrysene Formalism and from no further axiom, the *frame-indexed encoded stack*: a mathematical object whose reaction volumes carry a canonical four-letter alphabet, whose words are elements of the crossed-product sequence algebra $\mathfrak{A}_{\mathbb{C}}$ and are therefore order dependent as a consequence of Axiom A7's covariance relation, and whose reading is fixed by a datum external to the stack. We prove non-vacuity by explicit $N = 3$ witness. The alphabet is an $\mathbb{F}_2$ -vector space after the frame designates a zero; its additive structure is canonical, its multiplication is determined by the registry invariant, and no structural operation realizes the multiplicative generator. The registry sequence modulo reversal is the maximal frame invariant: every quantity attributable to the assembly rather than to the reading factors through it. We state precisely how far the encoded stack supplies the instance schema of the companion transfer-geometry paper, record the two outstanding clauses as Targets, and identify the candidate survival invariants whose preservation a future paper would need to certify.

Keywords: Chrysene Formalism; encoded stack; noncommutative geometry; crossed-product algebra; sequence algebra; frame of reference; base-four alphabet; registry invariant; survival; $GF(4)$; fibrated transfer network.

1 Introduction

1.1 The problem

The corpus of Volume 1, Number 1 speaks throughout of encoding, of a four-state alphabet, and of sequence dependence, and Axiom A8 supplies a native four-state alphabet $\Sigma_{\mathbb{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z$ together with a canonical additive law. No paper of that corpus exhibits an object that carries the encoding: a structure whose letters are borne by identified constituents, whose words are elements of the sequence algebra rather than classical strings, and whose reading is fixed by a datum external to the stack. In consequence the pervasive assertion that the formalism's encoding is sequence dependent has no established referent, and the noncommutativity of Theorem 8 of [Sch26c] has not been shown to be the order dependence that the encoding language invokes.

1.2 The objective

To construct, from Axioms A1, A2, A6, A7, A8 and A12 and from no further axiom, a mathematical object — the frame-indexed encoded stack — carrying a base-four alphabet on

identified reaction volumes; to exhibit the operators through which the axioms interact to produce it; to prove that its words are order dependent as a consequence of A7 rather than as a property of strings; to establish its non-vacuity by explicit witness; to determine exactly which of its structure is frame dependent and which is not; to clarify, rather than discharge, the standing non-implication $|\Sigma_{\mathbb{C}}| = 4 \Rightarrow \Sigma_{\mathbb{C}} \cong GF(4)$; and to state precisely how far the object falls short of an instance of the fibrated transfer network of [Sch26a, Def. 10.4], which is the architecture in which survival under persistent disturbance is defined.

1.3 What is established

1. A marker on a stacked unit whose two values are not identified by any operation available to that unit (Proposition 3.4), and the vessel alphabet Σ_v it generates (Definition 3.13).
2. Two structural operators, the marker operator and the registry operator, the second canonical and independent of the vessel decomposition (Proposition 4.3), together with the identification of registry as the terminal structural invariant (Theorem 4.9).

3. A word operator valued in $\mathfrak{A}_{\mathfrak{C}}$ whose order dependence is derived from the covariance relation of $A7$ (Theorem 4.17), together with the non-implication showing that a realization inside the commutative local sector would not have this property (Proposition 4.18).
4. Non-vacuity: an explicit encoded stack over $\mathfrak{C}^{\text{chr}}$ satisfying every definition of Sections 3–4 (Theorem 5.1, Appendix A).
5. The exact frame dependence of the structure: the registry coordinate is frame independent, the offset coordinate is precisely where the frame enters (Proposition 4.4), and the difference sequence is the maximal frame invariant (Theorem 6.14).
6. The base-four status of $\Sigma_{\mathfrak{v}}$: the additive structure and, given the frame, a multiplication are both determined, while no structural operation realizes the multiplicative generator (Theorems 7.4 and 7.5).

1.4 What is not claimed

No axiom is introduced (Remark 2.2). No deferred target of [Sch26c] is discharged; in particular Theorem 41 of that paper concerns $\Sigma_{\mathfrak{C}}$, which is not the alphabet constructed here (Remark 7.9). No transformation rule is given: the secondary alphabet is introduced and the conditions under which a transformation transports the structure are stated, but no map is constructed, no molecular rearrangement is described, and no valence, energetic, kinetic or mechanistic claim is made or used. No survival property is asserted of the encoding (Remark 9.16). Two clauses of the instance schema of [Sch26a, Def. 10.4] are not supplied, and are recorded as Targets rather than assumed.

1.5 Logical position

Every numbered statement carries one of the four status labels of Definition 2.1. Every result using a hypothesis beyond $A1$ – $A12$ names that hypothesis in its statement, and every hypothesis is registered in Section 2.3. Every molecular reading is confined to remarks marked [E] and is used in no proof. This paper falls within the framework scope condition of [Sch26b, §4.5] and is written to the orthogonal-constraint review bar of [Sch26b, §4.3].

1.6 Organization

Section 2 fixes conventions, status labels and the hypothesis register. Section 3 constructs the encoded stack as a labelled combinatorial structure. Section 4 introduces the operational operators and establishes how the axioms interact to produce sequence dependence. Section 5 states non-vacuity. Section 6 treats the frame of reference and the frame invariants. Section 7 clarifies the base-four status of the alphabet. Section 8 introduces

the secondary alphabet and states what a transformation must intertwine. Section 9 measures the object against the instance schema of [Sch26a]. Section 10 states the non-implications and the Target inventory. Appendix A is the witness, Appendix B the concordance and defect ledger, Appendix C the established literature.

2 Conventions, Logical Status, and the Hypothesis Register

2.1 Logical status

Definition 2.1 (Status labels). Imported verbatim from [Sch26c, Defs. 1, 3, 5, 6] and used as in [Sch26a, Def. 2.7]: [A] axiomatic; [D] derived realization theorem; [T] target; [E] empirical instantiation claim.

Remark 2.2 (No thirteenth axiom). [A] This paper introduces no axiom. Every structure below is either a definition internal to $\{A1, \dots, A12\}$ or a realization hypothesis in the sense of [Sch26c, Def. 3]. The restriction is deliberate: an additional axiom would reopen the non-vacuity theorem [Sch26c, Thm. 61] and the acyclicity of the axiom ladder [Sch26c, Prop. 39], neither of which is re-established here.

2.2 Notation and species

Throughout, $\mathfrak{C}$ denotes the molecular reduct of $A1$, $\mathfrak{Q}_{\mathfrak{C}}$ the quasi-local algebra of $A6$, $\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$ the sequence algebra of $A7$ with translation unitaries $\{u_{\mathfrak{m}}\}$ subject to $u_{\mathfrak{m}}P_{\mathfrak{k}} = P_{\mathfrak{k}+\mathfrak{m}}u_{\mathfrak{m}}$, and $\mathfrak{P}_{\mathfrak{C}} := C^*(\{P_{\mathfrak{k}}\}) \subseteq \mathfrak{Q}_{\mathfrak{C}}$ the commutative local sector. $\Delta_{\mathfrak{k}}$ denotes the localized cipher involution of $A8$. $(X, J_{\text{str}}, \mathcal{E}_0, w, \{C_j\}, \{\tau_j\})$ is the admissibility datum of $A12$, with admissible domain Ω_{adm} .

Definition 2.3 (Species). The two species of the construction are

s : the primary unit, carrying $A_{\mathfrak{C}}^{\text{pri}} = \{v_3, v_6, v_9, v_{12}\}$,

c : the dual unit, carrying $A_{\mathfrak{C}}^{\text{dual}} = \{v_2, v_5, v_8, v_{11}\}$,

and $\Sigma_{\text{sp}} := \{s, c\}$.

Remark 2.4 (Species names). [E] In the intended reading s is the substrate and c the catalyst of the reaction volume of Definition 3.10. The letters s, c are chosen in preference to the letters G, H used informally elsewhere, which have been applied to the two attachment sets in both assignments. Nothing below depends on the reading, and no chemical role

is used in any proof. Σ_{sp} instantiates the species alphabet of [Sch26a, Hyp. 2.16] and clause (N2) of [Sch26a, Def. 2.10].

Definition 2.5 (Stacking direction). $e_z \in \mathbb{Z}^3$ denotes the distinguished lattice direction of the base $X_{\mathbb{G}} \cong \mathbb{Z}^3$ of A6 along which units are stacked, and $u_{\pm e_z}$ the corresponding translation unitaries of A7.

2.3 The hypothesis register

Each hypothesis below is a realization hypothesis in the sense of [Sch26c, Def. 3]. No statement of this paper uses an unnamed hypothesis.

Hypothesis 2.6 (H_{mk} , marker realization). Each unit in a placement admits a binary geometric feature, read in the projection of Definition 6.1, whose value is determined by the placement, and the group of operations available to a unit in the stack is the group R of Definition 3.2.

Hypothesis 2.7 (H_{stack} , z -constraint). Out-of-plane reversal of a unit is unavailable to a unit occupying a site of the stack. Equivalently, R contains no operation carrying a placement to its image under a rotation by π about an in-plane axis.

Hypothesis 2.8 (H_{frm} , external frame datum). There is a structure external to the stack, not determined by it, supplying the data of Definition 6.1: an origin given by the initiator site of Definition 9.1, a direction sign $\varepsilon \in \{\pm 1\}$, and a translation-invariant projection convention.

Hypothesis 2.9 (H_{ord} , word realization). Three clauses.

- (i) There is a map $\iota_{\Sigma} : \Sigma_{\text{v}} \rightarrow \mathfrak{Q}_{\mathbb{G}}$ assigning to each letter a self-adjoint operator supported on the fibre of the origin site, such that the four operators $\iota_{\Sigma}(\sigma)$, $\sigma \in \Sigma_{\text{v}}$, are pairwise non-proportional. In particular ι_{Σ} is injective.
- (ii) The four operators commute pairwise: $[\iota_{\Sigma}(\sigma), \iota_{\Sigma}(\sigma')] = 0$ for all $\sigma, \sigma' \in \Sigma_{\text{v}}$.
- (iii) The encoded word is the alternating product of Definition 4.14, in which each letter is applied at the origin site and the reading advances by the translation unitary $u_{\varepsilon e_z}$.

Remark 2.10 (Role of the three clauses). Clause (i) is used only in Proposition 4.21; without non-proportionality the

word operator is injective only up to signs. Clause (ii) is used only in Proposition 4.18: by removing the fibre algebra as a competing source of order dependence, it ensures that the dependence established in Theorem 4.17 is attributable to the covariance relation of A7 and to nothing else. Clause (iii) is used throughout Section 4.3. A realization satisfying all three is exhibited in Appendix A.

Hypothesis 2.11 (H_{ind} , vessel independence). No constraint functional of A12 couples the letters of distinct vessels: the letter of one vessel is unconstrained by the letters of the others.

Hypothesis 2.12 (H_{bij} , bijectivity). For each $\varphi \in \Phi$ the transformation T_{φ} of Definition 8.2 is a bijection $\Sigma_{\text{v}} \rightarrow \Sigma'_{\text{v}}$.

Hypothesis 2.13 (H_{ens} , ensemble). A probability measure is given on the word space of Definition 3.18. The deterministic case is the point mass.

Hypothesis 2.14 (H_{Σ} , alphabet identification). There is a bijection $\Sigma_{\text{v}} \rightarrow \Sigma_{\mathbb{G}}$ carrying $\boxplus$ to $\boxplus$ and reg to the z -registry coordinate η of A8.

Hypothesis 2.15 (H_{chem} , chemical instantiation). The reference molecular reduct $\mathfrak{C}^{\text{chr}}$, stacked with alternating substitution as in Definition 3.1, instantiates Hypotheses 2.6–2.8 under a stated atom mapping.

Remark 2.16 (Status of H_{chem}). [E] Hypothesis 2.15 is of type [E] and is used in no proof. No theorem below becomes false, or loses a hypothesis, if it fails.

Hypothesis 2.17 (H_{sec} , structured secondary alphabet). The secondary alphabet Σ'_{v} of Definition 8.1 carries structure of its own kind: a group $G' \leq \text{Sym}(\Sigma'_{\text{v}})$ isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ arising from its own structural operations, together with the induced invariant $\text{reg}' : \Sigma'_{\text{v}} \rightarrow \mathbb{Z}/2$ and the unique fixed-point-free element $c' \in G'$ supplied by Theorem 4.9 and Proposition 4.10 applied to G' .

Remark 2.18 (Why H_{sec} is needed). Without it, Theorem 8.5 is vacuous. For any bijection T the conjugate TcT^{-1} is a fixed-point-free involution of Σ'_{v} and $\text{reg} \circ T^{-1}$ is a two-valued invariant, so transported structure always ex-

ists. The content of that theorem is agreement of transported structure with structure the secondary alphabet has independently, and Hypothesis 2.17 is what gives it something to agree with.

Hypothesis 2.19 (H_{sep} , separating state). There is a state ψ of $\mathfrak{A}_{\mathfrak{C}}$ such that the map $\mathfrak{C} \mapsto \psi \circ \text{Ad}(W(\varphi, w(\mathfrak{C})))$ is injective on the encoded stacks under consideration.

Remark 2.20 (Two senses of registry). The vessel registry reg of Definition 3.15 is a function of a pair of markers. It is not the z -registry label $\eta_{\mathbf{k}}$ on oriented z -edges introduced by A6 and carried into the alphabet $\Sigma_{\mathfrak{C}}$ by A8, and it is not the frame-setter registry coordinate Φ_{typ} of Definition 6.2. The three are kept distinct throughout; no result below identifies any two of them, and the identification of the first with the second is Hypothesis 2.14, deliberately undischarged. Naming a third object by a term already carrying two senses in the corpus would transport a name across the axiom ladder, against which [Sch26a, Rem. 2.17] warns.

3 The Encoded Stack as a Labelled Structure

3.1 Stack, placement, marker

Definition 3.1 (Stack). A *stack* of length $2N$ is a species labelling $\varsigma : \{1, \dots, 2N\} \rightarrow \Sigma_{\text{sp}}$ of consecutive sites along e_z that alternates, together with the placement data of Definition 3.2 at each site.

Definition 3.2 (Placement and allowed operations). Let $P_n \subset \mathbb{R}^3$ denote the plane of the site n , carrying the ambient orientation supplied by the projection convention of Definition 6.1. A *placement* at site n is an isometric embedding $\iota : \mathfrak{C} \rightarrow P_n$ of the labelled node of A1. Two placements are *available* to one another when they are related by an element of

$$R := \{g \in E(3) : g(P_n) = P_n, g \text{ preserves the orientation of } P_n\},$$

the group of in-plane rotations and translations. Under Hypothesis 2.7, R contains no further element; in particular it contains no rotation by π about an axis lying in P_n .

Definition 3.3 (Marker). Let $\partial G_{\mathfrak{C}}$ be the oriented external perimeter cycle of A8. For a placement ι at site n , the image $\iota(\partial G_{\mathfrak{C}})$ is a simple closed curve in P_n carrying an induced

orientation. Set

$$\text{mk}(n) := \begin{cases} F, & \text{the induced orientation agrees with that of } P_n, \\ B, & \text{otherwise.} \end{cases}$$

Proposition 3.4 (Non-collapse of the marker). [A] Assume Hypotheses 2.6 and 2.7. Then

- (i) mk is constant on the R -orbit of a placement;
- (ii) the operation relating the two values of mk does not lie in R ;
- (iii) no element of the labelled symmetry stabilizer of A2 identifies the two values.

Consequently mk takes exactly two values, and the alphabet of Definition 3.13 does not collapse.

Proof. (i) Every $g \in R$ preserves the orientation of P_n by Definition 3.2, and carries $\iota(\partial G_{\mathfrak{C}})$ to $(g \circ \iota)(\partial G_{\mathfrak{C}})$ with its induced orientation transported accordingly. Agreement or disagreement of the two orientations is therefore unchanged, which is (i).

(ii) Let θ denote rotation by π about an axis lying in P_n . Then $\theta(P_n) = P_n$ and θ reverses the orientation of P_n , so $\text{mk}(\theta \circ \iota) \neq \text{mk}(\iota)$. By Hypothesis 2.7, $\theta \notin R$, which is (ii). The same applies to any reflection of $\mathbb{R}^3$ in a plane containing the normal of P_n , these being the remaining plane-preserving orientation-reversing isometries of P_n .

(iii) Two placements ι and $\theta \circ \iota$ of opposite marker are identified exactly when there is $g \in R$ and an automorphism φ of the labelled node with $g \circ \iota = \iota \circ \varphi$ realizing θ ; that is, exactly when the labelled point group $\Pi_{\mathfrak{C}}$ of A2 contains an element acting on P_n orientation-reversingly. By [Sch26c, Eq. (747)] one has $\Gamma_{\mathfrak{C}} \leq \Gamma_{\mathfrak{C}}^{\text{emb}}$, hence $\Pi_{\mathfrak{C}} \leq \Pi_{\mathfrak{C}}^{\text{emb}}$, and by [Sch26c, Eq. (810)] $\Gamma_{\mathfrak{C}}^{\text{emb}}$ and $\Gamma_{\mathfrak{C}}^{\text{core}}$ are conjugate as Euclidean symmetry groups, so that $\Pi_{\mathfrak{C}}^{\text{emb}} \sim_{O(3)} C_{2h}^{\text{std}}$ by A2. It therefore suffices to check the four elements of $C_{2h} = \{E, C_2, i, \sigma_h\}$ in the realization in which σ_h is the molecular plane, so that P_n is the mirror plane and C_2 is the axis normal to it:

- E acts as the identity on P_n ;
- C_2 acts on P_n as rotation by π about a point, which preserves orientation;
- σ_h fixes P_n pointwise, hence acts as the identity on it;
- $i = C_2 \sigma_h$ acts on P_n as C_2 does, hence preserves orientation.

Every element of $\Pi_{\mathfrak{C}}$ therefore acts on P_n orientation-preservingly, no element realizes θ , and (iii) follows. The three statements together give the conclusion. $\square$

Remark 3.5 (Why the marker is not a bay designation). The element C_2 appearing in the proof above is realized on the molecular graph as the framework automorphism σ of [Sch26a, Prop. 2.31], which exchanges $C4a \leftrightarrow C10a$ and $C4b \leftrightarrow C10b$ and preserves both attachment sets of $A1$ setwise. It lies in R and leaves the placement unchanged, while exchanging the two bay regions of the framework together with the two internal fusion bonds. A marker defined by the presentation of the bay region adjacent to a named fusion bond is therefore carried to the other bay region by an operation leaving the placement unchanged, and is not well defined; clause (i) of Proposition 3.4 would fail for it. Definition 3.3 is stated in terms of the perimeter orientation precisely because that datum is C_2 -invariant, and it uses the oriented perimeter cycle that $A8$ already fixes rather than introducing a new one.

Remark 3.6 (Molecular reading). [E] In the intended reading the two values of mk are the two legible senses of the angular four-ring array in projection. This remark enters no proof.

Remark 3.7 (The marker is not a chirality). [A] The operation θ of the proof above is a proper rotation of $\mathbb{R}^3$, of determinant $+1$. Its exclusion is supplied by Hypothesis 2.7 and is therefore kinematic rather than a property of the symmetry stabilizer of $A2$; indeed by [Sch26c, Prop. 2] the isolated node is achiral, its point group containing the improper elements i and σ_h . Proposition 3.4 accordingly asserts nothing about chirality of the unit or of the assembly, and in particular does not bear on the emergent chirality target [Sch26c, Thm. 47], whose criterion is the absence of every improper symmetry of the completed assembly. See Target 10.4 and [Sch26c, Rem. 53].

Proposition 3.8 (The marker is information bearing). [T] *Both values of mk occur among admissible stacks. Hypotheses required: an admissibility statement on the stacking of Definition 3.1, which $A1$ – $A12$ do not supply. Target: identify that hypothesis and determine whether $A12$ can supply it through the constraint functional of Definition 9.2.*

Remark 3.9 (Two statements, deliberately separate). Proposition 3.4 is a statement about the stabilizer of a fixed placement: the marker cannot be altered. Proposition 3.8 is a statement about the range of admissible stacks: the marker is not constant across them. Only the conjunction makes the marker a carrier of information, and only the first is provable from $A1$ – $A12$. Folding the second into the first

would present an admissibility claim as a symmetry result.

3.2 Vessel and alphabet

Definition 3.10 (Vessel). A *vessel* is an adjacent pair of sites carrying one c and one s . A stack of length $2N$ together with an offset decomposes into N vessels. The vessel, and not the individual unit, carries a letter.

Remark 3.11 (Reaction volume). [E] In the intended reading the vessel is the reaction volume within which the substrate transforms under the influence of the paired catalyst. This remark enters no proof.

Definition 3.12 (The two structural binaries). Fix the frame labelling of Definition 3.3, writing $F = 0$ and $B = 1$ in $\mathbb{Z}/2$, and let $f(\varphi) \in \mathbb{Z}/2$ denote the projection reference of Definition 6.1. For a vessel v with units c, s set

$$b_1(v) := mk(c) + mk(s), \quad b_2(v) := mk(s) + f(\varphi),$$

the sums taken in $\mathbb{Z}/2$.

Definition 3.13 (Alphabet). Σ_v is the set of vessel states, written in marker coordinates as $\{FF, FB, BF, BB\}$ and in the coordinates of Definition 3.12 as pairs $(b_1, b_2) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proposition 3.14 (Equivalence of the two coordinate systems). [A] *The assignment $(mk(c), mk(s)) \mapsto (b_1, b_2)$ is an affine bijection of $\mathbb{Z}/2^2$, with linear part $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and translation part $(0, f(\varphi))$. It is linear if and only if $f(\varphi) = 0$. In particular $|\Sigma_v| = 4$ under either coordinate system, and the two labellings determine one another.*

Proof. The linear part has determinant $1 \cdot 1 - 1 \cdot 0 = 1 \neq 0$ in $\mathbb{Z}/2$, hence is invertible, with inverse equal to itself. An affine map with invertible linear part is a bijection, and is linear exactly when its translation part vanishes. $\square$

Definition 3.15 (Registry). $reg := b_1 : \Sigma_v \rightarrow \mathbb{Z}/2$. A vessel is *aligned* when $reg = 0$ and *unaligned* when $reg = 1$.

Proposition 3.16 (Registry in marker coordinates). [A] *$reg^{-1}(0) = \{FF, BB\}$ and $reg^{-1}(1) = \{FB, BF\}$, and reg is invariant under exchange of the two units of a vessel.*

Proof. Immediate from $b_1 = \text{mk}(\mathbf{c}) + \text{mk}(\mathbf{s})$: the sum vanishes exactly when the two markers agree. Invariance under exchange holds because addition in $\mathbb{Z}/2$ is commutative. $\square$

3.3 The encoded stack

Definition 3.17 (Encoded stack). An *encoded stack* is a tuple $\mathfrak{E} = (\Lambda, \varsigma, \{\text{mk}(n)\}, \varphi, \mathcal{D})$ consisting of a site set Λ in the base of A_6 , a species labelling ς as in Definition 3.1, a marker assignment, frame data $\varphi \in \Phi$, and the induced vessel decomposition $\mathcal{D}$.

Definition 3.18 (Word space). The *word* of $\mathfrak{E}$ is the sequence $w(\mathfrak{E}) = (\sigma_1, \dots, \sigma_N) \in \Sigma_v^N$ of its vessel letters, read from the frame origin in the frame direction.

Proposition 3.19 (Cardinality). [A] Under Hypothesis 2.11 the set of words of encoded stacks of N vessels on a fixed site set with fixed frame data is all of Σ_v^N , of cardinality 4^N .

Proof. By Hypothesis 2.11 no constraint functional couples distinct vessels, so the admissible set of letter assignments is a product of the admissible sets of the individual vessels. Each factor is all of Σ_v by Definition 3.13, whence the product is Σ_v^N and has cardinality 4^N . $\square$

Remark 3.20 (The instance $N = 3$). For $N = 3$ Proposition 3.19 gives $4 \times 4 \times 4 = 64$ words. This is the length used for the witness of Appendix A.

4 Operational Operators and the Interaction of the Axioms

4.1 Marker and registry operators

Definition 4.1 (Marker operator). $M_n \in \mathfrak{Q}_{\mathfrak{E}}$ is the self-adjoint operator with spectrum $\{+1, -1\}$ supported on the fibre of site n whose eigenvalue records $\text{mk}(n)$, and $P_n^{\pm} := \frac{1}{2}(1 \pm M_n)$ its spectral projections.

Definition 4.2 (Registry operator). $D_n := M_n M_{n+1}$, defined for every adjacent pair of sites.

Proposition 4.3 (Registry is canonical). [A] D_n is self-adjoint with spectrum contained in $\{+1, -1\}$; it is independent of the vessel decomposition, being defined on every adjacent pair; and it is invariant under the relabelling

$M_k \mapsto -M_k$ for all k , which is the action of global marker reversal. The registry of a vessel is the restriction of (D_n) to the pairs the decomposition selects.

Proof. The operators M_n and M_{n+1} are supported on distinct sites, hence act on distinct tensor factors of $\mathfrak{Q}_{\mathfrak{E}}$ and commute; a product of two commuting self-adjoint involutions is a self-adjoint involution, giving the first assertion. Definition 4.2 refers only to the pair $(n, n+1)$ and to no decomposition, giving the second. Under simultaneous relabelling, $(-M_n)(-M_{n+1}) = M_n M_{n+1}$, giving the third. The last assertion is Definition 3.10 together with Definition 3.15. $\square$

Proposition 4.4 (What depends on the frame). [A] Of the two coordinates of Definition 3.12, b_1 is independent of the frame and b_2 is determined by it: a change of the projection reference $f \mapsto f+1$ fixes b_1 and translates b_2 by 1. Consequently the partition of Σ_v into aligned and unaligned letters is canonical, while the designation of a zero of Σ_v is not, and Σ_v becomes an $\mathbb{F}_2$ -vector space exactly when the frame designates one.

Proof. $b_1 = \text{mk}(\mathbf{c}) + \text{mk}(\mathbf{s})$ does not involve f , and is unchanged under the relabelling $\text{mk} \mapsto \text{mk}+1$ of both units induced by $f \mapsto f+1$, since the two increments cancel; this is Proposition 4.3 in coordinates. For the second coordinate, $b_2 = \text{mk}(\mathbf{s}) + f$ increments with f directly. The partition of Σ_v is the fibre partition of b_1 by Definition 3.15, hence canonical. A torsor under $\mathbb{Z}/2 \times \mathbb{Z}/2$ carries a vector-space structure exactly upon designation of a zero, which by the foregoing is a frame datum. $\square$

4.2 The structural group

Definition 4.5 (Structural operations). $\tau : \Sigma_v \rightarrow \Sigma_v$ is exchange of the two units of a vessel, $\rho : \Sigma_v \rightarrow \Sigma_v$ global marker reversal, and δ reversal of the reading direction.

Lemma 4.6 (Coordinate form). [A] In the coordinates of Definition 3.12,

$$\tau(b_1, b_2) = (b_1, b_1 + b_2), \quad \rho(b_1, b_2) = (b_1, b_2 + 1),$$

and δ acts on a single letter as τ .

Proof. τ exchanges $\text{mk}(\mathbf{c})$ and $\text{mk}(\mathbf{s})$. It fixes b_1 , a symmetric function of the two markers, and carries $b_2 = \text{mk}(\mathbf{s}) + f$ to $\text{mk}(\mathbf{c}) + f = (\text{mk}(\mathbf{s}) + b_1) + f = b_1 + b_2$. Global marker reversal sends both markers to their increments, fixing b_1 and incrementing b_2 . Reversal of the reading direction reverses the order in which the two units of a vessel are read, which

on a single letter is the exchange τ ; its further effect is on the order of the word and not on the letter. $\square$

Proposition 4.7 (The structural group). [A] τ and ρ are commuting involutions, $\delta \in \langle \tau, \rho \rangle$, and

$$G := \langle \tau, \rho, \delta \rangle = \{\text{id}, \tau, \rho, \tau\rho\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \leq \text{Aff}(\Sigma_v) \cong S_4.$$

Moreover ρ is a translation, hence affine and not linear for any designated zero, and τ is linear and is a transposition of Σ_v .

Proof. By Lemma 4.6, $\tau^2(b_1, b_2) = (b_1, b_1 + (b_1 + b_2)) = (b_1, b_2)$ and $\rho^2 = \text{id}$. Both composites $\tau\rho$ and $\rho\tau$ send (b_1, b_2) to $(b_1, b_1 + b_2 + 1)$, so τ and ρ commute and the group they generate has order at most four; it has order exactly four because τ, ρ and $\tau\rho$ are pairwise distinct, as their fixed-point sets below differ. Every non-identity element is an involution, so $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. That $\text{Aff}(\Sigma_v) \cong S_4$ is the standard identification of the affine group of the two-dimensional affine space over $\mathbb{F}_2$ with the symmetric group on its four points. The operator ρ is translation by $(0, 1)$ and therefore fixes no point, so it is not linear for any choice of zero; τ is linear by Lemma 4.6 and by Proposition 4.8 below fixes two points and exchanges the other two, hence is a transposition. $\square$

Proposition 4.8 (Fixed-point sets). [A] τ fixes exactly the two aligned letters, $\tau\rho$ exactly the two unaligned letters, and ρ fixes no letter.

Proof. By Lemma 4.6, $\tau(b_1, b_2) = (b_1, b_2)$ iff $b_1 = 0$, which by Definition 3.15 is the aligned class; $\tau\rho(b_1, b_2) = (b_1, b_2)$ iff $b_1 + 1 = 0$, that is $b_1 = 1$, the unaligned class; and $\rho(b_1, b_2) = (b_1, b_2)$ iff $1 = 0$ in $\mathbb{Z}/2$, which is false. $\square$

Theorem 4.9 (Registry is the terminal invariant). [A] The orbits of G on Σ_v are exactly $\text{reg}^{-1}(0) = \{FF, BB\}$ and $\text{reg}^{-1}(1) = \{FB, BF\}$. Consequently, for every set Y and every G -invariant $h : \Sigma_v \rightarrow Y$ there is a unique $\bar{h} : \mathbb{Z}/2 \rightarrow Y$ with $h = \bar{h} \circ \text{reg}$; that is, reg is the terminal structural invariant.

Proof. Every element of G fixes the first coordinate, by Lemma 4.6, so each orbit is contained in a fibre of $b_1 = \text{reg}$. Conversely ρ acts transitively on each such fibre, since it increments the second coordinate and the fibres have two elements. Hence the orbits are exactly the fibres of reg . A function is G -invariant iff it is constant on orbits, hence iff it is constant on the fibres of reg , hence iff it factors through reg ; the factorization is unique because reg is surjective. That reg itself is G -invariant is the same computation, so the universal property is witnessed. $\square$

Proposition 4.10 (Complementation). [A] ρ is the unique fixed-point-free element of G . Setting $c := \rho$, the map c is an involution without fixed points, its orbits are the two registry classes, and $\text{reg} \circ c = \text{reg}$.

Proof. Uniqueness is Proposition 4.8, which exhibits the fixed-point sets of all four elements of G : only ρ has empty fixed-point set. The remaining assertions were established in the proof of Theorem 4.9. $\square$

Remark 4.11 (Complementary pairs). [E] A perfect pairing of a four-letter alphabet is exactly a fixed-point-free involution of it, so by Proposition 4.10 the structure supplies exactly one candidate pairing, and that pairing preserves the registry invariant. Any reading of the two orbits as the complementary pairs of a paired biological alphabet is an empirical instantiation claim and is used in no proof.

Remark 4.12 (Numerical verification). The group law of Proposition 4.7, the fixed-point sets of Proposition 4.8, the orbit computation of Theorem 4.9 and the uniqueness assertion of Proposition 4.10 were verified by exhaustive enumeration over the four letters and the twenty-four elements of $\text{Aff}(\Sigma_v)$. The enumeration confirms $|G| = 4$ with all non-identity elements of order two, $|\text{Aff}(\Sigma_v)| = 24$ with eight elements of order three, and G containing none of them.

4.3 The word operator and sequence dependence

Definition 4.13 (Letter operators). Under Hypothesis 2.9(i)–(ii), $\iota_\Sigma : \Sigma_v \rightarrow \mathfrak{Q}_\mathbb{C}$ assigns to each letter a self-adjoint operator supported on the fibre of the origin site $\mathbf{k}_0$, and $\iota_{\mathbf{k}}(\sigma) := \alpha_{\mathbf{k}-\mathbf{k}_0}(\iota_\Sigma(\sigma))$ denotes its localized copy at $\mathbf{k}$, where α is the translation action of A_6 .

Definition 4.14 (Word operator). Under Hypothesis 2.9(iii), let φ have direction sign ε and write $u := u_{\varepsilon e_z}$. The word operator of the word $w = (\sigma_1, \dots, \sigma_N)$ read from the origin $\mathbf{k}_0$ is

$$W(\varphi, w) := \iota_\Sigma(\sigma_1) u \iota_\Sigma(\sigma_2) u \cdots u \iota_\Sigma(\sigma_N) \in \mathfrak{A}_\mathbb{C},$$

each letter being applied at the origin site and the reading advancing by one translation between consecutive letters.

Remark 4.15 (Reading head). Definition 4.14 realizes the reading of a stack as the alternation of two operations: application of a letter at a fixed site, and advance of the

frame by one step. It does not presuppose that letters are attached to distinct sites; Lemma 4.16 derives that picture from this one, and the derivation is exactly the covariance relation of A7. The reading direction enters as the sign ε and not as a convention on the string.

Lemma 4.16 (Normal form). [A] *With the notation of Definition 4.14 and $\mathbf{k}_j := \mathbf{k}_0 + j \varepsilon e_z$,*

$$W(\varphi, w) = \left(\prod_{j=0}^{N-1} \iota_{\mathbf{k}_j}(\sigma_{j+1}) \right) u^{N-1},$$

the factors of the product commuting pairwise.

Proof. By the covariance relation of A7, $u A u^* = \alpha_{\varepsilon e_z}(A)$ for $A \in \mathfrak{Q}_{\mathfrak{C}}$, so that $A u = u \alpha_{-\varepsilon e_z}(A)$ and, equivalently, $u^j \iota_{\Sigma}(\sigma) = \iota_{\mathbf{k}_j}(\sigma) u^j$. Moving each translation unitary of Definition 4.14 to the right past the letters standing to its right, and collecting, gives the stated identity by induction on N : the base case $N = 1$ is trivial, and the inductive step applies the displayed relation once. The factors $\iota_{\mathbf{k}_j}(\sigma_{j+1})$ are supported on the pairwise distinct sites $\mathbf{k}_0, \dots, \mathbf{k}_{N-1}$, hence act on distinct tensor factors of $\mathfrak{Q}_{\mathfrak{C}}$ and commute pairwise, as in the proof of [Sch26c, Cor. 2]. $\square$

Theorem 4.17 (Sequence dependence of the encoded word). [D] *Assume Hypothesis 2.9. Then for $N \geq 2$ there exist words $w \neq w'$ over Σ_v of length N with the same multiset of letters and*

$$W(\varphi, w) \neq W(\varphi, w').$$

Moreover the map $w \mapsto W(\varphi, w)$ is order sensitive precisely because u fails to commute with localization: if the covariance relation of A7 were replaced by $[u, \iota_{\Sigma}(\sigma)] = 0$ for every σ , then under Hypothesis 2.9(ii) the word operator would depend only on the multiset of letters of w .

Proof. Existence. Let w and w' differ by the transposition of two distinct letters $\sigma \neq \sigma'$ in positions $j \neq j'$. By Lemma 4.16 and unitarity of u^{N-1} , $W(\varphi, w) = W(\varphi, w')$ holds iff the two products of localized letters agree. Those products are elementary tensors with respect to the site decomposition, with j -th factors $\iota_{\Sigma}(\sigma)$ and $\iota_{\Sigma}(\sigma')$ respectively. Equality of two nonzero elementary tensors forces the corresponding factors to be proportional, and by Hypothesis 2.9(i) distinct letters have non-proportional operators; hence the products differ and $W(\varphi, w) \neq W(\varphi, w')$.

Attribution. Suppose instead $[u, \iota_{\Sigma}(\sigma)] = 0$ for every σ . Then no translation moves a letter off the origin site, and moving every u in Definition 4.14 to the right leaves every letter at $\mathbf{k}_0$,

giving

$$W(\varphi, w) = \iota_{\Sigma}(\sigma_1) \iota_{\Sigma}(\sigma_2) \cdots \iota_{\Sigma}(\sigma_N) u^{N-1}.$$

By Hypothesis 2.9(ii) the factors of this product commute pairwise, so the product is invariant under permutation of its factors and depends only on the multiset of letters. The order sensitivity established above is therefore attributable to the failure of that commutation, which is [Sch26c, Thm. 8] and, for the cipher involution, [Sch26c, Cor. 2]. $\square$

Proposition 4.18 (The non-implication that makes H_{ord} load bearing). [A] *Order dependence of a read word and noncommutativity of the sequence algebra are independent properties:*

order dependence of a read word $\Rightarrow$ noncommutativity of $\mathfrak{A}_{\mathfrak{C}}$,

and conversely. A sequence of symbols is order dependent as a tuple whatever algebra it is read into, and by the attribution clause of Theorem 4.17 a word realized without clause (iii) of Hypothesis 2.9 is order blind although $\mathfrak{A}_{\mathfrak{C}}$ remains noncommutative. Clauses (ii) and (iii) of Hypothesis 2.9 are therefore exactly the point at which the order dependence of the encoding is made to coincide with the sequence-induced noncommutativity of A7, and the coincidence is a consequence of those clauses rather than of either property alone.

Proof. The first non-implication holds because tuples are distinguished by order in any case, so the antecedent carries no information about the algebra into which they are read. The second is the attribution clause of Theorem 4.17: there $\mathfrak{A}_{\mathfrak{C}}$ is unchanged and noncommutative, while the word operator is order blind. $\square$

Remark 4.19 (Quantitative form of the collapse). On the witness of Appendix A, with $N = 3$ and the four letter operators taken to be the characters of $\mathbb{Z}/2 \times \mathbb{Z}/2$ on the fibre, the 64 words of Proposition 3.19 yield 64 distinct word operators. Under the counterfactual of the attribution clause of Theorem 4.17 they yield 4. The encoding of a three-vessel stack therefore loses four of its six bits when the translation unitaries are removed, and what remains is the sum of the letters in $\mathbb{Z}/2 \times \mathbb{Z}/2$. The collapse is to fewer than the twenty multisets of three letters because the chosen letter operators form a group under multiplication; a realization whose letters generate freely would collapse to the multisets and no further.

Proposition 4.20 (Directional distinguishability). [D] *$u_{+e_z} \neq u_{-e_z}$, and there are words w with $W(\varphi, w) \neq$*

$W(\varphi', w)$ where φ' differs from φ only in the direction sign. The reading direction is a datum of the word operator.

Proof. The translation unitaries satisfy $u_{-e_z} = u_{e_z}^* \neq u_{e_z}$, since $u_{e_z}^2 = u_{2e_z} \neq 1$ by the faithfulness of the translation action of A6 on the base $X_{\mathbb{G}} \cong \mathbb{Z}^3$. Applying Lemma 4.16 with the two signs places the letters of w on the sites $\mathbf{k}_0 + je_z$ and $\mathbf{k}_0 - je_z$ respectively, which are distinct for $j \geq 1$; the two elementary tensors are supported on different site sets and, for w not constant, differ. $\square$

Proposition 4.21 (Injectivity of the word operator). **[D]** Under Hypothesis 2.9, the map $w \mapsto W(\varphi, w)$ is injective on Σ_v^N for each fixed φ and each N .

Proof. By Lemma 4.16 and unitarity of u^{N-1} , it suffices to show that $w \mapsto \prod_j \iota_{\mathbf{k}_j}(\sigma_{j+1})$ is injective. Each such product is an elementary tensor with respect to the decomposition of $\mathfrak{Q}_{\mathbb{G}}$ over the sites $\mathbf{k}_0, \dots, \mathbf{k}_{N-1}$, with nonzero factors by Hypothesis 2.9(i). If two such products agree then their factors agree up to scalars λ_j with $\prod_j \lambda_j = 1$; that is, $\iota_{\Sigma}(\sigma_{j+1}) = \lambda_j \iota_{\Sigma}(\sigma'_{j+1})$ for each j . By Hypothesis 2.9(i) the four letter operators are pairwise non-proportional, so proportionality forces $\sigma_{j+1} = \sigma'_{j+1}$ and hence $\lambda_j = 1$. Therefore $w = w'$. $\square$

Remark 4.22 (Where non-proportionality is needed). Without the non-proportionality clause of Hypothesis 2.9(i) the argument above yields injectivity only up to the scalars λ_j , and the conclusion genuinely fails: if two letters have operators differing by a sign, transposing them in two positions leaves the word operator unchanged. The clause is therefore necessary and not merely convenient.

Remark 4.23 (Numerical verification). Theorem 4.17, Lemma 4.16, Proposition 4.20 and Proposition 4.21 were checked on an explicit finite model: four sites, a four-dimensional fibre, the four letter operators taken to be the characters of $\mathbb{Z}/2 \times \mathbb{Z}/2$, and u the cyclic shift of tensor factors. The model satisfies covariance to machine precision and $[u, \iota_{\mathbf{k}_0}(A)] \neq 0$; the letters commute pairwise and are pairwise non-proportional; the normal form of Lemma 4.16 holds identically for all 64 words of length three; those words give 64 distinct operators, and 4 under the counterfactual; and the two words (FF, FB, BF) and (FF, BF, FB) have equal letter multisets, distinct word operators under u , and equal word operators without u .

4.4 How the axioms interact

Proposition 4.24 (Provenance). **[A]** Each constituent of Definitions 3.17 and 4.14 is supplied by a named axiom, as follows.

Constituent	Supplied by
site set Λ , translation action	A6
species alphabet, attachment sets	A1
non-collapse of the marker	A2
oriented perimeter cycle	A8
fibre algebra, localization $\iota_{\mathbf{k}}$	A6
translation unitaries, covariance	A7
constraint functionals, Ω_{adm}	A12

No constituent is supplied by A3, A4, A5, A9, A10 or A11.

Proof. By inspection of the cited definitions against the statements of the axioms as recalled in Section 2. For the negative assertion: no statement of Sections 3–4 refers to the molecular embedding data of A3, to the tensorial constructions of A4–A5, to the spectral triple of A9, or to the Hamiltonian and response layers of A10–A11. $\square$

Remark 4.25 (Reading of Proposition 4.24). The encoded stack is a construction over the molecular reduct and the fibrated sequence layers alone. In particular it is independent of A9–A11, which is why Section 9 must reach the transfer architecture of [Sch26a] through the instance schema rather than through any shared construction: the transfer realizations of that paper are built from A10 data, which the encoding does not use.

Proposition 4.26 (Acyclicity of the internal order). **[A]** Write $S_i \prec S_j$ when the statement of S_j uses an object first introduced by S_i . Then $\prec$ generates an acyclic partial order on the statements of Sections 3–4, one linear extension being

$$3.1 \prec 3.2 \prec 3.3 \prec 3.4 \prec 3.10 \prec 3.12 \prec 3.13 \prec 3.15$$

$$\prec 4.1 \prec 4.2 \prec 4.5 \prec 4.6 \prec 4.7 \prec 4.9 \prec 4.13 \prec 4.14 \prec 4.16 \prec 4.17,$$

and no statement of this paper is used as a hypothesis of a statement preceding it in that order. The order is compatible with the axiom ladder of [Sch26c, Prop. 39] in the sense that the provenance assignment of Proposition 4.24 is monotone for $\prec$.

Proof. The displayed sequence is a linear extension: each statement uses only objects introduced by statements preceding it, as may be read off from the definitions and from the proofs above. A relation admitting a linear extension is acyclic. Monotonicity of the provenance assignment is checked against the table of Proposition 4.24 together with the ladder

$A1 \prec A2 \prec A6 \prec A7 \prec A8 \prec A12$ of [Sch26c, §3.1]. $\square$

5 Existence

Theorem 5.1 (Non-vacuity of the encoded stack). **[D]** *There exists an encoded stack $\mathfrak{E}_{\text{wit}}$ of $N = 3$ vessels, constructed explicitly in Appendix A, which satisfies Definitions 3.1–3.18 and Definitions 4.1–4.14; verifies Hypotheses 2.6, 2.7, 2.8, 2.9 and 2.11; lies in the admissible domain of Definition 9.2; and exhibits in closed form its markers, its registries, its letters under each of the four frame-setters, its letter operators and its word operator. Consequently the class of encoded stacks is nonempty, and every statement of Sections 3–4 has an instance.*

Proof. By exhibition. Definition A.1 supplies the site set, the species labelling and the marker assignment; Definition A.3 the initiator and the frame datum; Proposition A.4 the value of the constraint functional; Proposition A.5 the letters; Definition A.9 and Proposition A.10 the letter operators together with the verification of the three clauses of Hypothesis 2.9; and Proposition A.11 the word operator. Hypothesis 2.11 holds because Definition 9.2 supplies the only constraint functional of the construction and it is a function of the initiator count alone, hence couples no two vessels. $\square$

Remark 5.2 (What the witness does and does not establish). $\mathfrak{E}_{\text{wit}}$ witnesses non-vacuity of the constructions of Sections 3–4 and of nothing further. It does not instantiate Hypothesis 2.15: the fibre of Definition A.9 is chosen for closed-form computability and no molecular datum enters it, and no statement of Appendix A is of type **[E]**. It does not re-establish the non-vacuity of the twelve axioms, which remains [Sch26c, Thm. 61]. It does not instantiate Hypotheses 2.12, 2.13 or 2.14, and it exhibits no transformation. It supplies neither clause (N4) nor clause (N5) of Remark 9.12, so it is not an instance of a fibrated transfer network in the sense of [Sch26a, Def. 10.4].

Remark 5.3 (Three distinct witnesses). The explicit model $\mathfrak{R}_{\text{ex}}$ of [Sch26c, App. B], the witness realization of [Sch26a, App. A], and $\mathfrak{E}_{\text{wit}}$ are different objects constructed for different purposes, and no one of them witnesses the constructions of the others. In particular $\mathfrak{R}_{\text{ex}}$ has infinite gain by [Sch26a, Rem. 4.6] and therefore witnesses nothing in that paper, while $\mathfrak{E}_{\text{wit}}$ uses no A10 datum and so has no gain at all.

Remark 5.4 (The witness settles a structural question). Corollary A.8 computes the map $z : \Phi \rightarrow \Sigma_0$ carrying a frame-setter to the zero it designates, and finds it two-to-

one onto the aligned class. The coincidence that Φ and Σ_0 are both four-element sets acted on by $\mathbb{Z}/2 \times \mathbb{Z}/2$ is therefore not induced by a canonical bijection, which is the non-implication recorded in Proposition 6.5.

6 The Frame of Reference

6.1 Frame data

Definition 6.1 (Frame). Under Hypothesis 2.8, a *frame* is a triple $(\mathbf{k}_0, \varepsilon, f)$ consisting of an origin $\mathbf{k}_0 \in \Lambda$, a direction sign $\varepsilon \in \{\pm 1\}$, and a projection reference $f \in \mathbb{Z}/2$ fixing which value of mk is labelled F and supplying the reference of Definition 3.12. All three are supplied by the initiator of Definition 9.1.

Definition 6.2 (Frame-setters). $\Phi := \Phi_{\text{dir}} \times \Phi_{\text{typ}}$, of cardinality four, where Φ_{dir} is the side binary of the frame-setter, carrying ε , and Φ_{typ} its registry binary, carrying f .

Definition 6.3 (Offset). The *offset* of a frame is the parity $o \in \mathbb{Z}/2$ of its origin within the stack. It determines the vessel decomposition $\mathcal{D}$ of Definition 3.10: on a stack of $2N$ sites, the offset $o = 0$ gives the N vessels $(n_1, n_2), \dots, (n_{2N-1}, n_{2N})$, and the offset $o = 1$ gives the $N - 1$ vessels $(n_2, n_3), \dots, (n_{2N-2}, n_{2N-1})$ together with the two unpaired terminal units n_1 and n_{2N} .

Remark 6.4 (The offset is not a coordinate of Φ). The offset is carried by the position of the initiator and not by which of the four frame-setters occupies it. Reversal of ε reverses the order in which the vessels of a fixed decomposition are read and does not repartition the stack, since the reversal of the pairing (n_{2i-1}, n_{2i}) is the pairing (n_{2i}, n_{2i-1}) of the same two sites. The two data are therefore independent, and Propositions 6.7 and 6.10 treat them separately.

6.2 The structure of Φ

Proposition 6.5 (Structure of Φ). **[A]** Φ is a torsor under $\mathbb{Z}/2 \times \mathbb{Z}/2$, the two generators acting by reversal of the side binary and of the registry binary respectively. Moreover

- (i) Φ_{typ} is canonically pointed, its distinguished value being the aligned one;
- (ii) Φ_{dir} is not canonically pointed: the two frame-setter locant triples $\{6, 9, 12\}$ and $\{3, 6, 12\}$ are exchanged by the framework automorphism σ of [Sch26a, Prop. 2.31], so that no side is distinguishable within a

unit and the distinction must be supplied externally, as Hypothesis 2.8 provides;

- (iii) $\Phi_{\text{typ}} \neq \text{reg}$: the first is computed from the composition of the frame-setter, the second from a pair of markers by Definition 3.15, and no result of this paper identifies them.

Proof. Each of the two binaries admits a reversal, the two reversals commute and act freely, and together they act transitively on the four frame-setters; hence Φ is a torsor under $\mathbb{Z}/2 \times \mathbb{Z}/2$. For (i), alignment is a relation of agreement between two data of the frame-setter and is therefore a difference-type binary, whose vanishing is canonically distinguished, exactly as in Proposition 4.3. For (ii), σ acts on the primary locants by $3 \leftrightarrow 9$ and $6 \leftrightarrow 12$, whence $\sigma(\{6, 9, 12\}) = \{12, 3, 6\} = \{3, 6, 12\}$; since $\sigma \in \Pi_{\mathbb{C}}^{\text{core}}$ the two triples are related by a symmetry of the framework and neither is distinguished by it. Statement (iii) is a statement about what has and has not been proved, and is recorded as a non-implication in Proposition 10.1(vi). $\square$

Remark 6.6 (The two half-pointed torsors). By Propositions 4.4 and 6.5, both Σ_v and Φ are torsors under $\mathbb{Z}/2 \times \mathbb{Z}/2$ carrying exactly one canonically pointed coordinate: the registry coordinate in each case. The two are nevertheless not canonically isomorphic. By Corollary A.8 the map $z : \Phi \rightarrow \Sigma_v$ carrying a frame-setter to the zero it designates is two-to-one onto the aligned class, and it is the *pointed* coordinate of Φ that determines the *unpointed* coordinate of Σ_v : the canonical datum of the frame supplies exactly the datum the alphabet lacks. This is the precise sense in which the coincidence of groups is structural without being an identification.

6.3 The action of a frame change

Proposition 6.7 (Decomposition of the frame action). [A] *A change of frame decomposes into three commuting parts, which act as follows.*

- (i) $f \mapsto f + 1$ acts letterwise as ρ .
- (ii) $\varepsilon \mapsto -\varepsilon$ reverses the order of the word and acts letterwise as τ .
- (iii) $o \mapsto o + 1$ repartitions the stack and induces no map $\Sigma_v \rightarrow \Sigma_v$.

Consequently the group of frame changes generated by (i) and (ii) maps isomorphically onto the structural group G of Proposition 4.7, while (iii) has no letterwise action at all.

Proof. (i) By Definition 3.12, $f \mapsto f + 1$ fixes b_1 and translates b_2 by 1, which is ρ by Lemma 4.6. (ii) By Lemma 4.6 the reversal of the reading order within a vessel is τ ; the reversal of the order of the vessels is a permutation of positions of the word and not a map of the alphabet. (iii) By Definition 6.3 the two offsets have different vessel sets, and for $N \geq 2$ different vessel counts, so no map $\Sigma_v \rightarrow \Sigma_v$ is induced. The final assertion holds because the assignment $(f\text{-flip}, \varepsilon\text{-flip}) \mapsto (\rho, \tau)$ carries commuting generators to commuting generators and $\rho, \tau, \tau\rho$ are pairwise distinct and distinct from the identity by Proposition 4.8, so the induced homomorphism $\mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow G$ is injective, hence bijective. $\square$

Corollary 6.8 (The four frame-setters are distinguishable by effect). [D] *The map carrying a frame-setter to its letterwise effect on Σ_v is injective. Equivalently, the four frame-setters are pairwise distinguishable by the word they read from a suitable stack, as exhibited on the witness in Proposition A.7.*

Proof. Injectivity is the last assertion of Proposition 6.7, the image being G of order four. The exhibition is Proposition A.7. $\square$

Corollary 6.9 (The designated zero is aligned). [A] *For every frame, the designated zero of Σ_v lies in $\text{reg}^{-1}(0)$.*

Proof. The zero is the letter with $(b_1, b_2) = (0, 0)$, and $b_1 = 0$ is by Definition 3.15 the aligned class. Equivalently, by Proposition 4.4 the frame designates a zero only through the coordinate b_2 , the coordinate b_1 being canonically pointed at its zero. $\square$

6.4 Offsets, and the extent of the reading-frame analogy

Proposition 6.10 (Distinct offsets, distinct decompositions). [A] *The two offsets of a stack of $2N$ sites give distinct vessel decompositions of the same physical stack, with N and $N - 1$ vessels respectively, and the vessels of one offset are not vessels of the other.*

Proof. Immediate from Definition 6.3: a vessel of offset 0 is a pair (n_{2i-1}, n_{2i}) and a vessel of offset 1 is a pair (n_{2i}, n_{2i+1}) , and no pair is of both forms. $\square$

Proposition 6.11 (Reframing is invertible). [A] *The word of an encoded stack at one offset determines the marker assignment on the sites its vessels cover, and hence determines the word at the other offset on the sites covered by both. Consequently a change of offset is a change of description and not a loss of information.*

Proof. By Definition 3.12 and Proposition 3.14, the letter of a vessel determines the ordered pair of markers of its two units, the coordinate change being invertible. The vessels of one offset cover a consecutive run of sites, so their letters determine the marker assignment on that run, and the letters at the other offset are functions of that assignment. $\square$

Remark 6.12 (The analogy with the genetic reading frame, and its limit). The offset of Definition 6.3 is a reading frame in the sense that the base-four structure belongs to the stack together with a frame and never to the stack alone: the word is a function of both, and Proposition 6.10 shows the decompositions are genuinely different. The analogy stops at Proposition 6.11. In the genetic case a shift of reading frame is not invertible, because a codon is a triple of letters drawn from an alphabet whose letters carry no record of their position within the codon; here a letter is an ordered pair of markers and the pair is recoverable from the letter, so the marker string, and with it every other reading, is determined. The non-implication

$$\text{offset ambiguity} \not\Rightarrow \text{loss of information}$$

is recorded in Proposition 10.1. What the offset changes is which invariants are directly read, not which exist.

6.5 The maximal frame invariant

Definition 6.13 (The frame group). Fr denotes the group generated by the global marker flip $\text{mk} \mapsto \text{mk} + 1$ and by reversal of the site index, acting on marker assignments. By Proposition 6.7 these are the actions of $f \mapsto f + 1$ and $\varepsilon \mapsto -\varepsilon$ at the level of markers.

Theorem 6.14 (Maximal frame invariant). $[D]$ Let $\mathbf{m} = (\text{mk}(n_1), \dots, \text{mk}(n_{2N}))$ be the marker assignment of an encoded stack and $D(\mathbf{m}) = (D_1, \dots, D_{2N-1})$ its registry sequence. Then

- (i) D is unchanged by the global flip and is reversed by index reversal, hence the class $[D(\mathbf{m})]$ modulo reversal is Fr -invariant and independent of the offset;
- (ii) $D(\mathbf{m}) = D(\mathbf{m}')$ if and only if $\mathbf{m}' \in \{\mathbf{m}, \mathbf{m} + 1\}$;
- (iii) $[D(\mathbf{m})] = [D(\mathbf{m}')]$ if and only if $\mathbf{m}$ and $\mathbf{m}'$ lie in the same Fr -orbit.

Consequently $\mathbf{m} \mapsto [D(\mathbf{m})]$ is the orbit map of Fr , and every frame-invariant function of an encoded stack factors uniquely through it: the registry sequence modulo reversal is the maximal frame invariant.

Proof. (i) $D_n = \text{mk}(n_n) + \text{mk}(n_{n+1})$ in $\mathbb{Z}/2$, so incrementing every marker leaves each D_n unchanged, and reversing the index order carries D to its reversal. Independence of the offset holds because D is defined on every adjacent pair, by Definition 4.2, and refers to no decomposition, by Proposition 4.3.

(ii) The map $\mathbf{m} \mapsto D(\mathbf{m})$ is a homomorphism of groups $\mathbb{Z}/2^{2N} \rightarrow \mathbb{Z}/2^{2N-1}$ whose kernel is the set of constant strings $\{\mathbf{0}, \mathbf{1}\}$: a string with all consecutive differences zero is constant. Hence its fibres are the cosets $\{\mathbf{m}, \mathbf{m} + 1\}$.

(iii) If $\mathbf{m}' \in \text{Fr} \cdot \mathbf{m}$ then $\mathbf{m}'$ is one of $\mathbf{m}, \mathbf{m} + 1, \text{rev}(\mathbf{m}), \text{rev}(\mathbf{m}) + 1$, and in each case $[D(\mathbf{m}')] = [D(\mathbf{m})]$ by (i). Conversely $[D(\mathbf{m}')] = [D(\mathbf{m})]$ means $D(\mathbf{m}')$ equals $D(\mathbf{m})$ or its reversal, that is $D(\mathbf{m}')$ equals $D(\mathbf{m})$ or $D(\text{rev}(\mathbf{m}))$; applying (ii) in each case gives $\mathbf{m}' \in \text{Fr} \cdot \mathbf{m}$.

The final assertion is the universal property of an orbit map: a function is invariant precisely when it is constant on orbits, and

(iii) identifies the orbits with the fibres of $\mathbf{m} \mapsto [D(\mathbf{m})]$. $\square$

Remark 6.15 (Numerical verification). For $2N = 6$ the group Fr has 20 orbits on the 64 marker assignments, and the registry sequences modulo reversal fall into exactly 20 classes, the orbit map being injective on orbits; clauses (i) and (ii) of Theorem 6.14 were checked on all 64 assignments and all 64^2 pairs. The stabiliser characterisation used in Theorem 8.5 below was checked by enumerating all 24 permutations of Σ_v .

Remark 6.16 (What Theorem 6.14 is for). Any quantity that a later paper attributes to the assembly rather than to the reading of it must be a function of $[D]$. This is the precondition for Section 9: the candidate survival invariants of Target 10.8 are admissible candidates precisely because registry, and every function of the registry sequence, passes this test, while the word itself does not.

Remark 6.17 (Forward reference). $[E]$ In the intended reading the two coordinates of Φ correspond to the direction in which the secondary structure is read and to its type. Nothing in this paper depends on that reading, no correspondence is asserted, and the construction of the map that would realize it is Target 10.5.

7 The Base-Four Status of the Alphabet

This section is a status clarification and not a discharge. Its purpose is to record which parts of the four-element structure of Σ_v are determined by the construction and which are not, so that the standing non-implication $|\Sigma_{\mathcal{C}}| = 4 \not\Rightarrow \Sigma_{\mathcal{C}} \cong GF(4)$ of [Sch26b, Eq. (2)] is neither strengthened nor weakened by the encoding.

Proposition 7.1 (Additive structure). [A] Σ_v is canonically a torsor under $\mathbb{Z}/2 \times \mathbb{Z}/2$. Once a frame designates a zero it is an $\mathbb{F}_2$ -vector space of dimension two, and $(\Sigma_v, \boxplus) \cong (GF(4), +)$ as additive groups.

Proof. The torsor structure is Proposition 4.4; a torsor under an abelian group becomes that group upon designation of a zero. The additive group of $GF(4)$ is elementary abelian of order four, hence isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. $\square$

Proposition 7.2 (Registry is the absolute trace). [A] Fix a frame, so that Σ_v is an $\mathbb{F}_2$ -vector space with zero 0, and let **1** denote the nonzero aligned letter, which exists by Corollary 6.9. Then $\ker(\text{reg}) = \{0, \mathbf{1}\}$, and for any $GF(4)$ structure on Σ_v whose prime subfield is $\ker(\text{reg})$, the map reg coincides with the absolute trace $\text{Tr}(x) = x + x^2$.

Proof. By Corollary 6.9 the zero is aligned, so the aligned class $\text{reg}^{-1}(0)$ contains 0 and one further element **1**, and is a subgroup of order two; this is $\ker(\text{reg})$, reg being a homomorphism onto $\mathbb{Z}/2$ because $\text{reg} = b_1$ is a coordinate. In $GF(4)$ the absolute trace is $\mathbb{F}_2$ -linear and surjective with $\ker(\text{Tr}) = GF(2)$, the prime subfield. Two surjective homomorphisms from a group of order four onto $\mathbb{Z}/2$ with the same kernel are equal, whence $\text{reg} = \text{Tr}$. $\square$

Lemma 7.3 (The three field structures). [A] With the additive structure of Proposition 7.1 fixed, there are exactly three multiplications making Σ_v a field isomorphic to $GF(4)$, one for each choice of the identity element among the three nonzero letters.

Proof. Let **1** be the intended identity and x, y the other two nonzero letters. In a field of order four the multiplicative group is cyclic of order three, so x has order three; thus $x^2 \notin \{1, x\}$, forcing $x^2 = y$, and symmetrically $y^2 = x$, and $xy = x \cdot x^2 = x^3 = \mathbf{1}$. The multiplication is therefore determined by the choice of **1**, and each of the three choices yields a structure satisfying the field axioms, as may be verified directly. Hence there are exactly three. $\square$

Theorem 7.4 (Determination of the multiplication). [D] Among the three multiplications of Lemma 7.3, exactly one has prime subfield equal to $\ker(\text{reg})$, namely the one with **1** the nonzero aligned letter. In that structure, position exchange τ is the nontrivial element of $\text{Gal}(GF(4)/GF(2))$, that is $\tau(x) = x^2$ for every $x \in \Sigma_v$.

Proof. The prime subfield of a $GF(4)$ structure is $\{0, \mathbf{1}\}$. By Proposition 7.2, $\ker(\text{reg}) = \{0, \mathbf{1}_{\text{al}}\}$ with $\mathbf{1}_{\text{al}}$ the nonzero aligned letter, so the requirement is exactly $\mathbf{1} = \mathbf{1}_{\text{al}}$, which

selects one of the three structures of Lemma 7.3 and excludes the other two. For the second assertion, Frobenius fixes the prime subfield pointwise and exchanges the two remaining nonzero elements, since these are x and x^2 . By Proposition 4.8, τ fixes exactly the two aligned letters, which are 0 and **1**, and exchanges the two unaligned letters. The two maps agree on all four elements, hence are equal. $\square$

Theorem 7.5 (The multiplicative generator is not structurally realized). [A] The structural group G contains no element of order three. In particular, for the multiplication of Theorem 7.4, multiplication by either unaligned letter is a permutation of Σ_v of order three and is not an element of G . No operation available to the encoded stack cycles the three nonzero letters, and the naturally selected generator satisfying $\omega^2 + \omega + 1 = 0$ sought by [Sch26c, App. C.10] does not exist among the structural operations.

Proof. By Proposition 4.7 every non-identity element of G is an involution, so G has no element of order three. Multiplication by a generator ω of $GF(4)^\times \cong \mathbb{Z}/2[3]$ is a permutation of order three of the nonzero letters, fixing 0. Since $\text{Aff}(\Sigma_v) \cong S_4$ contains elements of order three while G contains none, no such permutation lies in G . $\square$

Corollary 7.6 (The residual bit). [A] The multiplication of Theorem 7.4 is determined as an operation on Σ_v . The identification of Σ_v with $GF(4)$ as labelled sets is determined only up to $\text{Gal}(GF(4)/GF(2)) \cong \mathbb{Z}/2$: the set of field isomorphisms $\Sigma_v \rightarrow GF(4)$ is a torsor under the Galois group, and the residual ambiguity is which unaligned letter is named ω .

Proof. Two field isomorphisms $\Sigma_v \rightarrow GF(4)$ differ by a field automorphism of $GF(4)$, and conversely; the set of isomorphisms is therefore a torsor under $\text{Aut}(GF(4)) = \text{Gal}(GF(4)/GF(2))$, which is generated by Frobenius and has order two. The two isomorphisms agree on 0 and on **1**, which are pinned by Theorem 7.4, and differ by the exchange of the two unaligned letters. $\square$

Proposition 7.7 (Additive, not $\mathbb{F}_4$ -linear). [A] Under Hypothesis 2.11 the word space is all of Σ_v^N , which is closed under $\boxplus$ and is therefore an additive $GF(4)$ code of length N . Closure under multiplication by ω holds set-theoretically but is not induced by any structural operation, by Theorem 7.5; the $\mathbb{F}_4$ -linearity of the word space is accordingly a property of the ambient space and not a structurally induced one.

Proof. Closure under $\boxplus$ and under scalars is automatic for a full Cartesian power. The second assertion is Theorem 7.5: the map realizing multiplication by ω is not in G , so no operation

of the stack effects it.

□

Remark 7.8 (Reading of Proposition 7.7). The distinction the proposition draws is between closure of a set and realization of an operation. Under Hypothesis 2.11 the word space is everything, so no closure property distinguishes anything; what distinguishes is which operations the construction supplies. It supplies $\boxplus$, through the frame-designated zero, and the multiplication of Theorem 7.4, through reg ; it does not supply the multiplicative generator. A construction in which A12 cut out a proper subset would make the closure question substantive as well, and identifying such a constraint is left open.

Remark 7.9 (Σ_v is not $\Sigma_{\mathcal{C}}$). The alphabet $\Sigma_{\mathcal{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^z$ of A8 has the primary–dual cipher bit as its first coordinate. In a stack alternating as in Definition 3.1 that bit is a function of the parity of the site index, hence constant on vessels, and carries no information; the natural map $\Sigma_v \rightarrow \Sigma_{\mathcal{C}}$ is therefore not a bijection. Nothing in this section discharges [Sch26c, Thm. 41], which concerns $\Sigma_{\mathcal{C}}$. The identification is Hypothesis 2.14 and is recorded as Target 10.3. What the two alphabets do share canonically is the registry coordinate: reg of Definition 3.15 and the z -registry η of A6 are both labels on adjacent pairs along the stacking direction, and an identification of those two coordinates is the natural first half of Hypothesis 2.14.

Remark 7.10 (Two forward connections). [T] Two downstream connections are noted and not developed. First, the Calderbank–Rains–Shor–Sloane correspondence identifies binary quantum stabilizer codes with additive $GF(4)$ codes self-orthogonal under the trace-Hermitian inner product, whose conjugation is Frobenius, the Pauli group modulo phases being the Klein four-group and the order-three element cycling the three nonzero operators being exactly the operation Theorem 7.5 shows to be unavailable. By Theorem 7.4 and Corollary 7.6 the construction supplies the Klein four-group canonically and pins the element corresponding to the prime subfield, leaving the other two determined only up to $\mathbb{Z}/2$. Second, the dual-cipher structure of A8 asserts an equality of commutants $\mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}$, whereas trace-Hermitian self-orthogonality is an inclusion $C \subseteq C^{\perp}$; an identification of the two would restrict to the self-dual case. Both are recorded as Target 10.10 and neither is asserted here.

8 The Secondary Alphabet and the Conditions on a Transformation

No transformation is constructed in this paper. This section fixes the target of a transformation and determines what such a map must satisfy in order to transport the structure of Sections 3–6.

Definition 8.1 (Secondary alphabet). Σ'_v is the set of states of the transformed substrate, indexed by the four vessel states of Definition 3.13; $|\Sigma'_v| = 4$. Under Hypothesis 2.17 it carries a structural group G' , an invariant reg' and a distinguished fixed-point-free involution c' .

Definition 8.2 (Transformation). For $\varphi \in \Phi$, $T_\varphi : \Sigma_v \rightarrow \Sigma'_v$ is the induced map on vessel states.

Remark 8.3 (What is and is not content). A bijection between two four-element sets exists trivially, and its existence is not a result; nor, by Remark 2.18, is the existence of transported structure on the target. The content of Theorem 8.5 is the agreement of transported structure with structure the secondary alphabet possesses independently.

Lemma 8.4 (G is the stabilizer of the registry partition). [A] G is exactly the group of permutations of Σ_v preserving each of the two registry classes setwise, and likewise G' for Σ'_v .

Proof. A permutation preserving each class setwise is determined by its restrictions to the two classes, each of which has two elements, so there are $2 \times 2 = 4$ such permutations and they form a group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. By Proposition 4.8 the four elements of G are: the identity; τ , which fixes the aligned class pointwise and exchanges the unaligned pair; $\tau\rho$, which does the reverse; and ρ , which exchanges both pairs. These are four distinct such permutations, hence all of them. The same argument applies to G' under Hypothesis 2.17. □

Theorem 8.5 (Transport conditions). [D] Assume Hypotheses 2.12 and 2.17. For $T := T_\varphi$ the following are equivalent:

- (i) T transports registry: $\text{reg}' \circ T = \text{reg}$ after a labelling of the two values of reg' ;
- (ii) T transports the structural group: $TGT^{-1} = G'$;
- (iii) T transports complementation: $TcT^{-1} = c'$.

Proof. (i) $\Rightarrow$ (ii). Condition (i) says T carries the registry partition of Σ_v onto that of Σ'_v , class by class. Conjugation by T therefore carries a permutation preserving each class of the first partition to one preserving each class of the second, so TGT^{-1} is contained in the stabilizer of the registry partition of Σ'_v , which is G' by Lemma 8.4; both groups have order four, so the inclusion is an equality.

(ii) $\Rightarrow$ (iii). Conjugation by a bijection preserves the property of having no fixed point, so TcT^{-1} is a fixed-point-free element of $TGT^{-1} = G'$. By Proposition 4.10 applied to G' , which is legitimate under Hypothesis 2.17, the group G' has exactly one such element, namely c' . Hence $TcT^{-1} = c'$.

(iii) $\Rightarrow$ (i). The orbits of $\langle c \rangle$ are the registry classes of Σ_v , by Proposition 4.10, and likewise for $\langle c' \rangle$ and Σ'_v . Condition (iii) makes T equivariant for the two actions, so it carries orbits to orbits, that is registry classes to registry classes; this is (i) after labelling. $\square$

Remark 8.6 (A single criterion for the companion paper). Theorem 8.5 reduces the structural adequacy of a transformation to one checkable condition: a transformation preserves the structure of the encoded stack if and only if it preserves registry. The apparently stronger requirements, that it commute with the structural operations and that it carry complementation to complementation, follow from it rather than having to be imposed. This is the form in which the result is to be used when T_φ is constructed.

Proposition 8.7 (The family is a groupoid). **[D]** Assume Hypotheses 2.12 and 2.17 and suppose the equivalent conditions of Theorem 8.5 hold for every $\varphi \in \Phi$. Then

- (i) $T_\psi^{-1}T_\varphi \in G$ for all $\varphi, \psi \in \Phi$, so the family $\{T_\varphi\}$ is a torsor under G ;
- (ii) $\{T_\varphi\}$ generates a connected groupoid on the two objects Σ_v, Σ'_v whose vertex groups are G and G' ;
- (iii) the family is a group action of Φ on Σ_v only after a base frame is chosen, and by Proposition 6.5(ii) no such choice is canonical.

Proof. (i) Each T_φ carries the registry partition of Σ_v to that of Σ'_v , by Theorem 8.5(i); hence $T_\psi^{-1}T_\varphi$ is a permutation of Σ_v preserving each registry class, which by Lemma 8.4 lies in G . The action of G on the family by composition is free and transitive because $|G| = |\Phi| = 4$ and the map $\varphi \mapsto T_\varphi^{-1}T_\varphi$ is injective by Corollary 6.8.

(ii) The morphisms generated by $\{T_\varphi\}$ and their inverses include G and G' by (i) and by conjugation, and connect the two objects; this is the stated groupoid.

(iii) A group action requires an identity element of the acting set, and Φ is a torsor with no canonical element by Proposi-

tion 6.5(ii); choosing one identifies Φ with G and turns the torsor of (i) into an action. $\square$

Proposition 8.8 (Stack transformation). **[D]** Under Hypotheses 2.11 and 2.12 the transformation of a stack of N vessels induced by T_φ is the N -fold letterwise product

$$T_\varphi^{\times N} : \Sigma_v^N \longrightarrow (\Sigma'_v)^N,$$

and is a bijection. For $N = 3$ both word spaces have 64 elements, and the witness of Appendix A exhibits the source side of the correspondence.

Proof. By Hypothesis 2.11 the letters of distinct vessels are unconstrained by one another, so the induced map on words is the product map; a product of bijections is a bijection. The cardinality is Proposition 3.19. $\square$

Proposition 8.9 (Independence does not trivialize the sequence). **[A]** Letterwise independence of the transformation does not imply order independence of the encoded word: two stacks whose words have the same multiset of letters in different orders have distinct word operators, by Theorem 4.17, and distinct transformed words, by Proposition 8.8. Independence is a property of how a letter transforms; the sequence is what carries the information.

Proof. The transformed words are distinct because $T_\varphi^{\times N}$ is injective and the source words are distinct. The word operators are distinct by Theorem 4.17, whose hypothesis is Hypothesis 2.9 and not Hypothesis 2.11; the two hypotheses are independent, the first concerning the realization of words and the second the constraint structure on letters. $\square$

Remark 8.10 (Ensembles). Under Hypotheses 2.13 and 2.12, the distribution on transformed words is the push-forward along $T_\varphi^{\times N}$ of the distribution on words, and is therefore determined by it; the deterministic case is the point mass. Whether a given assembly can be made to realize a selected word is an assertion of type **[E]**, is not a consequence of any statement of this paper, and enters no proof.

9 The Set-Up for Survival

Survival in [Sch26a] is the conjunction of two statements certified there separately: robust stability of a form against the environment [Sch26a, Thm. 5.9], and invariance of the admissible domain under the disturbance [Sch26a, Thm. 8.13]. Neither implies the other [Sch26a, Prop. 9.1(ii)], and no single certificate for the conjunction is supplied there [Sch26a, Rem. 9.6]. Both are properties of transfer realizations, which are A10 data.

By Proposition 4.24 the encoded stack uses no A10 datum. This section therefore does not establish a survival property and does not approach one; it determines exactly how much of the instance schema [Sch26a, Def. 10.4] the encoded stack supplies, and states the remainder as Targets.

9.1 The admissibility datum

Definition 9.1 (Initiator). A site of a stack is an *initiator* when its local marker and substituent pattern deviates from the frame-regular pattern of the remainder of the stack, in the sense that its attachment set is neither $A_{\mathfrak{C}}^{\text{pri}}$ nor $A_{\mathfrak{C}}^{\text{dual}}$ of A1.

Definition 9.2 (Frame-consistency functional). $C_1(\mathfrak{C}) := |\#\{\text{initiators of } \mathfrak{C}\} - 1|$, with threshold $\tau_1 := 1$, in the sense of the constraint functionals of A12.

Definition 9.3 (Transition relation and weights). X denotes the set of encoded stacks on a fixed finite site set Λ with a fixed species labelling. Two elements of X are related by $\mathcal{E}_0$ when their marker assignments differ at exactly one site. The weight function is $w \equiv 1$ on $\mathcal{E}_0$.

Remark 9.4 (These are constructions). Definitions 9.2 and 9.3 supply data that A12 requires and does not determine, exactly as the witness datum of [Sch26a, Def. A.1] does. They are recorded so that each may be checked and, if a different choice is preferred, replaced without disturbing Sections 3–8, none of whose statements uses them.

Proposition 9.5 (The word operator is invisible to a tracial state). [A] *Let ω be a tracial state of $\mathfrak{A}_{\mathfrak{C}}$ and W any unitary of $\mathfrak{A}_{\mathfrak{C}}$. Then $\omega \circ \text{Ad}(W) = \omega$. In particular, if the state of A7 is tracial, the assignment $\mathfrak{C} \mapsto \omega \circ \text{Ad}(W(\varphi, w(\mathfrak{C})))$ is constant, and carries none of the information that Proposition 4.21 shows the word operator to carry.*

Proof. For $A \in \mathfrak{A}_{\mathfrak{C}}$, traciality gives $\omega(W^*AW) = \omega(AWW^*) = \omega(A)$, the last equality because W is unitary. The word operator is unitary by Proposition A.11, so the assignment is constant. $\square$

Remark 9.6 (Consequence for the spectral realization). A12 requires an *injective* spectral realization $J_{\text{str}} : X \rightarrow S(\mathfrak{A}_{\mathfrak{C}})$. Proposition 9.5 shows that the obvious construction from the word operator fails for the tracial state, and the explicit model of [Sch26c, Thm. 61] is built on a tracial GNS

representation. Injectivity of $w \mapsto W$ therefore does not by itself yield an injective spectral realization: the two injectivity statements are of different type, and conflating them would be an error of the kind recorded at [Sch26a, Rem. B.2]. Under Hypothesis 2.19 a realization built from the word operator exists; a generic vector state in a faithful representation separates the word operators of a finite set of encoded stacks, while a vector fixed by every letter operator separates none. Since Hypothesis 2.19 is not discharged here, Definition 9.7 supplies a realization by another device, and replacing it by one built from W is Target 10.11.

Definition 9.7 (Spectral realization). Let ω be the state of A7 and let ϕ be any state of $\mathfrak{A}_{\mathfrak{C}}$ with $\phi \neq \omega$. Enumerate the letters of Σ_0 as 0, 1, 2, 3 by the coordinates of Definition 3.12, and for $\mathfrak{C} \in X$ with word $w(\mathfrak{C}) = (\sigma_1, \dots, \sigma_N)$ set

$$s(\mathfrak{C}) := 4^{-N} \sum_{j=1}^N \sigma_j 4^{j-1} \in [0, 1), \quad J_{\text{str}}(\mathfrak{C}) := (1 - s(\mathfrak{C})) \omega + s(\mathfrak{C}) \phi.$$

Proposition 9.8 (The datum satisfies A12). [D] *With the data of Definitions 9.2, 9.3 and 9.7, the tuple $(X, J_{\text{str}}, \mathcal{E}_0, w, \{C_1\}, \{\tau_1\})$ is an admissibility datum in the sense of A12. Moreover J_{str} is injective; C_1 is integer valued, so that $C_1 < \tau_1$ holds exactly when $\mathfrak{C}$ carries exactly one initiator; and consequently*

$$\Omega_{\text{adm}} = \{ \mathfrak{C} \in X : \mathfrak{C} \text{ has a unique frame-setting site } \},$$

the initiator of an admissible stack being simultaneously the origin of its frame, the source of its direction sign, and its unique admissible inconsistency.

Proof. Each $J_{\text{str}}(\mathfrak{C})$ is a convex combination of states, hence a state. For injectivity, s is injective because the word is recovered from $s(\mathfrak{C}) 4^N$ by base-four expansion, using Proposition 3.19; and if $s(\mathfrak{C}) \neq s(\mathfrak{C}')$ then evaluating the difference of the two states on any A with $\phi(A) \neq \omega(A)$ gives $(s(\mathfrak{C}) - s(\mathfrak{C}'))(\phi(A) - \omega(A)) \neq 0$, so the states differ. The relation $\mathcal{E}_0$ is symmetric by construction and w is symmetric and positive. The functional C_1 takes values in $\mathbb{Z}_{\geq 0} \subset (-\infty, \infty]$ and $\tau_1 = 1$, so $C_1 < \tau_1$ is equivalent to $C_1 = 0$, that is to an initiator count of exactly one; this gives Ω_{adm} as stated. The last clause is Definition A.3 read in general: the frame data of Definition 6.1 are supplied by the initiator. $\square$

Proposition 9.9 (Admissibility is frame invariant). [A] *C_1 is invariant under the frame group Fr of Definition 6.13, and hence Ω_{adm} is a set of encoded stacks determined by the assembly and not by the reading of it.*

Proof. By Definition 9.1 the property of being an initiator is determined by the attachment set of the unit at a site, which the global marker flip and the index reversal do not alter; reversal permutes the sites and therefore permutes the initiators without changing their number. Hence the initiator count, and with it C_1 , is Fr-invariant. $\square$

Remark 9.10 (Constraint functionals are not energies). [E] In the intended reading the initiator is the site at which the transformation of the stack to a secondary structure begins. The threshold τ_1 of Definition 9.2 is a constraint threshold in the sense of A12 and is not an energy, and C_1 is a count and not a quantity carrying physical dimension. Accordingly

$$C_1(\mathfrak{E}) < \tau_1 \Rightarrow \text{energetic sufficiency for initiation,}$$

in the manner of the standing restraint $d_C^* = +\infty \Rightarrow$ autonomous physical scission of [Sch26b, Eq. (3)] and of the type mismatch recorded at [Sch26a, Rem. B.2], where a bound on a gain was read as a bound on a constraint functional. No energetic, kinetic or mechanistic claim is made or used anywhere in this paper.

9.2 Against the instance schema

Proposition 9.11 (Partial discharge of the instance schema). [D] *The encoded stack supplies clauses (N1), (N2) and (N3) of [Sch26a, Def. 2.10], and the admissibility clause of that definition, as follows.*

- (N1) *The site lattice is the base $X_{\mathfrak{E}} \cong \mathbb{Z}^3$ of A6 with its translation action, of rank $d = 3$.*
- (N2) *The species alphabet is $\Sigma_{\text{sp}} = \{\mathfrak{s}, \mathfrak{c}\}$ of Definition 2.3 with the labelling ς of Definition 3.1. These are the primary and dual attachment sets of A1, so Σ_{sp} is the alphabet of [Sch26a, Hyp. 2.16] and ς its species labelling.*
- (N3) *The set of forms is Ω_{adm} of Proposition 9.8, with transition relation $\mathcal{E}_{\text{adm}}$ obtained from $\mathcal{E}_0$ of Definition 9.3 by restriction to pairs of admissible stacks, and with the constraint functional and threshold of Definition 9.2.*

Proof. (N1) is A6. (N2) is Definition 2.3 together with the alternation of Definition 3.1; the identification with [Sch26a, Hyp. 2.16] is the observation that the two species are the two attachment sets of A1, which is how that hypothesis defines its alphabet. (N3) is Proposition 9.8, the restriction of a symmetric relation to a subset being again symmetric, and the admissibility clause of [Sch26a, Def. 2.10] being exactly restriction in the manner of A12. $\square$

Remark 9.12 (What is not supplied). Clauses (N4) and (N5) are not supplied, and by [Sch26a, Rem. 10.5] these are the substantive ones.

Clause (N4) requires, for each form, a transfer realization $G_s : \mathbb{C}_+ \times \mathbb{T}^d \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{Y})$ with input and output spaces common to all forms, analytic in the energy variable and continuous in the quasi-momentum. By Proposition 4.24 the encoded stack uses no datum of A10, from which such realizations are built. Without (N4) the unitary-orbit distance δ_u of [Sch26a, Def. 5.1] is undefined, hence so is the transformation metric $d_{\mathcal{H}_{\infty}}$, and there is no transformation geometry at all.

Clause (N5) requires a dissipation datum: a decomposition $\mathcal{H}_s = \mathcal{H}_s^{\text{pri}} \oplus \mathcal{H}_s^{\text{sec}}$ for each form together with coupling operators between the summands, the second summand being the second fibration of [Sch26a, Hyp. 2.15]. Nothing in Sections 3–8 exhibits such a decomposition. Without (N5), [Sch26a, Prop. 4.5] gives $\Gamma \equiv 0$, and every result of Section 4 of that paper is vacuous for the encoded stack.

Asserting either clause on the strength of (N1)–(N3) is exactly the inference [Sch26a, Rem. 10.5] warns against, namely that a system is a fibrated transfer network because it is a network with dynamics. The non-implication is recorded at Proposition 10.1(ix).

9.3 Which quantities could carry a survival claim

Proposition 9.13 (Candidate invariants are functions of the registry sequence). [D] *Let q be a function of the marker assignment of an encoded stack, with species labelling held fixed. If q is to be attributed to the assembly rather than to the reading of it, in the sense of being invariant under the frame group Fr, then q factors uniquely through the registry sequence modulo reversal. In particular the registry reg of a vessel, the number of unaligned adjacent pairs, and every symmetric function of $[D]$ are admissible candidates, while the word $w(\mathfrak{E})$ is not.*

Proof. This is Theorem 6.14: $\mathbf{m} \mapsto [D(\mathbf{m})]$ is the orbit map of Fr, and an invariant function is one constant on orbits, hence one factoring through the orbit map, uniquely because the orbit map is surjective. The word is not invariant: by Proposition A.7 the four frame-setters read four distinct words from one and the same stack. $\square$

Remark 9.14 (The shape of a future survival claim). By Proposition 9.13 the encoding supplies candidate invariants; by Remark 9.12 it supplies no transfer realization against which their preservation could be certified. A survival claim in the sense of [Sch26a] would therefore require, in order: clause (N4), so that forms carry comparable realizations and

$d_{\mathcal{H}_{\infty}}$ is defined; clause (N5), so that the finiteness criterion [Sch26a, Thm. 4.17] is not vacuous; then a proof that a candidate invariant of Proposition 9.13 is preserved along admissible transformations; and finally both halves of the survival conjunction, which [Sch26a, Prop. 8.15] shows cannot be obtained from a single bound on the worst-case norm. Four steps, of which this paper takes none. They are Targets 10.6, 10.7 and 10.8.

Remark 9.15 (The initiator and the second fibration). The initiator of Definition 9.1 is the only object in this construction that separates the stack, in the sense that it is the unique site at which the frame-regular pattern fails and the unique site from which the reading of Definition 4.14 begins. Clause (N5) requires a sector into which the active region hands off, disjoint from the primary network as a site set. The two roles are formally similar and are not here identified: separation of a reading and orthogonal decomposition of a state space are statements about different objects, and an identification would require a construction of the second summand and its coupling, neither of which is supplied. This is Target 10.7, and it is recorded as the most promising of the Targets of this paper rather than as a result.

Remark 9.16 (No survival property is asserted). No quantity constructed in this paper is shown to be preserved under admissible transformation, and no survival claim is made of the encoding, of the alphabet, of the word operator, or of any code over Σ_v . The candidate invariants are named in Proposition 9.13 and the hypotheses under which they might carry such a claim in Remark 9.14; the claim itself belongs to a paper that has discharged Targets 10.6 and 10.7.

Target 9.17 (Spectral realization from the word operator). [T] *Discharge Hypothesis 2.19, thereby replacing the interpolation realization of Definition 9.7 by one built from the word operator, and determine for which states of $\mathfrak{A}_{\mathfrak{C}}$ the map $\mathfrak{C} \mapsto \psi \circ \text{Ad}(W)$ is injective. By Proposition 9.5 no tracial state will serve, so the target includes identifying a non-tracial state compatible with the explicit model of [Sch26c, Thm. 61], whose GNS representation is tracial.*

10 Non-Implications and Targets

The formalism's governing restraint is that restoring a foundational object never licenses restoring an unproved implication of that object [Sch26b, Eq. (2)]. This section states, in that spirit, what the constructions of this paper do not establish, and collects the open targets in one place. Each non-implication is refuted by a witness named in its proof, so the section adds no new

argument; its purpose is to make the list checkable.

10.1 Non-implications

Proposition 10.1 (Non-implications). [A] *None of the following implications holds in general; each is refuted by the witness named in its proof.*

- (i) *Order dependence of a read word $\Rightarrow$ noncommutativity of $\mathfrak{A}_{\mathfrak{C}}$, and conversely.*
- (ii) *The two values of the marker binary $\Rightarrow$ chirality of the unit or of the assembly; and conversely, assembly chirality $\Rightarrow$ a nontrivial marker binary.*
- (iii) *$|\Sigma_v| = 4 \Rightarrow$ a structurally induced field structure on Σ_v .*
- (iv) *Determination of a multiplication on $\Sigma_v \Rightarrow$ realization of the multiplicative generator by a structural operation.*
- (v) *$\Sigma_v \cong \Sigma_{\mathfrak{C}}$ as four-element sets $\Rightarrow$ any relation between the results of Section 7 and [Sch26c, Thm. 41].*
- (vi) *$\Phi_{\text{typ}} = \text{reg}$.*
- (vii) *Bijectivity of $T_{\varphi} \Rightarrow$ structural determination of T_{φ} , or transport of complementation, or transport of registry.*
- (viii) *Letterwise independence of the transformation $\Rightarrow$ order independence of the encoded word.*
- (ix) *Offset ambiguity $\Rightarrow$ loss of information.*
- (x) *Clauses (N1)–(N3) of the instance schema $\Rightarrow$ the encoded stack is a fibrated transfer network.*
- (xi) *$C_1(\mathfrak{C}) < \tau_1 \Rightarrow$ energetic sufficiency for initiation.*
- (xii) *The tracial state of A7 distinguishes word operators.*

Proof. (i) Both implications fail. The forward implication fails because classical strings are distinguished by order in any case, regardless of the algebra into which they are read; in particular it fails for the bare-product realization of the counterfactual of Theorem 4.17. The reverse implication fails because $\mathfrak{A}_{\mathfrak{C}}$ is noncommutative and yet, under the counterfactual, the word operator depends only on the multiset of letters (Proposition A.13).

(ii) The marker binary is not a chirality: its source is the kinematic exclusion of Hypothesis 2.7, not the absence of improper symmetry of the assembly as required by [Sch26c, Thm. 47] and Remark 3.7. Assembly chirality does not imply the marker binary because an achiral assembly can still be read under a projection convention assigning F or B .

- (iii) Theorem 7.5: the structural group contains no element of order three, hence no structural operation cycles the three nonzero letters as $GF(4)^\times$ requires.
- (iv) Theorem 7.4 determines the multiplication; Theorem 7.5 shows multiplication by either unaligned letter has order three and lies outside the structural group G .
- (v) Remark 7.9: the natural map $\Sigma_v \rightarrow \Sigma_\mathbb{C}$ is not a bijection, since the cipher bit of $\Sigma_\mathbb{C}$ is constant on vessels.
- (vi) Proposition 6.5(iii): Φ_{typ} is computed from the frame-setter's substituent composition, reg from a pair of markers; no result of this paper identifies them.
- (vii) Theorem 8.5: bijectivity places the transported structure on Σ'_v but does not make it agree with structure Σ'_v carries independently; all three conditions are necessary and are not implied by bijectivity alone.
- (viii) Proposition 8.9: the letters of distinct vessels transform independently by Hypothesis 2.11, yet two stacks with the same multiset of letters in different orders have distinct word operators by Theorem 4.17.
- (ix) Proposition 6.11: the word at one offset determines the marker assignment on its sites, hence every other reading; the decompositions are genuinely different but neither loses information.
- (x) Remark 9.12: clauses (N4) and (N5) are not supplied; without them the unitary-orbit distance is undefined (no (N4)) and the dissipation is zero (no (N5)).
- (xi) Remark 9.10: C_1 is an integer count and τ_1 is a constraint threshold in the sense of A12; neither carries the dimension of energy, and the threshold is not an activation barrier.
- (xii) Proposition 9.5: for any unitary W and any tracial state ω , $\omega(W^*AW) = \omega(A)$; the conjugated state is identical to the original. $\square$

Remark 10.2 (The two that matter most for reading this paper). Items (i) and (x) are the ones on which a reader is most likely to over-read. Item (i) states that the connection between the reading order of Section 3 and the algebraic noncommutativity of A7 is created by Hypothesis 2.9, not by either property alone: if either the fibre operators commuted with translation (failing clause (iii) of H_{ord}) or the letters landed in the commutative local sector (failing clause (ii)), the word operator would be order blind despite $\mathfrak{A}_\mathbb{C}$ remaining noncommutative. Item (x) states that exhibiting (N1)–(N3) does not make the encoded stack a fibrated transfer network in the sense of [Sch26a, Def. 10.4]: that designation requires all five clauses, of which the two missing ones are the substantive ones by [Sch26a, Rem. 10.5], and nothing in this paper weakens that assertion.

10.2 Targets introduced here

Target 10.3 (Alphabet identification). [T] *Discharge Hypothesis 2.14, or refute it, and thereby determine whether the results of Section 7 bear on [Sch26c, Thm. 41]. Hypotheses required: a construction of the bijection $\Sigma_v \rightarrow \Sigma_\mathbb{C}$, which must in particular assign a cipher bit to a vessel state. A natural first step is identifying the registry coordinate reg of Definition 3.15 with the z -registry η of A6, which is the natural common datum.*

Target 10.4 (Marker binary and emergent chirality). [T] *Determine the relation, if any, between the marker binary of Definition 3.3 and the emergent assembly chirality target [Sch26c, Thm. 47]. Two binary choices exist; that is not an identification, and Proposition 10.1(ii) records the non-implication in both directions. If discharged, the target would show which registry sequences remove every improper symmetry of the completed assembly, in the sense of [Sch26c, Def. 84], and two deferred targets of the corpus would collapse into one.*

Target 10.5 (The transformation). [T] *Construct T_φ from the structural data of Sections 3–6, and verify the conditions of Theorem 8.5: that it carries each registry class of Σ_v to the corresponding class of Σ'_v , equivalently that it commutes with complementation c . Determine which element of Φ produces which secondary alphabet type. In particular, determine whether Φ_{typ} corresponds to the distinction between two secondary sequence types and Φ_{dir} to the distinction between two reading directions.*

Target 10.6 (Common channels). [T] *Supply clause (N4) of [Sch26a, Def. 2.10] for the encoded stack: a common input space $\mathcal{K}$ and output space $\mathcal{Y}$, and for each form a transfer realization $G_s : \mathbb{C}_+ \times \mathbb{T}^3 \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{Y})$ with the required analyticity. Without (N4) the unitary-orbit distance δ_u of [Sch26a, Def. 5.1] is undefined and there is no transformation geometry. With (N4) the data of Sections 3–9 are available to the framework of [Sch26a] by its Theorem 10.2.*

Target 10.7 (The initiator as separator). [T] *Determine whether the separating role of the initiator of Definition 9.1 supplies the second fibration of [Sch26a, Hyp. 2.15], and thereby clause (N5) of the instance schema. The separating role here is that the initiator is the unique site at which the frame-regular pattern fails and the unique site from which the reading begins. Clause (N5) requires an orthogonal*

decomposition $\mathcal{H}_s = \mathcal{H}_s^{\text{pri}} \oplus \mathcal{H}_s^{\text{sec}}$ together with coupling operators; two roles are formally similar and are not here identified. Together with Target 10.6 this would make [Sch26a, Thm. 10.2] apply verbatim to the encoded stack, at which point the robust-stability and admissibility-invariance results of that paper become available.

Target 10.8 (Candidate survival invariants). [T] Show that reg, the frame-invariants of Theorem 6.14, and the domain of T_φ are preserved under admissible transformation in the sense of A12. Hypotheses required: Targets 10.6 and 10.7, so that a transfer realization exists whose admissible transformations can be certified by the methods of [Sch26a]; and the transport conditions of Theorem 8.5, so that registry is preserved by the transformation. By Proposition 9.13 every frame-invariant quantity is a function of the registry sequence modulo reversal, and the candidate invariants are exactly those; by Remark 9.16 no such claim is asserted in this paper.

Target 10.9 (Injectivity in general). [T] Extend Proposition 4.21 beyond the witness of Appendix A, or identify the hypotheses under which it fails. The proof of Proposition 4.21 uses non-proportionality of the letter operators at one site, which the witness fibre supplies; a general argument requires recovering each letter from the word operator by a localization procedure, and its scope depends on the class of realizations admitted by Hypothesis 2.9(i).

Target 10.10 (Stabilizer codes and A8). [T] Determine whether the dual-cipher commutant of A8, $\mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}$, can be identified with a trace-Hermitian dual code in the sense of the Calderbank–Rains–Shor–Sloane correspondence. Two obstructions are on record. First, A8 asserts an equality $\mathfrak{M}_{\text{dual}} = \mathfrak{M}'_{\text{pri}}$, whereas trace-Hermitian self-orthogonality of a code is an inclusion $C \subseteq C^\perp$; an identification would restrict to the self-dual case. Second, $\mathfrak{M}_{\text{pri}}$ and $\mathfrak{M}_{\text{dual}}$ act on $\mathcal{H}_\omega$, while a code over Σ_v is a subset of Σ_v^N , and a connecting map between them requires an intertwiner that is not supplied here, in the manner of [Sch26a, Target 8.25]. In the periodic case the target is likely provable and is expected to give A8 a coding-theoretic meaning it currently lacks.

Target 10.11 (Spectral realization from the word operator). [T] Discharge Hypothesis 2.19, thereby replacing the interpolation realization of Definition 9.7 by one built from the word operator. By Proposition 9.5 no tracial state of $\mathfrak{A}_\mathbb{C}$ will

serve, so the target includes identifying a non-tracial state compatible with the explicit model of [Sch26c, Thm. 61], whose GNS representation is tracial. On the witness of Appendix A a generic vector state in a faithful representation separates all 64 word operators, while a vector fixed by every letter operator separates none.

Remark 10.12 (Inventory of Targets). This paper introduces nine targets. Of these, Targets 10.6, 10.7 and 10.8 are load-bearing in the sense that the paper’s connection to the survival programme of [Sch26a] is incomplete without them: the first two supply the missing clauses of the instance schema, and the third is the survival claim whose discharge they make possible. Target 10.5 is the first obligation of the companion paper constructing the bijective map. Target 10.4 is potentially the strongest result the programme could produce: if discharged, two deferred targets of the twelve-axiom corpus collapse into one.

11 Conclusion

The question this paper set out to answer was whether the Chrysene Formalism’s informal language of encoding, sequence dependence, and four-state alphabet refers to a well-defined mathematical object. The answer is affirmative and constructive: the frame-indexed encoded stack exists, is non-vacuous, and its properties are derived rather than postulated.

The object and its non-vacuity. The encoded stack of Definition 3.17 is constructed from Axioms A1, A2, A6, A7, A8 and A12 without further axiom (Proposition 4.24), and is therefore a consequence of the twelve-axiom system rather than an extension of it. Non-vacuity is established by the explicit witness $\mathfrak{C}_{\text{wit}}$ of Theorem 5.1: an $N = 3$ stack over $\mathbb{C}^{\text{chr}}$ with all markers, registries, letters, frames, and word operators in closed form, and all twelve propositions of Sections 3–4 verified on it.

Sequence dependence is derived, not asserted. The central result of Section 4 is that the word operator $W(\varphi, w)$ is order sensitive as a consequence of A7’s covariance relation $u_{\mathbf{m}} P_{\mathbf{k}} = P_{\mathbf{k}+\mathbf{m}} u_{\mathbf{m}}$ and of nothing else (Theorem 4.17 with its attribution clause). The non-implication is as important as the theorem: Proposition 4.18 shows that order dependence of a classical string and algebraic noncommutativity of $\mathfrak{A}_\mathbb{C}$ are independent properties, and that the Hypothesis 2.9 is where and only where the two are made to coincide. Without clause (iii) of H_{ord} , the word operator is order blind; without clause (ii), fibre-level noncommutativity could masquerade as sequence-level noncommutativity. Three findings make this precise on the witness: the 64 words of an $N = 3$ stack give 64 distinct operators; removing the translation unitaries collapses them to

4; and a generic vector state separates all 64 while the tracial state of A_7 separates none (Propositions A.12, A.13, 9.5).

The frame and what depends on it. Of the two coordinates of a vessel letter, the registry b_1 is canonical and independent of the frame (Proposition 4.3); the offset b_2 is determined by the frame's projection reference (Proposition 4.4). The maximal frame-invariant is the registry sequence modulo reversal: every quantity attributable to the assembly rather than to the reading factors through it uniquely (Theorem 6.14). Admissibility is frame-invariant (Proposition 9.9). The word is not: the four frame-setters read four distinct words from one stack (Proposition A.7).

Base-four status, clarified. The alphabet Σ_v is canonically a $(\mathbb{Z}/2)^2$ -torsor; the frame makes it an $\mathbb{F}_2$ -vector space; and requiring the prime subfield to equal $\ker(\text{reg})$ determines a unique multiplication, with position exchange τ acting as the Frobenius automorphism (Theorems 7.4). The structural group contains no element of order three, so no structural operation cycles the three nonzero letters (Theorem 7.5); the residual ambiguity is a single Galois bit, namely which unaligned letter is named ω (Corollary 7.6). The result clarifies the standing non-implication $|\Sigma_{\mathbb{C}}| = 4 \not\Rightarrow \Sigma_{\mathbb{C}} \cong GF(4)$ of [Sch26b, Eq. (2)] without discharging [Sch26c, Thm. 41], which concerns $\Sigma_{\mathbb{C}}$ and not Σ_v (Remark 7.9).

Survival: two clauses short. The encoded stack supplies clauses (N1), (N2) and (N3) of the instance schema [Sch26a, Def. 10.4] and does not supply (N4) or (N5) (Proposition 9.11, Remark 9.12). By [Sch26a, Rem. 10.5] these are the substantive clauses: without (N4) the transformation geometry of that paper is undefined, and without (N5) its finiteness criterion is vacuous. The candidate survival invariants are the functions of the registry sequence modulo reversal (Proposition 9.13), and the shape of a future survival claim is Remark 9.14. No survival property is asserted here.

What is not done. No deferred target of [Sch26c] is discharged. Twelve non-implications are recorded (Proposition 10.1), of which items (i) and (x) are the most likely to be over-read. Nine targets are opened (Section 10), of which Targets 10.6, 10.7 and 10.8 are load-bearing for the survival programme, and Target 10.5 is the first obligation of the companion paper. Six defects in the drafting process are recorded in the concordance appendix; one correction to the molecular marker definition, from a bay designation to a perimeter-orientation criterion, is the only change in a Part I claim that had been advertised.

A The Witness Encoded Stack

By Remark 5.3 neither witness already present in the corpus instantiates the constructions of this paper. This appendix supplies one. Everything below is a construction: the choices made are not fixed by A1–A12, and are recorded so that each may be checked.

A.1 Site set, species, markers

Definition A.1 (The witness stack). Let $\mathbf{k}_0 \in \mathbb{Z}^3$ and let the site set be

$$\Lambda := \{n_1, \dots, n_6\}, \quad n_j := \mathbf{k}_0 + (j-1)e_z,$$

together with the initiator site $n_0 := \mathbf{k}_0 - e_z$. The species labelling alternates,

$$s(n_j) = \begin{cases} \mathbf{c}, & j \text{ odd}, \\ \mathbf{s}, & j \text{ even}, \end{cases}$$

and the marker assignment, written in the convention $F = 0, B = 1$ of Definition 3.12, is

$$(\text{mk}(n_1), \dots, \text{mk}(n_6)) = (F, F, F, B, B, F) = (0, 0, 0, 1, 1, 0).$$

The vessel decomposition is $\mathcal{D} = \{(n_1, n_2), (n_3, n_4), (n_5, n_6)\}$, so that $N = 3$.

Remark A.2 (Both marker values occur). The assignment of Definition A.1 takes both values of mk . This does not prove Proposition 3.8, which is a statement about the admissible class and remains a Target; it shows only that the witness is not degenerate in the way a constant marker assignment would be, and that the letters of Proposition A.5 are not all equal.

Definition A.3 (Initiator and frame datum). The site n_0 carries a unit whose attachment set is neither $A_{\mathbb{C}}^{\text{pri}}$ nor $A_{\mathbb{C}}^{\text{dual}}$ but one of the four three-element frame-setter sets, so that its local pattern deviates from the frame-regular pattern of $n_1, \dots, n_6$ in the sense of Definition 9.1. The frame datum $\varphi = (\varepsilon, f)$ supplied by n_0 consists of

- the origin $\mathbf{k}_0 = n_1$, the first site of the regular stack;
- the direction sign $\varepsilon \in \{\pm 1\}$, carried by Φ_{dir} ;
- the projection reference $f \in \mathbb{Z}/2$, carried by Φ_{typ} .

The default frame of the witness is $\varphi_1 := (\varepsilon, f) = (+1, 0)$.

Proposition A.4 (The witness is admissible). [D] $\mathfrak{C}_{\text{wit}}$ has exactly one initiator, namely n_0 ; hence $C_1(\mathfrak{C}_{\text{wit}}) = |1-1| = 0 < 1 = \tau_1$ and $\mathfrak{C}_{\text{wit}} \in \Omega_{\text{adm}}$.

Proof. Each of $n_1, \dots, n_6$ carries one of the two four-element attachment sets of A1 in the alternating pattern of Definition A.1, and is therefore frame-regular; n_0 carries a three-element frame-setter set by Definition A.3 and is therefore not. The initiator count is one and Definition 9.2 gives the stated value. $\square$

A.2 Letters, frames, and the designated zero

Proposition A.5 (The letters in closed form). [D] Under the default frame $\varphi_1 = (+1, 0)$ the three vessels of $\mathfrak{E}_{\text{wit}}$ carry

$$\begin{aligned} v_1 = (n_1, n_2) : & \quad (\text{mk}(\mathbf{c}), \text{mk}(\mathbf{s})) = (F, F), & (b_1, b_2) = (0, 0), & \text{aligned}, \\ v_2 = (n_3, n_4) : & \quad (F, B), & (b_1, b_2) = (1, 1), & \text{unaligned}, \\ v_3 = (n_5, n_6) : & \quad (B, F), & (b_1, b_2) = (1, 0), & \text{unaligned}, \end{aligned}$$

so that the word is $w(\mathfrak{E}_{\text{wit}}) = (FF, FB, BF)$ and the registry sequence of Definition 4.2 is

$$(D_1, \dots, D_5) = (+1, +1, -1, +1, -1).$$

Proof. Direct computation from Definition 3.12 with $f = 0$: $b_1 = \text{mk}(\mathbf{c}) + \text{mk}(\mathbf{s})$ and $b_2 = \text{mk}(\mathbf{s})$. For the registry sequence, $D_n = M_n M_{n+1}$ takes the value $+1$ when consecutive markers agree and -1 when they differ, and the marker string $(0, 0, 0, 1, 1, 0)$ has agreements at $n = 1, 2, 4$ and disagreements at $n = 3, 5$. $\square$

Remark A.6 (The registry sequence carries the vessel registries). The registries of v_1, v_2, v_3 are D_1, D_3, D_5 , the odd-indexed terms of the sequence of Proposition A.5. The even-indexed terms D_2, D_4 are the registries of the vessels of the opposite offset, and are read by no frame of Definition A.3. This is the witness form of Theorem 6.14: the full sequence is frame independent, and a frame reads an alternate subsequence of it.

Proposition A.7 (The four frame-setters are distinguishable by effect). [D] On $\mathfrak{E}_{\text{wit}}$ the four elements of Φ produce the four pairwise distinct words

$$\begin{aligned} \varphi_1 = (+1, 0) : & \quad (FF, FB, BF) = ((0, 0), (1, 1), (1, 0)), \\ \varphi_2 = (+1, 1) : & \quad (BB, BF, FB) = ((0, 1), (1, 0), (1, 1)), \\ \varphi_3 = (-1, 0) : & \quad (FB, BF, FF) = ((1, 1), (1, 0), (0, 0)), \\ \varphi_4 = (-1, 1) : & \quad (BF, FB, BB) = ((1, 0), (1, 1), (0, 1)). \end{aligned}$$

In particular $\Phi \rightarrow \{\text{read words}\}$ is injective on this stack, which is Proposition 6.8.

Proof. A change $f \mapsto f + 1$ translates b_2 by 1 in every vessel, by Proposition 4.4, hence applies ρ letterwise; this carries φ_1 to φ_2 . A change $\varepsilon \mapsto -\varepsilon$ reverses the order in which the vessels are read and, by Lemma 4.6, applies τ within each letter; this carries φ_1 to φ_3 and φ_2 to φ_4 . The four resulting triples are as displayed and are pairwise distinct by inspection. $\square$

Corollary A.8 (The designated zero). [D] For every $\varphi \in \Phi$ the zero designated by φ is the letter with $(b_1, b_2) = (0, 0)$, namely FF when $f = 0$ and BB when $f = 1$. It is aligned in all four cases, confirming Corollary 6.9, and the map $z : \Phi \rightarrow \Sigma_{\mathbf{v}}$ is two-to-one onto $\text{reg}^{-1}(0)$, its fibres being the pairs sharing a value of f .

Proof. The zero is by definition the letter whose coordinates vanish. By Definition 3.12 this is the vessel state with $\text{mk}(\mathbf{c}) = \text{mk}(\mathbf{s}) = f$, which is FF for $f = 0$ and BB for $f = 1$; both have $b_1 = 0$. The map z depends only on f and not on ε , whence the fibres. $\square$

A.3 The operator realization

Definition A.9 (Fibre and letter operators). Let the fibre at each site be $\mathcal{K} := \mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2] \cong \mathbb{C}^4$, with basis $\{e_g\}$ indexed by $g \in \mathbb{Z}/2 \times \mathbb{Z}/2$, and define for $\sigma = (b_1, b_2) \in \Sigma_{\mathbf{v}}$

$$\iota_{\Sigma}(\sigma) := \chi_{\sigma}, \quad \chi_{\sigma} e_g := (-1)^{\langle \sigma, g \rangle} e_g, \quad \langle \sigma, g \rangle = b_1 g_1 + b_2 g_2 \in \mathbb{Z}/2,$$

the character of $\mathbb{Z}/2 \times \mathbb{Z}/2$ at σ , supported on the fibre of the origin site. Let $u := u_{\varepsilon e_z}$ be the translation unitary of A7.

Proposition A.10 (H_{ord} holds on the witness). [D] *The assignment of Definition A.9 satisfies all three clauses of Hypothesis 2.9:*

- (i) *each χ_σ is self-adjoint with $\chi_\sigma^2 = 1$, and the four are pairwise non-proportional;*
- (ii) *the four commute pairwise;*
- (iii) *the word operator of Definition 4.14 is formed with u as displayed there.*

Proof. Each χ_σ is diagonal with entries ± 1 in the basis $\{e_g\}$, hence self-adjoint and involutive, and any two diagonal operators commute, giving (ii). For non-proportionality, suppose $\chi_\sigma = \lambda \chi_{\sigma'}$ with $\sigma \neq \sigma'$. Evaluating at $g = 0$ gives $1 = \lambda$, and evaluating at any g with $\langle \sigma - \sigma', g \rangle = 1$, which exists because $\sigma - \sigma' \neq 0$ and the pairing is non-degenerate, gives $1 = -1$, a contradiction. Clause (iii) is Definition 4.14. $\square$

Proposition A.11 (The witness word operator). [D] *With $\varphi_1 = (+1, 0)$ and $w(\mathfrak{E}_{\text{wit}}) = (FF, FB, BF)$,*

$$W(\varphi_1, w) = \chi_{(0,0)} u \chi_{(1,1)} u \chi_{(1,0)} = \iota_{\mathbf{k}_0}(\chi_{(0,0)}) \iota_{\mathbf{k}_1}(\chi_{(1,1)}) \iota_{\mathbf{k}_2}(\chi_{(1,0)}) u^2,$$

the second equality being Lemma 4.16. The operator is unitary, being a product of self-adjoint involutions and translation unitaries, and lies in $\mathfrak{A}_{\mathfrak{E}} \setminus \mathfrak{Q}_{\mathfrak{E}}$.

Proof. The first equality is Definition 4.14 evaluated on the word of Proposition A.5; the second is Lemma 4.16. Unitarity holds because each χ_σ is a self-adjoint involution, hence unitary, and u is unitary. Membership in $\mathfrak{A}_{\mathfrak{E}} \setminus \mathfrak{Q}_{\mathfrak{E}}$ holds because the normal form contains the factor u^2 with a nonzero exponent and $\mathfrak{Q}_{\mathfrak{E}}$ contains no translation unitary $u_{\mathbf{m}}$ with $\mathbf{m} \neq 0$. $\square$

Proposition A.12 (Injectivity on the witness). [D] *For $N = 3$ and the realization of Definition A.9, the $4^3 = 64$ words of Proposition 3.19 give 64 pairwise distinct word operators. This is Proposition 4.21 in the present instance.*

Proof. Proposition 4.21 applies, its hypothesis being Proposition A.10(i). The count is confirmed independently in Remark A.17. $\square$

Proposition A.13 (The counterfactual collapse). [D] *If u is replaced by the identity in Definition 4.14, the same 64 words give exactly 4 distinct operators, namely $\chi_{\sigma_1+\sigma_2+\sigma_3}$; the word operator then depends only on the sum of its letters in $\mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof. With $u = 1$ every letter is applied at the origin site and the product is $\chi_{\sigma_1} \chi_{\sigma_2} \chi_{\sigma_3}$, which equals $\chi_{\sigma_1+\sigma_2+\sigma_3}$ because $\sigma \mapsto \chi_\sigma$ is a group homomorphism from $\mathbb{Z}/2 \times \mathbb{Z}/2$ to the diagonal unitaries. The sum takes each of the four values of $\mathbb{Z}/2 \times \mathbb{Z}/2$, so the image has exactly four elements. $\square$

Example A.14 (A multiset pair). [D] *The words $w = (FF, FB, BF)$ and $w' = (FF, BF, FB)$ have the same multiset of letters. By Proposition A.12 their word operators differ; by Proposition A.13 the two operators obtained with u deleted are equal, both being $\chi_{(0,0)+(1,1)+(1,0)} = \chi_{(0,1)}$. This is the witness for Theorem 4.17 and for its attribution clause.*

Remark A.15 (Reading of Propositions A.12 and A.13). *The encoding of a three-vessel stack carries six bits. Deleting the translation unitaries leaves two. The four bits so lost are exactly the positional information, and what survives is the sum of the letters, an invariant of the multiset and indeed coarser than it: the twenty multisets of three letters collapse to four sums, because the letter operators of Definition A.9 form a group under multiplication. A realization whose letters generate freely would collapse to the twenty multisets and no further. In either case the collapse is total in the sense that matters: without clause (iii) of Hypothesis 2.9, no order information survives.*

Hypothesis	Status on $\mathfrak{E}_{\text{wit}}$
H_{mk}	verified, Def. A.1
H_{stack}	verified, by construction of R
H_{frm}	verified, Def. A.3
H_{ord}	verified (i)–(iii), Prop. A.10
H_{ind}	verified, Prop. A.4
H_{bij}	not instantiated
H_{ens}	not instantiated
H_{Σ}	not instantiated
H_{chem}	not instantiated

Table 1: Status of the hypothesis register on the witness encoded stack, in the manner of [Sch26a, Tab. 3].

A.4 Status of the hypothesis register on the witness

Remark A.16 (Register status). Table 1 records, for each hypothesis of Section 2.3, whether $\mathfrak{E}_{\text{wit}}$ verifies it. The hypotheses marked not instantiated are used by no statement proved on the witness; they are required only by Sections 7–9 and by the transformation of the companion paper.

A.5 Independent numerical verification

Remark A.17 (Numerical verification). The closed forms of Propositions A.5, A.7, A.11, A.12 and A.13 were checked against direct computation in an explicit finite model: four sites, the four-dimensional fibre $\mathcal{K}$ of Definition A.9, the four characters as letter operators, and u realized as the cyclic shift of tensor factors. The model satisfies the covariance relation $u \iota_{\mathbf{k}}(A) u^* = \iota_{\mathbf{k}+e_z}(A)$ to machine precision while $[u, \iota_{\mathbf{k}_0}(A)] \neq 0$; the four letter operators commute pairwise and are pairwise non-proportional; the normal form of Lemma 4.16 holds identically for all 64 words; the 64 words give 64 distinct operators, and 4 under the counterfactual of Proposition A.13; the four frame-setters of Proposition A.7 give four distinct word operators; and the pair of Example A.14 differs with u present and agrees with u deleted. All comparisons were made to nine decimal places.

B Concordance and Defect Ledger

B.1 Scope and method

This appendix records, for every prior statement of the Chrysene Formalism corpus that this paper touches, whether the statement is preserved (used unchanged), extended (used with a further hypothesis, without alteration of its content), restated (reformulated here in a manner the reviewer must check is equivalent), corrected (found to be false or type-incorrect as written, with the repair given), not applicable (outside the scope of the objects considered here), or noted (recorded for information without change of status). The classification is made against the criterion C of [Sch26b, Def. 2]: a statement of this paper is concordant when it neither contradicts a prior statement of the formalism nor silently weakens a structure that a prior statement relies upon.

B.2 Concordance with the twelve-axiom paper

B.3 Concordance with the transfer-geometry paper

B.4 Chemistry and domain items

The following items are those on which the chemistry review of [Sch26b, §4.3] is invited to rule. All are of type [E] or are statements about labelled graphs; none is used in a proof.

Remark B.1 (The perimeter-orientation marker). [E] Definition 3.3 reads the orientation of the external perimeter cycle $\partial G_{\mathfrak{C}}$ of A8 in projection. In the intended reading the two values are the two legible senses of the angular four-ring array. The chemistry

Prior statement	Content relied upon	Disposition
A1, A2, A6, A7, A8, A12	The twelve axioms in full	Preserved throughout; no axiom added (Rem. 2.2)
[Sch26c, Eq. (2)] $ \Sigma_{\mathfrak{C}} =4 \Rightarrow \Sigma_{\mathfrak{C}} \cong GF(4)$	Standing non-implication	Preserved; Prop. 10.1(v) ensures the result does not weaken it
[Sch26c, Thm. 8] Sequence-induced noncommutativity	$[u_{\mathbf{m}}, P_{\mathbf{k}}] \neq 0$	Preserved; used in Thm. 4.17 attribution clause
[Sch26c, Cor. 2] $[u_{\mathbf{m}}, \Delta_{\mathbf{k}}] \neq 0$	Cipher involution does not commute with translation	Preserved; invoked in proof of Thm. 4.17
[Sch26c, Prop. 2] C_{2h} implies achirality	Isolated node is achiral	Preserved; used in proof of Prop. 3.4(iii)
[Sch26c, Rem. 20] No $GF(4)$ field law yet	$(\Sigma_{\mathfrak{v}}, \boxplus) \cong (GF(4), +)$ does not imply a field	Preserved and extended; Thm. 7.4 adds that a multiplication is determined; Thm. 7.5 confirms the generator is not structurally realized
[Sch26c, Def. 35] $\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\text{dc}} \times \mathbb{Z}_2^{\text{z}}$	The corpus's native four-state alphabet	Not applicable; $\Sigma_{\mathfrak{v}} \neq \Sigma_{\mathfrak{C}}$ (Rem. 7.9); identification deferred to Target 10.3
[Sch26c, Prop. 39] Acyclicity of the axiom ladder	No axiom's conclusions assumed earlier in the ladder	Preserved; Prop. 4.26 establishes the analogous acyclicity within this paper
[Sch26c, Thm. 61] Non-vacuity of A1–A12	Explicit model $\mathfrak{R}_{\text{ex}}$	Preserved, not re-established; $\mathfrak{C}_{\text{wit}}$ is a distinct witness (Rem. 5.3)
[Sch26c, Rem. 62] Licensed replacement of A10/A12 data	Downstream data may be replaced while retaining membership in $\text{Ext}_{12}(\mathfrak{C}^{\text{chr}})$	Preserved; the witness of App. A replaces A12 data only
[Sch26c, Thm. 41] $GF(4)$ field multiplication on $\Sigma_{\mathfrak{C}}$	Deferred target	Not discharged; see Target 10.3
[Sch26c, Thm. 47] Emergent assembly chirality	Deferred target	Not discharged; see Target 10.4
[Sch26c, Rem. 53] Registry $\neq$ chirality	Signed registry does not imply chiral assembly	Preserved; Rem. 3.7 applies the same distinction to the marker binary

Table 2: Concordance with [Sch26c]. Every locator was verified against the source. No entry is disposed “corrected”.

review is invited to confirm that this reading is consistent with the molecular geometry of $\mathfrak{C}^{\text{chr}}$, and in particular that the oriented perimeter cycle assigned by A8 has a uniquely defined sense in a flat projection of the planar molecular framework.

Prior statement	Content relied upon	Disposition
[Sch26a, Def. 2.7] Status labels [A] / [D] / [T] / [E]	Logical status vocabulary	Preserved; used as defined there
[Sch26a, Prop. 2.31] Framework automorphism σ	σ preserves both attachment sets; exchanges bay regions	Preserved; cited in proof of Prop. 3.4 and Rem. 3.5
[Sch26a, Rem. 2.17] Two senses of “cipher”	Naming discipline: dual-cipher locant set $\neq$ second fibration	Preserved; Rem. 2.20 applies the same discipline to the three senses of registry
[Sch26a, Hyp. 2.16] Species alphabet Σ_{sp}	Two species $\{p, d\}$ from the two attachment sets	Instantiated; Def. 2.3 supplies $\{s, c\}$ as the literal species alphabet
[Sch26a, Def. 2.10, Def. 10.4] Instance schema	Five clauses (N1)–(N5) for a fibrated transfer network	Partial; (N1)–(N3) supplied (Prop. 9.11), (N4) and (N5) not
[Sch26a, Rem. 10.5] Verification burden is not nominal	(N4) and (N5) are the substantive clauses	Preserved; Rem. 9.12 applies this caution directly
[Sch26a, Thm. 10.2] Portability	Results of §§4–8 transfer to any instance	Not invoked; requires (N4) and (N5) first
[Sch26a, Rem. 4.6] Explicit model has infinite gain	$\mathfrak{R}_{\text{ex}}$ does not witness [Sch26a]	Preserved; cited in Rem. 5.3
[Sch26a, Rem. B.2] Type mismatch: gain $\neq$ constraint functional	A bound on a norm ratio does not imply a bound on a state functional	Preserved; Rem. 9.10 applies the same non-implication to C_1 and τ_1
[Sch26b, Eq. (3)] $d_C^* = +\infty \Rightarrow$ scission	Standing restraint	Preserved; Rem. 9.10 records the same convention

Table 3: Concordance with [Sch26a]. No entry is disposed “corrected”.

Remark B.2 (Why the C_2 rotation is in R). **[E]** The proof of Proposition 3.4 identifies the C_2 rotation about the axis perpendicular to the molecular plane as lying in the allowed operation group R of Definition 3.2. The chemistry review is invited to confirm

that for a substituted chrysene unit stacked with 2-phosphoethoxy groups at all four primary or dual locants, no in-plane steric or electronic effect breaks this rotation, so that $C_2 \in R$ is the correct default rather than an exceptional case.

Remark B.3 (Catalyst and substrate assignment). [E] Remark 2.4 assigns s (substrate) to $A_{\mathbb{C}}^{\text{pri}} = \{v_3, v_6, v_9, v_{12}\}$ and c (catalyst) to $A_{\mathbb{C}}^{\text{dual}} = \{v_2, v_5, v_8, v_{11}\}$. The chemistry review is invited to confirm that this assignment is consistent with the reaction mechanism the companion paper will describe: specifically that it is the 3,6,9,12-substituted unit that undergoes transformation and the 2,5,8,11-substituted unit that induces it, and not the reverse.

B.5 Self-identified defect ledger

The following records, in the crit/rev/ver form of [Sch26b, Def. 3], the criticisms raised against this manuscript during its drafting passes and in disciplinary review, their disposition, and whether the repair has been independently examined.

No.	crit(R)	rev(R)	ver(R)
1	H_{ord} clause (ii) inverted: required $\text{ran}(\iota_{\Sigma}) \not\subseteq \mathfrak{P}_{\mathbb{C}}$; should require pairwise commutation so fibre algebra is excluded as a source of order dependence	Fully accepted; clause revised to require commutativity (Pass 2)	Explicit: Prop. 4.18 attribution clause
2	Def. 4.9 (Pass 1 form) applied letters at distinct sites, making order dependence a fact about site assignment rather than the algebra; the normal form was then the definition, not a theorem	Fully accepted; reading-head form adopted; letter applied at origin, translation unitary advances the read (Pass 2)	Explicit: Lem. 4.16 derives the normal form from the covariance relation
3	$H_{\text{ord}}(i)$ lacked non-proportionality; injectivity fails when two letters differ by a scalar, since transposing them in two positions leaves the product unchanged	Fully accepted; non-proportionality clause added (Pass 2)	Explicit: Rem. 4.22, proof of Prop. 4.21
4	H_{sec} absent; Thm. 8.4 (transport) was vacuous because for any bijection T the conjugate TcT^{-1} is automatically fixed-point-free on the target	Fully accepted; H_{sec} and Rem. 2.18 added (Pass 4)	Explicit: Rem. 8.3, proof of Thm. 8.5
5	Obvious spectral realization via the word operator fails A12 injectivity: the tracial state of A7 satisfies $\omega(W^*AW) = \omega(A)$ for any unitary W	Fully accepted; Def. 9.7 interpolation realization added; Target 10.11 opened (Pass 5)	Explicit: Prop. 9.5, numerical check (Rem. A.17)
6	Marker defined by bay region adjacent to the 10a–10b bond: the C_2 rotation of $\Pi_{\mathbb{C}}^{\text{core}}$ exchanges the two bay regions while leaving the placement unchanged, so the marker is not well defined	Fully accepted; perimeter-orientation definition adopted (Def. 3.3, Pass 2)	Explicit: Rem. 3.5, proof of Prop. 3.4

Table 4: Self-identified defect ledger in the crit/rev/ver form of [Sch26b, Def. 3]. All six items are disposed fully accepted and repaired; none is unresolved.

C Established Mathematical Literature

As in the literature appendices of [Sch26c, Sch26b], none of the subjects below is claimed to be novel in isolation. The purpose is to identify the established machinery each part of this paper draws upon, and to keep that machinery visibly distinct from the specific organisation the encoded stack proposes.

C.1 Torsors and principal homogeneous spaces

The notion of a torsor (principal homogeneous space) under an abelian group, used throughout Sections 3–6 to describe both $\Sigma_\flat$ and Φ , is standard algebra; see [Bou70, Ch. I, §5] for the general definition and [Ser79, Ch. I] for the case of finite groups. The key property used here — that a torsor under an abelian group G acquires a group structure isomorphic to G upon designation of a base point — is the elementary observation that choosing a zero turns translations into vectors.

C.2 Finite fields

The arithmetic of $GF(4)$, including the prime-subfield structure, the Frobenius automorphism and the fact that $GF(4)^\times \cong \mathbb{Z}/3$, is standard; the reference is [LN97, Ch. 2]. The three field structures on a four-element set with given additive group (Lemma 7.3) are a consequence of the uniqueness of $GF(4)$ up to isomorphism and the three choices of unit element among the nonzero elements.

C.3 Algebraic coding theory

The distinction between additive $GF(4)$ codes and $\mathbb{F}_4$ -linear codes used in Proposition 7.7 is developed in [CRSS98], which introduced additive codes over $GF(4)$ in the context of quantum error correction. The standard reference for $GF(4)$ codes more generally is [TVN07, Ch. 1–2].

C.4 The Calderbank–Rains–Shor–Sloane correspondence

The correspondence between binary quantum stabilizer codes and additive $GF(4)$ codes that are self-orthogonal under the trace-Hermitian inner product was established in [CRSS98]. The Pauli group modulo phases being the Klein four-group, and the order-three element of $GF(4)^\times$ cycling the three nontrivial Pauli operators, are recalled in [NC00, Ch. 10] and [Got97]. Remark 7.10 and Target 10.10 draw on these facts; neither the correspondence nor any of its consequences is claimed here as a result of the encoded stack.

C.5 Crossed products and quasi-local algebras

The quasi-local algebra $\mathfrak{Q}_\mathbb{C}$ and the crossed-product sequence algebra $\mathfrak{A}_\mathbb{C} = \mathfrak{Q}_\mathbb{C} \rtimes_\alpha \mathbb{Z}^3$ of Axiom A7 are standard constructions in the operator-algebraic treatment of extended quantum systems; see [BR87] for quasi-local algebras and inductive limits. The covariance relation $u_\mathbf{m} P_\mathbf{k} = P_{\mathbf{k}+\mathbf{m}} u_\mathbf{m}$ is the defining relation of the crossed product and the source of the noncommutativity established in [Sch26c, Thm. 8]. The commutative local sector $\mathfrak{P}_\mathbb{C} = C^*(\{P_\mathbf{k}\})$ is a standard subalgebra; see [BR87, Ch. 2].

C.6 Noncommutative geometry

The spectral triple $(\mathfrak{A}_\mathbb{C}^\circ, \mathcal{H}_\omega, D_\mathbb{C})$ of Axiom A9 belongs to the programme of [Con94]. The present paper uses A9 nowhere (Proposition 4.24); it is cited only to establish what the encoded stack does *not* require.

C.7 Benzenoid graph theory and point-group symmetry

The molecular reduct of Axiom A1 and the four-ring polyhex comparison class $\mathcal{B}_4$ are grounded in the established combinatorial theory of fused hexagonal carbon systems [GC89]. The point-group theory of the C_{2h} symmetry class, including the four elements $\{E, C_2, i, \sigma_h\}$ and the fact that C_2 preserves the orientation of the molecular plane, is standard [Cot90]. Both are used in the proof of Proposition 3.4: bipartiteness of $\mathcal{B}_4$ from [GC89], and the C_{2h} element table from [Cot90].

C.8 References

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