

The Twenty Registry Classes of a Chrysene Stack: Frame Orbits, Descent of a Letter Transformation, and a Variational Criterion

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Abstract

The frame group $\text{Fr} = \langle \text{flip}, \text{rev} \rangle$ acts on the binary marker assignments of a chrysene stack of $2N$ sites. We prove that the orbit map is the registry sequence modulo reversal and that the orbits number $2^{2N-2} + 2^{N-1}$, which is twenty for $2N = 6$; that an orbit has size 2 or 4, never 1; that its stabiliser is a subgroup of Fr determined by the class, the two families of size-2 orbits being distinguished by the central entry of the registry sequence; and that by the maximal-frame-invariant theorem of the predecessor every candidate assembly invariant factors through this classification. We then determine when a letterwise transformation into a secondary alphabet transports the classification. Under a single structural clause on the target — a distinguished partition into two two-element classes — transport of the registry invariant, conjugation of the structural group, conjugation of complementation, intertwining of the two frame-group actions, and descent to the orbit sets are equivalent; exactly eight of the twenty-four bijections satisfy them, and a transformation failing them carries a single orbit onto a set meeting four distinct orbits. The eight induce only four maps of orbit sets, so no orbit-level quantity separates more than four; the residual ambiguity is exactly the projection reference the frame supplies. On an ensemble of classes and a weight on the target classes we define a functional, exhibit the exact algebraic locus on which its maximiser fails to be unique, and prove uniqueness off that locus.

Two coincidences of counting are refuted rather than used. Agreement of counts between source and target is evidence of nothing: the target's own classification has twenty classes with the same eight-and-twelve split whatever the transformation. The twenty classes agree with the twenty three-letter multisets only at $N = 3$; at $N = 4$ the counts are seventy-two and thirty-five. No correspondence with the genetic code is asserted or used: our surjection of the sixty-four words onto the twenty classes has every fibre of size two or four, in the profile eight and twelve, whereas the standard genetic code has fibre sizes one through six over twenty amino acids together with three further stop codons, so the two surjections $64 \rightarrow 20$ are not the same map. That the functional is maximised by any process is a named hypothesis used in no proof, and that its maximisation is a survival property in the sense of the transfer-geometry paper is a Target with its hypotheses named.

Keywords: Chrysene Formalism; encoded stack; frame group; Burnside orbit count; registry sequence; descent; equivariance; torsor; variational criterion; non-implication.

1 Introduction

1.1 The problem

The encoded stack of the predecessor paper [Sch26d] carries a four-letter alphabet on its vessels, a canonical registry invariant, and a frame of reference external to it; its Theorem 6.14 establishes that the registry sequence modulo reversal is the maximal frame invariant, so that every quantity attributable to the assembly rather than to the reading of it factors through that class. What the predecessor does not do is say how many such

classes there are, what distinguishes them from one another, or which of them a transformation of the stack can carry onto which. Its Proposition 9.13 accordingly names the candidate assembly invariants without exhibiting the objects that carry them, and its Theorem 8.5 states the conditions under which a transformation transports the structure of a single letter without saying what transport does to a stack.

Three gaps follow. The elementary objects of the classification are not enumerated, so no statement of the corpus is about them. The transport conditions of Theorem 8.5 are conditions on

a map of four-element sets, and it is not established that a map satisfying them acts on the classification at all. And no quantity has been exhibited whose optimisation distinguishes one admissible transformation from another, so the standing language of a selected or preferred transformation has no referent.

1.2 The objective

To classify the marker assignments of a stack of $2N$ sites under the frame group, and to count and characterise the classes; to determine exactly when a letterwise transformation into a secondary alphabet descends to the classification, and to exhibit the failure when it does not; to determine how many distinct actions on the classification the descending transformations induce; to construct, from an ensemble and a weight on the target classes, a functional on the admissible transformations, and to determine exactly when its maximiser is unique; and, throughout, to state at the point of use which coincidences of counting are evidence and which are not.

1.3 What is established

1. The orbit count $2^{2N-2} + 2^{N-1}$ for stacks of even site number $2N$, by three independent arguments, together with the statement of the count as a theorem about encoded stacks under named hypotheses (Lemma 3.1, Theorem 3.3).
2. The stabiliser refinement: every orbit has size 2 or 4; the stabiliser is a subgroup of the frame group determined by the class; and the two families of size-2 classes are separated by the central entry of the registry sequence (Proposition 3.6).
3. The descent theorem: five conditions on a letterwise transformation, equivalent under one structural clause on the target, the last two being new; and the refinement that the induced bijection of classes preserves size but not stabiliser type (Theorem 4.9).
4. The count of the descending transformations, exactly eight of twenty-four, and an explicit non-descending transformation carrying one class of size four onto a set meeting four distinct classes.
5. The structure of the induced action: the eight form a torsor, the induced maps of class sets number four, and the two-element indeterminacy is exactly the projection reference supplied by the frame (Proposition 5.3, Theorem 5.8).
6. The variational criterion: its well-posedness, the exact algebraic locus of non-uniqueness of its maximiser, and uniqueness off that locus (Definition 6.6, Theorem 6.12).

1.4 What is not claimed

No axiom is introduced (Remark 2.2). No deferred target of [Sch26c] is discharged, and no target of [Sch26d] is discharged;

in particular no transformation is constructed here, which remains Target 10.5 of that paper. No survival property is asserted of any quantity constructed here. Axiom A_7 is used nowhere (Proposition 2.7), so no statement below is a statement about sequence dependence, and the order-dependence theorem of the predecessor is not invoked.

Two coincidences of counting are refuted here rather than used, and they are stated at the outset because both invite a reading that the body of the paper would otherwise be taken to support.

The first concerns the transformation. The secondary alphabet, carrying its own structure, has its own classification, and that classification has twenty classes with the same split into eight of size two and twelve of size four *whatever the transformation is*, and indeed whether or not any transformation is given. Agreement between the number of source classes and the number of target classes is therefore a fact about the two alphabets separately, and carries no information about a map between them. The content of Section 4 is descent, not counting.

The second concerns the number twenty itself, and the number sixty-four standing beside it. At $N = 3$ the classes number twenty and the words number sixty-four; the three-letter multisets over a four-letter alphabet also number twenty; and the genetic code is a surjection of sixty-four codons onto twenty amino acids. Neither coincidence is used, and both are refuted: the multiset count and the class count agree at $N = 3$ and at no other length, being seventy-two and thirty-five at $N = 4$; and our surjection of the sixty-four words onto the twenty classes has every fibre of size two or four, in the profile eight and twelve, whereas the genetic code has fibre sizes one, two, three, four and six over its twenty amino acids, together with three codons lying in no fibre at all, so that the two surjections $64 \rightarrow 20$ differ in their fibre-size profiles and in the presence of a distinguished null class, and are not the same map. No identification of the classes of this paper with any twenty-element set, combinatorial or biological, is asserted; any such identification would be an empirical instantiation claim in the sense of [Sch26c, Def. 6], and none is used in any proof. The full statement is Proposition 3.15.

Finally, that any process maximises the functional of Section 6 is the hypothesis H_{var} of Section 2, which is registered and used in no proof; and that maximising it maximises persistence under persistent disturbance, in the sense of [Sch26a], is a Target whose hypotheses include the two clauses of the instance schema that the predecessor records as not supplied.

1.5 Logical position

Every numbered statement carries one of the four status labels of Definition 2.1. Every result using a hypothesis beyond A_1 – A_{12} names that hypothesis in its statement, and every hypothesis is registered in Section 2.4. Every molecular reading is confined to remarks marked [E] and is used in no proof. This paper falls within the framework scope condition of [Sch26b, §4.5] and

is written to the orthogonal-constraint review bar of [Sch26b, §4.3].

Two features of the review record are stated here rather than left to the appendix. First, the defect ledger of Appendix B is populated from the pre-drafting pass onward, in the crit/rev/ver form of [Sch26b, Def. 3], and therefore records criticisms raised before any section was written as well as those raised during review; a criticism accepted before drafting is recorded as such and not retrospectively presented as a design decision. Second, targeted verification for this manuscript re-runs the mathematical stage rather than employing an independent reviewing system. By [Sch26b, §9.4], independence of pass is not independence of reviewing system, and the majority of the Volume 1, Number 1 corpus is subject to exactly that limitation; the verification recorded in Appendix B inherits it, and the reader should discount it accordingly. The concordance and chemistry stages are not affected, being assigned distinct questions in the sense of [Sch26b, §4.3].

1.6 Organization

Section 2 fixes conventions, status labels, the imported objects, the offset convention, and the hypothesis register, and records the notation concordance. Section 3 classifies the marker assignments and proves the count, the stabiliser refinement, and the non-implications of the number twenty. Section 4 fixes the abstract finite setting in which all counting statements of the paper are proved, states the first trap, and proves the descent theorem with its counterexample and its count. Section 5 determines the induced action on classes. Section 6 states the invariance trap, constructs the functional, and settles its genericity. Section 7 collects the non-implications and the Targets. Appendix A tabulates the twenty classes; Appendix B is the concordance, the chemistry item and the defect ledger; Appendix C is the established literature.

2 Conventions, Logical Status, and the Hypothesis Register

2.1 Logical status

Definition 2.1 (Status labels). Imported verbatim from [Sch26c, Defs. 1, 3, 5, 6] and used as in [Sch26a, Def. 2.7]: [A] axiomatic; [D] derived realization theorem; [T] target; [E] empirical instantiation claim.

Remark 2.2 (No thirteenth axiom). [A] This paper introduces no axiom. Every structure below is either an object imported from the predecessor, a definition internal to $\{A_1, \dots, A_{12}\}$, or a realization hypothesis in the sense

of [Sch26c, Def. 3]. The restriction is that of [Sch26d, Rem. 2.2]: an additional axiom would reopen the non-vacuity theorem [Sch26c, Thm. 61] and the acyclicity of the axiom ladder [Sch26c, Prop. 39], neither of which is re-established here.

Remark 2.3 (Two counting conventions). Statements about a purely finite object — a set of binary strings, a group of permutations of a four-element set — are proved in this paper without reference to any axiom, and are labelled [A] in accordance with Definition 2.1, under which a statement following solely from the definitions needed to interpret the axioms is axiomatic. Each such statement is instantiated once, by a corollary labelled [D], as a statement about encoded stacks under the hypotheses named there. The two are kept apart throughout, in the manner of [Sch26d, Rem. 3.9], so that a count of strings is never presented as a statement about a stack.

2.2 Imported objects

Throughout, $\mathfrak{C}$ denotes the molecular reduct of A_1 , $\mathfrak{Q}_{\mathfrak{C}}$ the quasi-local algebra of A_6 , $\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$ the sequence algebra of A_7 , and $(X, J_{\text{str}}, \mathcal{E}_0, w, \{C_j\}, \{\tau_j\})$ the admissibility datum of A_{12} , with admissible domain Ω_{adm} . The species alphabet is $\Sigma_{\text{sp}} = \{\mathfrak{s}, \mathfrak{c}\}$ of [Sch26d, Def. 2.3], and e_z is the stacking direction of [Sch26d, Def. 2.5].

The following are imported unchanged from the predecessor and are used below without restatement: the marker mk of its Definition 3.3, with values written $F = 0$ and $B = 1$ in $\mathbb{Z}/2$; the vessel of its Definition 3.10; the coordinates b_1, b_2 of its Definition 3.12 and the alphabet Σ_v of its Definition 3.13; the registry $\text{reg} = b_1 : \Sigma_v \rightarrow \mathbb{Z}/2$ of its Definition 3.15, with classes $\text{reg}^{-1}(0) = \{FF, BB\}$ and $\text{reg}^{-1}(1) = \{FB, BF\}$; the structural operations τ, ρ and the structural group $G = \{\text{id}, \tau, \rho, \tau\rho\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of its Proposition 4.7; the fixed-point-free element $c = \rho$ of its Proposition 4.10; the fact that G is the group of permutations of Σ_v preserving each registry class setwise, which is its Lemma 8.4; and the frame (k_0, ε, f) and the frame-setters $\Phi = \Phi_{\text{dir}} \times \Phi_{\text{typ}}$ of its Definitions 6.1 and 6.2.

Definition 2.4 (The normaliser of the structural group). $\mathcal{N}$ denotes the group of permutations of Σ_v carrying the registry partition $\{\text{reg}^{-1}(0), \text{reg}^{-1}(1)\}$ to itself, the two classes being permitted to be exchanged.

Lemma 2.5 (Elementary properties of $\mathcal{N}$). [A] $\mathcal{N} = N_{\text{Sym}(\Sigma_v)}(G)$ has order 8; it contains G as the subgroup of index two preserving each class setwise, the other coset consisting of the four permutations exchanging the two

classes; and ρ lies in the centre of N .

Remark 2.6. N is isomorphic to the dihedral group of order eight. The isomorphism is recorded here and used nowhere; the letter D is reserved throughout for the registry sequence and for the derived-realization status label, and $\mathfrak{D}$ for the vessel decomposition.

Proposition 2.7 (Provenance). [A] *Each constituent of the constructions of this paper is supplied by a named axiom, as follows: the site set and the translation action, together with the stacking direction, by A_6 ; the two attachment sets, hence the species alternation, by A_1 ; the non-collapse of the marker by A_2 ; the oriented external perimeter cycle read by the marker, by A_8 ; and the constraint functionals and the admissible domain by A_{12} . No constituent is supplied by $A_3, A_4, A_5, A_7, A_9, A_{10}$ or A_{11} .*

Proof. By inspection of the cited definitions of the predecessor against the statements of the axioms, exactly as in its Proposition 4.24, from which the present list differs in one entry only. That paper records A_7 as supplying the translation unitaries and the covariance relation, which enter its word operator; no statement of the present paper refers to the word operator, to the sequence algebra, or to any translation unitary, and the negative assertion follows for A_7 as for the remaining six. $\square$

Remark 2.8 (Consequence of Proposition 2.7). The present paper is a combinatorial construction over the alphabet and the site set. In particular the sequence dependence established at [Sch26d, Thm. 4.17] and the non-implication of its Proposition 4.18 are not used, and no conclusion below may be read as a statement about the order dependence of an encoded word. The word space enters only as a set of 4^N elements carrying a group action.

2.3 Stacks, the offset convention, and the two frame groups

Definition 2.9 (Standing setting). Throughout, a stack is a stack of $2N$ sites $n_1, \dots, n_{2N}$ in the sense of [Sch26d, Def. 3.1], with $N \geq 1$; the site number is even, and odd site numbers are not treated. A marker assignment is an element $m = (m_1, \dots, m_{2N}) \in \{0, 1\}^{2N}$, $m_j = \text{mk}(n_j)$. The offset is fixed at $o = 0$ throughout, so that the vessel decomposition is $\mathfrak{D} = \{(n_1, n_2), \dots, (n_{2N-1}, n_{2N})\}$ and the stack carries N vessels. Statements requiring $N \geq 2$ say so.

Remark 2.10 (Why the offset is fixed). By [Sch26d, Def. 6.3, Prop. 6.10] the two offsets give distinct vessel decompositions, of N and $N - 1$ vessels, and by its Remark 6.4 the offset is carried by the position of the initiator and not by the frame-setter, so that it is not acted on by the frame group. Fixing $o = 0$ is therefore a convention and not a hypothesis: it selects which of two descriptions of one stack is read, and by [Sch26d, Prop. 6.11] no information is lost in the selection. The classification of Section 3 is nevertheless carried out on marker assignments and not on words, and is by [Sch26d, Thm. 6.14(i)] independent of the offset; the offset convention is needed only for the comparison of the two group actions in Lemma 2.15, and the extension of Sections 4 and 6 to $o = 1$ is Target 3.14.

Definition 2.11 (Registry sequence, class, central entry). For a marker assignment m of a stack of $2N$ sites, the registry sequence is

$$D(m) = (D_1, \dots, D_{2N-1}), \quad D_n = m_n + m_{n+1} \in \mathbb{Z}/2,$$

its class $\overline{D}(m)$ is its orbit under reversal, and its central entry is $D_N(m) = D_N(m)$, the middle term of a sequence of odd length $2N - 1$.

Remark 2.12 (Notation for the class). The predecessor writes $[D(m)]$ for the class of the registry sequence modulo reversal. We write $\overline{D}(m)$, reserving square brackets for the status labels of Definition 2.1. The objects are the same and the concordance records the substitution.

Definition 2.13 (The frame group on markers). $\text{Fr} = \langle \text{flip}, \text{rev} \rangle$ acts on marker assignments by

$$\text{flip}(m) = (1 - m_1, \dots, 1 - m_{2N}), \quad \text{rev}(m) = (m_{2N}, \dots, m_1).$$

This is the frame group of [Sch26d, Def. 6.13]; by its Proposition 6.7 the two generators are the actions of $f \mapsto f + 1$ and of $\varepsilon \mapsto -\varepsilon$ at the level of markers.

Definition 2.14 (The frame group on words). For a four-element set Σ carrying commuting involutions ρ and τ , Fr acts on Σ^N by

$$\text{flip} = \rho^{\times N}, \quad \text{rev} = (\text{reversal of positions}) \circ \tau^{\times N}.$$

Lemma 2.15 (The two actions agree). [D] *Assume H_{mk} , H_{stack} and H_{frm} , and fix a frame with projection reference f . Let W carry a marker assignment to the word of its*

vessels under the offset convention of Definition 2.9. Then $W : \{0, 1\}^{2N} \rightarrow \Sigma_v^N$ is a bijection, and

$$W \circ \text{flip} = \rho^{\times N} \circ W, \quad W \circ \text{rev} = (\text{reversal}) \circ \tau^{\times N} \circ W.$$

Consequently W is Fr-equivariant for the actions of Definitions 2.13 and 2.14, and the two actions have the same orbits under the identification it supplies.

Proof. Bijectivity is [Sch26d, Prop. 3.14] applied vesselwise: the letter of a vessel determines the ordered pair of markers of its two units, the coordinate change being invertible, and the vessels of the offset-0 decomposition partition the sites. For the first identity, incrementing every marker fixes $b_1 = \text{mk}(\mathbf{c}) + \text{mk}(\mathbf{s})$ and increments $b_2 = \text{mk}(\mathbf{s}) + f$, which is ρ by [Sch26d, Lem. 4.6], applied in each vessel independently. For the second, reversal of the site index carries the vessel (n_{2i-1}, n_{2i}) to the pair of the same two sites read in the opposite order, by [Sch26d, Rem. 6.4], and carries the i -th vessel of the stack to the $(N + 1 - i)$ -th position; the effect on the order of the letters is therefore the reversal of positions, and the effect within a letter is the exchange of the two units, which is τ by [Sch26d, Lem. 4.6] and is the letterwise part of $\varepsilon \mapsto -\varepsilon$ by its Proposition 6.7(ii). The final assertion is immediate, the generators being carried to the generators. $\square$

Remark 2.16 (What reversal does and does not exchange). Reversal of the site index exchanges the positions of the two units of a vessel and therefore exchanges which of them is read second. It does not alter the species of any site, and no step of Lemma 2.15 uses the species labelling beyond the alternation that makes each vessel carry one unit of each species. This is consistent with the predecessor's Proposition 3.16, by which reg is invariant under exchange of the two units; it is the second coordinate b_2 , and only that coordinate, on which the exchange acts.

Remark 2.17 (Verification). Lemma 2.15 was checked on the witness stack $\mathfrak{E}_{\text{wit}}$ of [Sch26d, App. A], whose marker assignment is (F, F, F, B, B, F) and whose word under the default frame is (FF, FB, BF) . The images of that marker assignment under flip and rev have words (BB, BF, FB) and (FB, BF, FF) , which are respectively $\rho^{\times 3}$ and reversal composed with $\tau^{\times 3}$ applied to (FF, FB, BF) , and which are the words read by the frame-setters φ_2 and φ_3 of [Sch26d, Prop. A.7]. The identification of the two computations is the content of the lemma.

hypothesis. The first five are imported; H_{sec} is imported in a weakened form and the weakening is recorded in the concordance; the last five are introduced here.

Hypothesis 2.18 (H_{mk} , marker realization). As [Sch26d, Hyp. 2.6].

Hypothesis 2.19 (H_{stack} , z -constraint). As [Sch26d, Hyp. 2.7].

Hypothesis 2.20 (H_{frm} , external frame datum). As [Sch26d, Hyp. 2.8]: a structure external to the stack, not determined by it, supplies the origin, the direction sign ε and the projection reference f .

Hypothesis 2.21 (H_{ind} , vessel independence). As [Sch26d, Hyp. 2.11]. Its consequence used here is [Sch26d, Prop. 3.19]: the set of words is all of Σ_v^N .

Hypothesis 2.22 (H_{bij} , bijectivity). As [Sch26d, Hyp. 2.12]: the transformation T_φ is a bijection $\Sigma_v \rightarrow \Sigma'_v$.

Hypothesis 2.23 (H_{sec} , structured secondary alphabet). The secondary alphabet Σ'_v carries, from its own structural operations and not by transport from the source, a distinguished partition into two two-element classes.

Remark 2.24 (The weakening, and what it costs). [Sch26d, Hyp. 2.17] posits on Σ'_v a group $G' \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the induced invariant reg' , and the fixed-point-free element c' . Hypothesis 2.23 posits the partition alone, and the three are recovered from it: G' is the group of permutations preserving each class setwise, by [Sch26d, Lem. 8.4]; c' is its unique fixed-point-free element, by [Sch26d, Prop. 4.10] applied to G' ; and reg' is the indicator of the partition, determined up to the labelling of its two values, by [Sch26d, Thm. 4.9] applied to G' . Conversely each of the three determines the partition. The four data are therefore equivalent, and Hypothesis 2.23 is the weakest of them. Nothing is lost: Remark 2.18 of the predecessor, that without a structural clause on the target the transport theorem is vacuous because transported structure always exists, applies verbatim to the weakened form and is the reason it cannot be weakened further.

2.4 The hypothesis register

Each hypothesis below is a realization hypothesis in the sense of [Sch26c, Def. 3]. No statement of this paper uses an unnamed

Hypothesis 2.25 (H'_{ind} , target independence). The words of the secondary realization are all of $(\Sigma'_0)^N$.

Remark 2.26. H'_{ind} is used only where the number of target classes is asserted, which is Proposition 4.6 and the size refinement of Theorem 4.9. It is the target-side analogue of H_{ind} , and the predecessor supplies no reason for it, the secondary realization being undescribed there.

Hypothesis 2.27 (H'_{ens} , frame-invariant ensemble). A probability measure μ is given on the set X of frame classes of Theorem 3.3.

Remark 2.28. The ensemble is required on classes and not on words. By Theorem 3.3 the classes are the frame-invariant objects, and a measure on words is a datum of the reading; the two are compared in Proposition 6.4, where the cost of the word-level alternative is computed. A measure on words induces one on classes by pushforward, so H'_{ens} is the weaker datum. H'_{ens} replaces [Sch26d, Hyp. 2.13], which is a measure on the word space; the concordance records the substitution.

Hypothesis 2.29 (H_{fit} , secondary weight). A function w' on the words of the secondary realization is given which is Fr' -invariant, equivalently a function on the set X' of target classes.

Remark 2.30 (Provenance stipulation, used in no proof). The intended reading of H_{fit} is that w' is supplied by the secondary realization's own structure and not by transport from the source. That stipulation is recorded here and is used in no proof: every statement of Section 6 uses the invariance clause of Hypothesis 2.29 and nothing else. The distinction matters because the provenance stipulation is what a physical reading of the functional would rest on, while the invariance clause is what its theorems rest on, and the two must not be allowed to borrow each other's warrant.

Hypothesis 2.31 (H_{gen} , non-degeneracy). The pair (μ, w') lies outside the locus Z of Definition 6.9.

Hypothesis 2.32 (H_{var} , variational selection). Let an encoded stack carry an ensemble μ in the sense of H'_{ens} and a secondary weight w' in the sense of H_{fit} . The transformation realized by the process under consideration lies among those at which the functional Φ of Definition 6.6

attains its maximum over the admissible transformations of Theorem 4.9.

Remark 2.33 (Role of H_{var} : none). H_{var} is used in no proof of this paper. It occurs in exactly two places: the interpretive definition closing Section 6, and Target 7.4. Every mathematical statement of Sections 3–6 — the classification, the descent theorem, the structure of the induced action, the well-posedness of Φ , the degeneracy locus, the uniqueness of the maximiser, and the determination of the transformation given the frame — is proved without it, and none of them becomes false, or loses a hypothesis, if H_{var} fails. H_{var} is a realization hypothesis in the sense of [Sch26c, Def. 3] and not an empirical instantiation claim; an assertion that a physically realized process satisfies it would be a further statement, of type [E], and none is made here. The non-implication that a unique maximiser is not a selection is Proposition 7.1(v).

Hypothesis 2.34 (H_{chem} , chemical instantiation). As [Sch26d, Hyp. 2.15].

Remark 2.35 (Status of H_{chem}). [E] As [Sch26d, Rem. 2.16]: H_{chem} is used in no proof, and no theorem below becomes false, or loses a hypothesis, if it fails.

Remark 2.36 (Three senses of registry, and a fourth name avoided). The predecessor's Remark 2.20 keeps apart the vessel registry reg, the z -registry η of A_6 carried into $\Sigma_{\mathfrak{C}}$ by A_8 , and the frame-setter registry coordinate Φ_{typ} . This paper adds no fourth sense: the registry sequence of Definition 2.11 is the sequence of values of the registry operator of [Sch26d, Def. 4.2] on adjacent pairs, and the registry partition of Definition 2.4 is the fibre partition of reg. The target invariant reg' is named only as a consequence of Hypothesis 2.23, per Remark 2.24.

2.5 Notation concordance

Table 1 records the symbols introduced or renamed here. Symbols imported unchanged from the predecessor are not listed.

Remark 2.37 (Caution on λ). The element κ of K is canonical, being the image of the structural group G . The coset $\{\lambda, \kappa\lambda\}$ is canonical; neither of its two elements is. Naming one of them λ requires the labelling of the two registry classes of the target, which is the same labelling that clause (i) of Theorem 4.9 carries explicitly, and which by

Symbol	Meaning
N	stabiliser of the registry partition, order 8
Fr, Fr'	frame groups on source and target words
$D(m)$	registry sequence, length $2N - 1$
$\overline{D}(m)$	its class modulo reversal; written $[D(m)]$ in [Sch26d]
D_N	central entry D_N of the registry sequence
X, X'	source and target class sets
K	$N/\langle \rho \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$
κ	canonical element of K , class of $\{\tau, \tau\rho\}$
λ	a class-inverting element of K ; see the caution below
θ_T	automorphism of Fr induced by an admissible T
$\Phi, \mu, w', \widehat{w}$	functional; ensemble; weight; w' in source coordinates
$Z, \mathcal{P}$	degeneracy locus; parameter space
W	the offset-0 marker-to-word bijection

Table 1: Symbols introduced or renamed in this paper.

[Sch26d, Prop. 6.5(ii)] is not supplied by the structure. Every statement below in which λ occurs is invariant under the substitution $\lambda \mapsto \kappa\lambda$, and this is checked where it is not evident.

3 The Orbit Classification

This section classifies the marker assignments of a stack of $2N$ sites under the frame group. Section 3.1 proves the count as a statement about binary strings; Section 3.2 instantiates it as a statement about encoded stacks; Section 3.3 refines it by stabiliser; Section 3.4 determines how much of the classification is realized within the admissible domain; and Section 3.5 states what the resulting number does not mean.

3.1 The count

Lemma 3.1 (Orbit count). [A] *Let $N \geq 1$ and let Fr act on $\{0, 1\}^{2N}$ as in Definition 2.13. The number of orbits is*

$$|\{0, 1\}^{2N}/\text{Fr}| = 2^{2N-2} + 2^{N-1}. \quad (1)$$

For $2N = 6$ the value is 20; for $2N = 4$ and $2N = 8$ it is 6 and 72.

Three proofs are given. They are independent in the sense that no one of them uses a step of another, and they are given because the number (1) carries the weight of the rest of the paper and because each proof exhibits a different part of the structure: the first the fixed-point sets, the second the orbit sizes, the third the registry sequence.

Proof 1 (Burnside). $\text{Fr} = \{\text{id}, \text{flip}, \text{rev}, \text{fliprev}\}$ is of order four, the two generators being commuting involutions with $\text{flip} \neq \text{rev}$ for $N \geq 1$. We compute the four fixed-point sets. $\text{Fix}(\text{id}) = \{0, 1\}^{2N}$, of cardinality 2^{2N} . A fixed point of flip would satisfy $m_j = 1 - m_j$ in $\mathbb{Z}/2$, so $\text{Fix}(\text{flip}) = \emptyset$. A fixed

point of rev satisfies $m_j = m_{2N+1-j}$ and is determined freely by $m_1, \dots, m_N$, the length being even so that no index is its own image; hence $|\text{Fix}(\text{rev})| = 2^N$. A fixed point of fliprev satisfies $m_{2N+1-j} = 1 - m_j$ and is likewise determined freely by $m_1, \dots, m_N$, so $|\text{Fix}(\text{fliprev})| = 2^N$. Burnside's lemma gives

$$\frac{2^{2N} + 0 + 2^N + 2^N}{4} = 2^{2N-2} + 2^{N-1}. \quad \square$$

Proof 2 (orbit sizes). Call m palindromic when $\text{rev}(m) = m$ and antipalindromic when $\text{rev}(m) = \text{flip}(m)$. The two conditions are exclusive: if both held then $m = \text{flip}(m)$, which is impossible. There are 2^N of each, as in Proof 1. If m is palindromic then $\text{Fr} \cdot m = \{m, \text{flip}(m)\}$, and the same holds if m is antipalindromic; either way the orbit has two elements, and each such orbit consists of two strings of the same kind, flip preserving both conditions. The palindromic strings therefore form 2^{N-1} orbits and the antipalindromic strings a further 2^{N-1} . Every remaining string has trivial stabiliser, since a nontrivial stabiliser would contain rev or fliprev , and its orbit has four elements; the remaining strings number $2^{2N} - 2^{N+1}$ and form $2^{2N-2} - 2^{N-1}$ orbits. The total is $2^{N-1} + 2^{N-1} + 2^{2N-2} - 2^{N-1} = 2^{2N-2} + 2^{N-1}$. $\square$

Proof 3 (registry sequences). The map D of Definition 2.11 is a homomorphism of groups $\{0, 1\}^{2N} \rightarrow \{0, 1\}^{2N-1}$ whose kernel is the set of constant strings $\{0, 1\}$, a string with all consecutive differences zero being constant; this is [Sch26d, Thm. 6.14(ii)]. Its image therefore has 2^{2N-1} elements and is all of $\{0, 1\}^{2N-1}$. By [Sch26d, Thm. 6.14(i)], flip acts trivially on registry sequences and rev carries D to its reversal, so the orbits of Fr on strings correspond to the orbits of the two-element group generated by reversal on $\{0, 1\}^{2N-1}$. A sequence of odd length $2N - 1$ fixed by reversal is determined freely by its first N entries, so there are 2^N of them, and Burnside's lemma for the two-element group gives $(2^{2N-1} + 2^N)/2 = 2^{2N-2} + 2^{N-1}$. $\square$

Remark 3.2 (The parity restriction). Formula (1) is for an even site number. The two places where evenness is used are the count of the fixed points of rev and fliprev in Proof 1, and the odd length $2N - 1$ of the registry sequence in Proof 3. For an odd site number the antipalindromic strings would be absent, since the central index would require $m = 1 - m$, and the count would differ. Odd stacks are outside the vessel decomposition of [Sch26d, Def. 3.10] and are not treated here.

3.2 The classification of encoded stacks

Theorem 3.3 (The frame classification). [D] Assume H_{mk} , H_{stack} and H_{frm} , and let the stack have $2N$ sites. Then

- (i) $m \mapsto \overline{D}(m)$ is the orbit map of Fr , and every frame-invariant function of the marker assignment factors uniquely through it;
- (ii) the set X of frame classes has $2^{2N-2} + 2^{N-1}$ elements, which is 20 for $2N = 6$;
- (iii) every quantity attributable to the assembly rather than to the reading of it is a function on X .

Proof. Clause (i) is [Sch26d, Thm. 6.14], whose hypotheses are those named here: H_{mk} and H_{stack} make the marker a two-valued function of the placement that no operation available to a unit alters, by [Sch26d, Prop. 3.4], so that the marker assignment is an element of $\{0, 1\}^{2N}$; and H_{frm} makes Fr the group of frame changes at the level of markers, by [Sch26d, Prop. 6.7]. Clause (ii) is Lemma 3.1 applied to the orbit set of (i). Clause (iii) is [Sch26d, Prop. 9.13] together with (i). $\square$

Remark 3.4 (Why the theorem is [D] and the lemma [A]). Lemma 3.1 is a statement about binary strings and uses no axiom and no hypothesis. Theorem 3.3 is a statement about encoded stacks, and the identification of their marker assignments with $\{0, 1\}^{2N}$, and of the frame changes with Fr , is what the three named hypotheses supply. The predecessor's Theorem 6.14 carries the label [D] for the same reason, and consistency with it is required.

Remark 3.5 (Citation). Clause (iii) rests on Proposition 9.13 of the predecessor, not on its Proposition 9.10, which is a different statement; drafts of the present paper circulated before Pass 1 cite the latter, and the correction is recorded in the defect ledger.

3.3 The stabiliser refinement

Theorem 3.3 counts the classes. The following distinguishes them by the symmetry of the stack, and does so in terms of the registry sequence alone, which by Theorem 3.3(iii) is the only kind of datum admissible for the purpose.

Proposition 3.6 (Stabiliser type and the central entry). [A] Let $m \in \{0, 1\}^{2N}$.

- (i) $\text{flip} \notin \text{Stab}_{\text{Fr}}(m)$, so no orbit has one element, and $\text{Stab}_{\text{Fr}}(m)$ is one of $\{\text{id}\}$, $\{\text{id}, \text{rev}\}$, $\{\text{id}, \text{fliprev}\}$, with orbit sizes 4, 2, 2 respectively.

- (ii) Stab_{Fr} is constant on orbits; the stabiliser subgroup, and not merely its order, is an invariant of the class.
- (iii) $|\text{Fr} \cdot m| = 2$ if and only if $D(m)$ is fixed by reversal, that is if and only if $\overline{D}(m)$ contains exactly one sequence.
- (iv) In that case $\text{Stab}_{\text{Fr}}(m) = \{\text{id}, \text{rev}\}$ if and only if $D_N(m) = 0$, and $\text{Stab}_{\text{Fr}}(m) = \{\text{id}, \text{fliprev}\}$ if and only if $D_N(m) = 1$.
- (v) D_N is fixed by reversal of the registry sequence and is therefore a function on X . Consequently the three types partition X into 2^{N-1} classes of size two with $D_N = 0$, 2^{N-1} classes of size two with $D_N = 1$, and $2^{2N-2} - 2^{N-1}$ classes of size four.

Proof. (i) flip has no fixed point, as in Proof 1 of Lemma 3.1; the three remaining subgroups of Fr not containing flip are those listed, and the orbit size is the index of the stabiliser. (ii) Fr is abelian, so stabilisers of points of one orbit, being conjugate, are equal. (iii) If $|\text{Fr} \cdot m| = 2$ then m is palindromic or antipalindromic by (i), and in either case $D(\text{rev}(m)) = D(m)$: for a palindrome because $\text{rev}(m) = m$, for an antipalindrome because $\text{rev}(m) = \text{flip}(m)$ and $D \circ \text{flip} = D$. Since $D(\text{rev}(m))$ is the reversal of $D(m)$ by [Sch26d, Thm. 6.14(i)], the sequence is fixed by reversal. Conversely, if $D(m)$ is fixed by reversal then $D(\text{rev}(m)) = D(m)$, so $\text{rev}(m) \in \{m, \text{flip}(m)\}$ by [Sch26d, Thm. 6.14(ii)], and the stabiliser is nontrivial. (iv) The registry sequence has odd length $2N - 1$ and its central entry is $D_N = m_N + m_{N+1}$. If m is palindromic then $m_{N+1} = m_{2N-N}$ reads $m_{N+1} = m_N$, so $D_N = 0$; if m is antipalindromic then $m_{N+1} = 1 - m_N$, so $D_N = 1$. The two cases are exclusive and exhaust the size-two orbits by (i), so each implication is an equivalence. (v) Reversal of a sequence of odd length fixes its central entry, so D_N is constant on $\overline{D}$ and descends to X ; the three cardinals are those computed in Proof 2 of Lemma 3.1. $\square$

Corollary 3.7 (The twenty classes of a six-site stack). [D] Assume H_{mk} , H_{stack} and H_{frm} , and let $2N = 6$. Of the twenty classes, eight have size two — four with $D_N = 0$ and four with $D_N = 1$ — and twelve have size four.

Remark 3.8 (Reading of Proposition 3.6). The size-two classes are those whose registry pattern is symmetric about the central adjacent pair, and the two families are separated by whether that central pair is an agreement or a disagreement. Neither statement refers to which marker value is called F , nor to which end of the stack is read first; both are properties of the pattern of agreement and disagreement alone, which is what Theorem 3.3(iii) requires of any statement made about the assembly.

Remark 3.9 (Molecular reading). [E] In the intended reading the registry sequence records, for each adjacent pair of stacked units, whether the two markers agree, and a size-two class is a stack whose pattern of agreement reads the same from either end. This remark enters no proof.

Remark 3.10 (What the refinement will cost later). Proposition 3.6(ii) makes the stabiliser subgroup an invariant of the class, and it is the finer of the two invariants available: orbit size records only its order. The distinction is not idle. By Theorem 4.9 the admissible transformations preserve orbit size and half of them do not preserve stabiliser type, so the eight-and-four refinement of Corollary 3.7 is not transported even when the eight-and-twelve split is.

3.4 How much of the classification is realized

Theorem 3.3 counts the classes of an abstractly given marker assignment. Whether each class contains an admissible stack is a separate question, and the answer depends on data that A_1 – A_{12} do not fix.

Proposition 3.11 (Realization under the constraint set of the predecessor). [D] Assume H_{mk} , H_{stack} , H_{frm} and H_{ind} , and let the constraint functionals of A_{12} be those of [Sch26d, Def. 9.2], namely the single frame-consistency functional C_1 with threshold $\tau_1 = 1$. Then every class of X contains the marker assignment of an admissible stack, and the number of realized classes is exactly $2^{2N-2} + 2^{N-1}$.

Proof. By [Sch26d, Prop. 9.8], Ω_{adm} is the set of stacks carrying exactly one initiator, so membership in Ω_{adm} is a function of the initiator alone and of no marker of the regular sites; C_1 is an initiator count, and couples no two vessels. By H_{ind} and [Sch26d, Prop. 3.19] the set of words of stacks on a fixed site set with fixed frame data is all of Σ_v^N , and by Lemma 2.15 the corresponding set of marker assignments is all of $\{0, 1\}^{2N}$. Every class therefore contains such an assignment, and by Theorem 3.3(ii) the classes number $2^{2N-2} + 2^{N-1}$. $\square$

Remark 3.12 (The count is exact for one constraint set and an upper bound in general). Three qualifications are attached to Proposition 3.11 and none of them is removable here.

First, [Sch26d, Def. 9.2] is a construction and not an axiom; the predecessor records at its Remark 9.4 that the datum may be replaced without disturbing its Sections 3–8. A replacement constraint functional coupling the markers of distinct sites would cut Ω_{adm} inside the classes, and $2^{2N-2} + 2^{N-1}$ would then be an upper bound on the number of realized classes rather than the exact value. Theorem 3.3

is unaffected by such a replacement; Proposition 3.11 is not. Second, Proposition 3.11 is a statement about the admissible set as the predecessor defines it, and not about which stacks occur. It does not discharge [Sch26d, Prop. 3.8], which asks for an admissibility statement on the stacking that A_1 – A_{12} do not supply and which remains [T]; nothing below bears on it, and no statement of this paper asserts that any class is occupied by a physically realized assembly.

Third, the distinction the predecessor draws at its Remark 3.9, between a statement about the stabiliser of a fixed placement and a statement about the range of admissible stacks, is the distinction between Proposition 3.6 and Proposition 3.11 here. Only the conjunction makes the classification a classification of anything, and only the first is provable without H_{ind} .

Target 3.13 (Marker-coupling constraint functionals). [T] Determine whether A_{12} admits a constraint functional coupling the markers of distinct sites, and if so which classes of X survive in Ω_{adm} and whether the surviving classes are still separated by $\bar{D}$. Hypotheses required: an admissibility statement on the stacking of [Sch26d, Def. 3.1], which is the hypothesis [Sch26d, Prop. 3.8] identifies and does not supply.

Target 3.14 (The opposite offset). [T] Extend Sections 4 and 6 to the offset $o = 1$ of [Sch26d, Def. 6.3], at which the stack carries $N - 1$ vessels together with two unpaired terminal units. Theorem 3.3 is already independent of the offset, by [Sch26d, Thm. 6.14(i)]; what is not available at $o = 1$ is Lemma 2.15, and with it the identification of the frame group on markers with the frame group on words on which the descent theorem is stated. Hypotheses required: a letter for the unpaired terminal units, which [Sch26d, Def. 3.13] does not supply.

3.5 What the number twenty does not mean

The classification of a six-site stack has twenty classes and its word space has sixty-four elements. Two other pairs of numbers in the neighbourhood are equal to these, and the second of them will be recognised by every reader of the companion paper. Neither is used anywhere below, and this subsection refutes both, in the manner of the standing non-implications [Sch26b, Eqs. (2)–(3)].

Proposition 3.15 (Non-implications of the number twenty). [A] Write $q : \Sigma_v^N \rightarrow X$ for the quotient map of Theorem 3.3 under the identification of Lemma 2.15, and Mult_N for the set of multisets of N letters drawn from Σ_v .

- (i) $|X| = |\text{Mult}_N|$ holds at $N = 3$, both being 20, and at no other N : the two sequences are 2, 6, 20, 72, ... and 4, 10, 20, 35, ... In particular $|X| = |\text{Mult}_N| \not\Rightarrow$ any map between the two sets, and no map between them is constructed here.
- (ii) Let g denote the standard genetic code, a map of the sixty-four codons onto twenty amino acids together with three codons designated stop, whose fibres over the amino acids have sizes 1, 2, 3, 4, 6 with multiplicities 2, 9, 1, 5, 3 [Cri68]. At $N = 3$ the map q is a surjection of sixty-four elements onto twenty, as is g restricted to its sixty-one coding codons; but every fibre of q has size two or four, with multiplicities eight and twelve, by Corollary 3.7, and q has no fibre of size one, three or six and no element outside a fibre. Since the multiset of fibre sizes is an invariant of a surjection up to bijections of source and target, q and g are not equivalent, and no pair of bijections carries one to the other. Hence $|\Sigma_0^3| = 64$ and $|X| = 20 \not\Rightarrow$ any correspondence with the genetic code.
- (iii) No identification of X with any twenty-element set, combinatorial or biological, is asserted in this paper. Any such identification would be an empirical instantiation claim in the sense of [Sch26c, Def. 6] and would be used in no proof.

Proof. (i) The two counts are $2^{2N-2} + 2^{N-1}$ by Lemma 3.1 and $\binom{N+3}{3}$ by the standard count of multisets. At $N = 1, 2, 3, 4$ they are 2, 6, 20, 72 and 4, 10, 20, 35; they agree at $N = 3$ only, since for $N \geq 4$ the first exceeds the second. Indeed the claim holds at $N = 4$; writing $a = 2^{2N-2}$ and $b = 2^{N-1}$, so that $a \geq 8b$ for $N \geq 4$, passing from N to $N + 1$ multiplies the first count by $(4a + 2b)/(a + b) \geq 34/9 > 3$, while it multiplies the second by $(N + 4)/(N + 1) \leq 2$. (ii) A pair of bijections of source and target carrying one surjection to the other carries fibres to fibres bijectively, hence preserves the multiset of fibre sizes; the two multisets differ, one containing the value two with multiplicity eight and the other not containing the value two with that multiplicity, and the second surjection is in addition defined only after three elements of its source are removed. The fibre sizes of q are Corollary 3.7; those of g are quoted. (iii) Is a statement about what this paper asserts, and is verified by inspection of its statements. $\square$

Remark 3.16 (Provenance and scope of the datum used in the refutation). [E] Three qualifications attach to clause (ii) of Proposition 3.15.

The datum is external. The code itself is cited to [Cri68]; the degeneracy profile quoted was recomputed from the standard codon table at the domain-review pass, and gives

fibres of sizes 1, 2, 3, 4 and 6 carrying the amino acids in multiplicities 2, 9, 1, 5 and 3, with sixty-one coding codons and three stop codons. It is used only to refute, and no statement of this paper depends on it: deleting clause (ii) would weaken the paper's warnings and no theorem.

The refutation is against that table and not against every conceivable assignment. The argument compares two fibre profiles, and a surjection of sixty-four elements onto twenty with the profile of q is arithmetically possible, $8 \cdot 2 + 12 \cdot 4 = 64$; what clause (ii) establishes is that the standard code is not such a surjection, not that none exists. Any other twenty-element target is covered by clause (iii) and by nothing else.

The clause is not an assertion about biological systems. It records a property of a published table, and the paper makes no claim about any organism, any molecule, or any process.

Remark 3.17 (Why the refutations are stated here and not later). A reader who arrives at Section 4 with the numbers 64 and 20 in hand will read the descent theorem as the construction of a code, and will do so whether or not any such construction is offered; the reading is invited by the reading-frame analogy of [Sch26d, Rem. 6.12] and by the biogenesis thesis of [Sch26b, §7.2], neither of which this paper uses. Both refutations are therefore placed before the transformation is introduced, and both are stated again in Section 7. Clause (ii) is decisive at $N = 3$ itself and needs no appeal to other lengths; clause (i) needs the comparison at $N = 4$ and is the weaker of the two. No reading of the secondary alphabet is asserted here and none is used, and what a companion paper may supply is not settled by anything below.

Remark 3.18 (Numerical verification). The orbit counts, the orbit sizes, the stabiliser types and the central entries were computed by exhaustive enumeration over $\{0, 1\}^{2N}$ for $2N = 4, 6, 8$. The enumeration confirms: orbit counts 6, 20, 72, agreeing with (1); size profiles $\{2 : 4, 4 : 2\}$, $\{2 : 8, 4 : 12\}$ and $\{2 : 16, 4 : 56\}$; that no orbit has size one; that at $2N = 6$ the eight size-two classes divide four and four according to the value of D_N , in agreement with Proposition 3.6(iv) on every class and not merely on representatives; that D_N is constant on each class; and that $\overline{D}$ separates the classes, the twenty classes carrying twenty distinct values. The multiset counts 10, 20, 35 of Proposition 3.15(i) were computed in the same run. The identification of the two frame-group actions asserted in Lemma 2.15 was checked on all 64 marker assignments at $2N = 6$ as well as on the witness of Remark 2.17.

4 Descent of a Letter Transformation

Section 3 classified the stacks. This section determines when a letterwise transformation into a secondary alphabet acts on the classification at all. Section 4.1 fixes the finite setting in which every counting statement of this paper is proved; Section 4.2 states what the agreement of two counts does and does not establish; Section 4.3 proves the descent theorem; Section 4.4 counts the transformations that descend and exhibits one that does not.

4.1 Structured alphabets

Definition 4.1 (Structured alphabet). A structured alphabet is a pair $(\Sigma, \mathcal{P})$ consisting of a set Σ of four elements and a partition $\mathcal{P}$ of Σ into two classes of two elements. Its structural group $G(\Sigma, \mathcal{P})$ is the group of permutations of Σ preserving each class of $\mathcal{P}$ setwise, and $N(\Sigma, \mathcal{P})$ is the group of permutations carrying $\mathcal{P}$ to itself, the two classes being permitted to be exchanged. A morphism of structured alphabets is a bijection carrying the one partition to the other.

Remark 4.2 (Two instances, and why the setting is abstract). The alphabet Σ_v with its registry partition $\{\text{reg}^{-1}(0), \text{reg}^{-1}(1)\}$ is a structured alphabet, and its structural group is the G of [Sch26d, Prop. 4.7], by [Sch26d, Lem. 8.4]. Under H_{sec} the secondary alphabet Σ'_v is a second structured alphabet, and by Remark 2.24 the derived data G' , reg' and c' are those supplied by Definition 4.1 and require nothing further. Every counting statement of Sections 4–5 is proved for an arbitrary structured alphabet and is therefore labelled [A]; each is instantiated once, by a corollary labelled [D] naming H_{bij} , H_{sec} or H'_{ind} as the case requires. Lemma 2.5 was stated for Σ_v and holds verbatim for any structured alphabet, its proof using only the partition.

Remark 4.3 (What is canonical in G and what is not). By [Sch26d, Prop. 4.8, Prop. 4.10] applied to $G(\Sigma, \mathcal{P})$, the group has three non-identity elements: one without fixed points, which is $c = \rho$ and is canonical; and two others, each fixing one class of $\mathcal{P}$ pointwise and exchanging the other. Naming one of the latter τ is the same act as designating one class of $\mathcal{P}$, and the structure supplies no such designation. Every statement below in which τ occurs is therefore checked for invariance under the substitution $\tau \mapsto \tau\rho$, and the first such check is Lemma 4.4.

Lemma 4.4 (The frame group of a structured alphabet). [A] Let $(\Sigma, \mathcal{P})$ be a structured alphabet, let $\rho, \tau \in G(\Sigma, \mathcal{P})$ be

as in Remark 4.3, and let Fr act on Σ^N as in Definition 2.14. Then, for every $N \geq 1$:

- (i) $\text{Fr} \leq \text{Sym}(\Sigma^N)$ is a Klein four-group, and it does not depend on which of the two non-fixed-point-free elements of $G(\Sigma, \mathcal{P})$ is named τ ;
- (ii) flip has no fixed point on Σ^N , while rev and fliprev have 2^N each; consequently flip is the unique fixed-point-free element of Fr , and is canonical.

Proof. (i) ρ and τ commute and are involutions, and a letterwise map commutes with the reversal of positions, so flip and rev are commuting involutions; they are distinct because ρ is fixed-point-free and τ is not, so Fr has order four and every non-identity element is an involution. Replacing τ by $\tau\rho$ replaces rev by fliprev , and $\langle \text{flip}, \text{fliprev} \rangle = \langle \text{flip}, \text{rev} \rangle$, which is (i). (ii) $\text{flip}(w) = w$ would give $\rho(w_j) = w_j$, excluded. For rev , the condition is $w_j = \tau(w_{N+1-j})$; when N is even the first $N/2$ letters are free and determine the rest, giving $4^{N/2} = 2^N$; when N is odd the same holds for the $(N-1)/2$ outer pairs and the central letter must satisfy $w_m = \tau(w_m)$, which has two solutions, τ fixing one class pointwise, giving $4^{(N-1)/2} \cdot 2 = 2^N$. The count for fliprev is the same with τ replaced by $\tau\rho$, which likewise fixes one class pointwise. Uniqueness of the fixed-point-free element follows, and canonicity from (i). $\square$

Lemma 4.5 (All structured alphabets are isomorphic). [A] Any two structured alphabets are isomorphic; the isomorphisms between two of them form a torsor under N acting by precomposition, of cardinality eight; and every isomorphism carries G to G , ρ to ρ , and the action of Fr on Σ^N to the action of Fr on $(\Sigma')^N$. Consequently the class set Σ^N/Fr of any structured alphabet has $2^{2N-2} + 2^{N-1}$ elements, of which 2^N are classes of size two and the remainder classes of size four.

Proof. A bijection carrying one partition to the other is obtained by matching the classes in either of two ways and the elements within them in either of two ways, giving $2 \cdot 2 \cdot 2 = 8$ such bijections and none other; that they form a torsor under N is immediate, N being by Definition 4.1 the group of self-isomorphisms. Such a bijection conjugates the stabiliser of one partition onto that of the other, hence carries G to G , and carries the unique fixed-point-free element of the first to that of the second. It therefore carries the generators of Definition 2.14 to generators of the same form, up to the substitution of Lemma 4.4(i), and so carries the two actions to one another. The count is Lemma 3.1 transported along the isomorphism, the case $\Sigma = \Sigma_v$ being Theorem 3.3 under Lemma 2.15. $\square$

4.2 Trap 1: counting is not evidence

Proposition 4.6 (The target classification is independent of the transformation). [D] Assume H_{sec} and H'_{ind} . Then the set X' of classes of the secondary realization has $2^{2N-2} + 2^{N-1}$ elements, of which 2^N have size two and the remainder size four. At $2N = 6$ these are twenty, eight and twelve. The statement holds for every secondary alphabet satisfying H_{sec} , whether or not any transformation $\Sigma_v \rightarrow \Sigma'_v$ is given, and is therefore independent of T .

Proof. Lemma 4.5 applied to $(\Sigma'_v, \mathcal{P}')$, the partition being that of H_{sec} ; H'_{ind} supplies that the words of the secondary realization are all of $(\Sigma'_v)^N$, so that the class set of the realization is the class set of the alphabet. $\square$

Remark 4.7 (Trap 1). Proposition 4.6 is a fact about the target alphabet alone. Consequently

$|X| = |X'|$ and equal size profiles $\not\Rightarrow$ any correspondence $X \rightarrow$

and the agreement of the two counts, and of the two eight-and-twelve splits, is evidence of nothing whatever about a transformation. A reader who takes the matching counts as support for a correspondence has been shown only that two four-element sets each carry a partition. The content of this section is descent: whether the image of a class is a class. That is Theorem 4.9(v), it fails for two thirds of the bijections by Proposition 4.14, and it is refuted for a specific bijection in Example 4.15.

Remark 4.8 (What Proposition 4.6 does not transport). The refinement of Proposition 3.6 does not appear in Proposition 4.6, and its absence is not an omission. The central entry D_N is computed from the registry sequence of a stack of sites; the secondary alphabet carries a partition and nothing else, and in particular no site structure and no registry sequence. The size profile of X' is therefore available, while the division of its size-two classes into two families of 2^{N-1} is available only after a class of $\mathcal{P}'$ has been designated, which by Remark 4.3 the structure does not do. This is the same labelling that Theorem 4.9(i) carries explicitly.

4.3 The descent theorem

Theorem 4.9 (Transport and descent). [D] Assume H_{bij} and H_{sec} , let $N \geq 1$, and let $T : \Sigma_v \rightarrow \Sigma'_v$ be the transformation of [Sch26d, Def. 8.2], the word spaces being Σ_v^N and $(\Sigma'_v)^N$ in full. The following are equivalent.

(i) T transports registry: $\text{reg}' \circ T = \text{reg}$, after a labelling

of the two values of reg' .

(ii) T transports the structural group: $TGT^{-1} = G'$.

(iii) T transports complementation: $TcT^{-1} = c'$.

(iv) T intertwines the two frame-group actions: there is an isomorphism $\theta_T : \text{Fr} \rightarrow \text{Fr}'$ with $T^{\times N} \circ \gamma = \theta_T(\gamma) \circ T^{\times N}$ for every $\gamma \in \text{Fr}$; equivalently $T^{\times N} \text{Fr} (T^{\times N})^{-1} = \text{Fr}'$ in $\text{Sym}((\Sigma'_v)^N)$.

(v) T descends: $T^{\times N}$ carries every Fr -class of Σ_v^N onto an Fr' -class of $(\Sigma'_v)^N$.

When they hold, θ_T is unique, $\theta_T(\text{flip}) = \text{flip}'$, and $\theta_T(\text{rev}) \in \{\text{rev}', \text{flip}'\text{rev}'\}$; the induced map $X \rightarrow X'$ is a bijection; and for every class O ,

$$\text{Stab}_{\text{Fr}'}(T \cdot O) = \theta_T(\text{Stab}_{\text{Fr}}(O)). \quad (2)$$

In particular the induced bijection preserves class size, carrying the 2^N classes of size two onto the 2^N classes of size two and the remainder onto the remainder; at $2N = 6$, eight onto eight and twelve onto twelve.

Proof. The equivalence of (i), (ii) and (iii) is [Sch26d, Thm. 8.5], imported with its hypotheses H_{bij} and H_{sec} , the latter in the weakened form of Remark 2.24, which suffices because the proof of that theorem uses only the registry partition of Σ'_v and the derived data Remark 2.24 recovers from it.

(ii) $\Rightarrow$ (iv). Write $S := T^{\times N}$. By (ii) and (iii), $T\rho T^{-1} = c' = \rho'$, so $S\text{flip}S^{-1} = (T\rho T^{-1})^{\times N} = \text{flip}'$. A letterwise bijection commutes with the reversal of positions, so $S\text{rev}S^{-1} = (\text{reversal}) \circ (T\tau T^{-1})^{\times N}$. By (ii), $T\tau T^{-1} \in G'$; it is not the identity and, being conjugate to τ , is not fixed-point-free, so it is one of the two elements of G' fixing a class pointwise and $S\text{rev}S^{-1} \in \{\text{rev}', \text{flip}'\text{rev}'\} \subset \text{Fr}'$. Hence $S\text{Fr}S^{-1} \leq \text{Fr}'$, and equality holds because both groups have order four by Lemma 4.4(i). Setting $\theta_T(\gamma) := S\gamma S^{-1}$ gives (iv); θ_T is unique because S is a bijection, and $\theta_T(\text{flip}) = \text{flip}'$ was computed above, in agreement with Lemma 4.4(ii), an isomorphism carrying the unique fixed-point-free element to the unique fixed-point-free element.

(iv) $\Rightarrow$ (v). If $S\gamma = \theta_T(\gamma)S$ for all $\gamma \in \text{Fr}$ then $S(\text{Fr} \cdot w) = \text{Fr}' \cdot S(w)$ for every w , which is (v). Equation (2) follows: γ stabilises w if and only if $\theta_T(\gamma)$ stabilises Sw , and stabilisers are constant on classes by Proposition 3.6(ii), which applies to Fr' as to Fr , both being abelian.

(v) $\Rightarrow$ (iii). Put $\pi := T\rho T^{-1} \in \text{Sym}(\Sigma'_v)$ and let $a \in \Sigma_v$. Apply (v) to the constant word $w = (a, \dots, a)$. Since $\text{flip}(w)$ lies in $\text{Fr} \cdot w$, its image $S(\text{flip}(w)) = (\pi Ta, \dots, \pi Ta)$ lies in $\text{Fr}' \cdot Sw$. The four elements of $\text{Fr}' \cdot Sw$ are the constant words with letters $Ta, \rho'Ta, \tau'Ta$ and $\rho'\tau'Ta$, the reversal of a constant word being itself; that is, $\text{Fr}' \cdot Sw$ consists of the constant words with letter in $G' \cdot Ta$, which by [Sch26d, Thm. 4.9]

applied to G' is the class of Ta in $\mathcal{P}'$. Hence $\pi(Ta)$ lies in the class of Ta , and as a runs over Σ_0 the element Ta runs over Σ'_0 , so π preserves each class of $\mathcal{P}'$ setwise and lies in G' by [Sch26d, Lem. 8.4]. Being conjugate to ρ , the permutation π is fixed-point-free, and c' is the unique such element of G' by [Sch26d, Prop. 4.10]; therefore $T\rho T^{-1} = c'$, which is (iii). The final assertions are (2) together with Lemma 4.5: an isomorphism of groups carries a subgroup of order two to a subgroup of order two, so the stabiliser order, and with it the class size, is preserved, and the counts are those of Proposition 4.6. $\square$

Remark 4.10 (Where the hypotheses are used, and where $N \geq 2$ is not needed). The implication (v) $\Rightarrow$ (iii) uses only the four constant words; it is valid for every $N \geq 1$, and it is the only implication that inspects a word at all. Consequently the theorem survives restriction of the source word space to any subset containing the four constant words and closed under Fr, clause (v) being read on the classes that subset contains; under H_{ind} the restriction is vacuous, by [Sch26d, Prop. 3.19], and no hypothesis on the word space is needed in the statement. An earlier plan for this paper proved the implication by exhibiting a class of size four and so required $N \geq 2$ together with the whole of H_{ind} ; the constant-word argument above removes both requirements, and the change is recorded in the defect ledger. The restriction $N \geq 2$ is genuinely needed in Section 5 and is imposed there, not here: at $N = 1$ the classes are the two registry classes and Proposition 5.3 is false.

Proposition 4.11 (The literal equivariance condition is strictly stronger). [A] *Fix a class of $\mathcal{P}$ and a class of $\mathcal{P}'$, so that τ and τ' are named, and consider*

$$(iv') \quad T^{\times N} \circ \text{flip} = \text{flip}' \circ T^{\times N} \text{ and } T^{\times N} \circ \text{rev} = \text{rev}' \circ T^{\times N}.$$

Then (iv') implies (iv), and the implication is strict: among the transformations satisfying the equivalent conditions of Theorem 4.9, exactly those carrying the designated class of $\mathcal{P}$ to the designated class of $\mathcal{P}'$ satisfy (iv'). The remainder satisfy $\theta_T(\text{rev}) = \text{flip}'\text{rev}'$ and therefore $T^{\times N} \circ \text{rev} = \text{flip}' \circ \text{rev}' \circ T^{\times N}$.

Proof. That (iv') implies (iv) is immediate with θ_T the isomorphism matching generators. For the converse, by the proof of Theorem 4.9 the value $\theta_T(\text{rev})$ is determined by $T\tau T^{-1}$, which is τ' if T carries the class fixed pointwise by τ to the class fixed pointwise by τ' , and is $\tau'\rho'$ otherwise; these are the two cases, since T carries classes to classes by Theorem 4.9(i). For a witness of strictness take $\Sigma = \Sigma' = \{0, 1, 2, 3\}$ with classes $\{0, 1\}$ and $\{2, 3\}$, $\tau = \tau'$ the transposition $(2\ 3)$, and $T = (0\ 2)(1\ 3)$. Then T carries the partition to itself, so Theorem 4.9(i) holds, while $T\tau T^{-1} = (0\ 1) = \tau'\rho'$. $\square$

Corollary 4.12 (The induced automorphism). [D] *Let α denote the unique non-trivial automorphism of Fr fixing flip, namely $\text{rev} \mapsto \text{fliprev}$. Under the hypotheses of Theorem 4.9, and after the labelling of Proposition 4.11, θ_T takes exactly two values: the generator-matching isomorphism on the transformations carrying the designated class to the designated class, and its composite with α on the others. The two values partition the admissible transformations into two sets of equal size, and the partition is canonical although neither of its parts is named canonically.*

Proof. Fr is a Klein four-group with flip distinguished by Lemma 4.4(ii), and an automorphism fixing one non-identity element either fixes or exchanges the other two, giving exactly two such automorphisms. The two cases are those of Proposition 4.11, and they are of equal size by Proposition 4.14. That the partition into two is canonical while its parts are not is Remark 4.3 applied to $\mathcal{P}'$. $\square$

Remark 4.13 (Size is preserved; stabiliser type is not). By (2) the stabiliser of a class is transported by θ_T , and by Corollary 4.12 half of the admissible transformations transport it by α , which exchanges $\{\text{id}, \text{rev}\}$ with $\{\text{id}, \text{fliprev}\}$. Such a transformation carries a size-two class of one stabiliser type onto a size-two class of the other. The refinement of Corollary 3.7, four and four, is therefore *not* transported by half of the admissible transformations, although the split eight and twelve is transported by all of them. The assertion *eight classes of size two onto eight classes of size two* is correct; the assertion *palindromic onto palindromic* is false for four of the eight, and by Remark 4.8 is not even well posed on the target until a class of $\mathcal{P}'$ is designated.

4.4 The count, and a transformation that does not descend

Proposition 4.14 (Exactly eight of twenty-four). [A] *Of the 24 bijections between two structured alphabets, exactly eight satisfy the equivalent conditions of Theorem 4.9, namely the isomorphisms of structured alphabets; they form a torsor under $\mathcal{N}$ by precomposition. Four of them carry each designated class to the designated class and four exchange the two classes, the division being that of Corollary 4.12. The remaining sixteen carry some class of Σ^N/Fr onto a set that is not a class.*

Proof. By Theorem 4.9(i) an admissible bijection is exactly one carrying the partition to the partition, and these number eight and form a torsor under $\mathcal{N}$ by Lemma 4.5; the division into two sets of four is Lemma 2.5, G being of index two in $\mathcal{N}$. A bijection not of this kind fails (i), hence fails (v) by

Theorem 4.9, which is the last assertion. The count $24 - 8 = 16$ is $|\text{Sym}(\Sigma)| - |\mathcal{N}|$. $\square$

Example 4.15 (A transformation that does not descend). [A] Let $\Sigma = \Sigma' = \Sigma_0$ with the registry partition, let $N = 3$, and let T fix FF and FB and exchange BB with BF . Then T carries the aligned class $\{FF, BB\}$ to $\{FF, BF\}$, which is not a class, so T is one of the sixteen. The class of $FF \cdot FB \cdot BF$ has four elements, and its image under $T^{\times 3}$ is the following set of four words.

Word of the class	Image under $T^{\times 3}$	Size of its class
$FF \cdot FB \cdot BF$	$FF \cdot FB \cdot BB$	2
$BB \cdot BF \cdot FB$	$BF \cdot BB \cdot FB$	2
$BF \cdot FB \cdot BB$	$BB \cdot FB \cdot BF$	4
$FB \cdot BF \cdot FF$	$FB \cdot BB \cdot FF$	4

The four images lie in four distinct classes, of sizes 2, 2, 4 and 4; in particular the class of the image of the first word has two elements and not four, so no bijection $X \rightarrow X'$ is induced.

Proof. By direct computation, the class of $FF \cdot FB \cdot BF$ being $\{FF \cdot FB \cdot BF, BB \cdot BF \cdot FB, BF \cdot FB \cdot BB, FB \cdot BF \cdot FF\}$, obtained by applying flip, rev and fliprev to the first word. For the image of the first word, flip and rev agree on it, so its class has two elements; for the image of the second, rev fixes it, so its class has two elements; the images of the third and fourth have trivial stabiliser. The four classes are pairwise disjoint by inspection of their eight and twelve elements respectively. The computation is confirmed by the enumeration of Remark 4.18. $\square$

Remark 4.16 (What the example shows). The target of Example 4.15 carries the same structure as the source and therefore has, by Proposition 4.6, twenty classes with the split eight and twelve. The failure is therefore not detectable by counting anything on either side, which is the point of Remark 4.7: the two class sets have the same cardinality and the same profile, and the map between the word spaces is a bijection, and nevertheless there is no induced map of class sets at all. The example also shows that the failure is not partial: it is not that the induced map is ill-behaved on some classes, but that the image of a single class of size four meets four distinct classes, of two different sizes.

Target 4.17 (Transformations that are not letterwise). [T] Theorem 4.9 concerns the letterwise map $T^{\times N}$ of [Sch26d, Prop. 8.8]. Determine which bijections $\Sigma_0^N \rightarrow (\Sigma_0')^N$ that are not letterwise descend to the class sets, and whether any of them arises from a structure the secondary realization supplies. Hypotheses required: a construction of the

transformation, which is [Sch26d, Target 10.5] and is not supplied here. The question is not vacuous: the descending bijections of word spaces form a group containing the image of $\mathcal{N}$ properly, since Fr' itself descends and acts trivially on classes.

Remark 4.18 (Numerical verification). The assertions of this section were checked by exhaustive enumeration at $N = 3$ over the 24 bijections and the 64 words. The enumeration confirms: that exactly eight bijections carry every class onto a class, and that these are exactly the eight carrying the registry partition to itself, four preserving each class and four exchanging them, as in Proposition 4.14; that the remaining sixteen each carry some class onto a set that is not a class; the four images of Example 4.15, their class sizes 2, 2, 4, 4 in the order displayed, and that the four classes so met are distinct; and the fixed-point counts of Lemma 4.4(ii), namely 0 for flip and 2^N each for rev and fliprev, verified for $N = 1, 2, 3, 4$.

5 The Induced Action on Classes

Theorem 4.9 produces, from each of the eight admissible transformations, a bijection $X \rightarrow X'$. This section determines how many distinct bijections are so produced, which is fewer than eight, and identifies exactly what is lost.

5.1 The action of $\mathcal{N}$ on the classes

Definition 5.1 (The induced map). For an admissible T in the sense of Theorem 4.9, write $\Psi(T)$ for the bijection $X \rightarrow X'$ induced by $T^{\times N}$, as supplied by clause (v) of that theorem. By Proposition 4.14 the admissible transformations form a torsor under $\mathcal{N}$ acting by precomposition, so Ψ is defined on a set of eight elements.

Lemma 5.2 ($\mathcal{N}$ acts on X). [A] Every $g \in \mathcal{N}$ is an isomorphism of the structured alphabet $(\Sigma, \mathcal{P})$ with itself, hence admissible in the sense of Theorem 4.9 with $\Sigma' = \Sigma$; the assignment $g \mapsto (O \mapsto g^{\times N}(O))$ is an action of $\mathcal{N}$ on X ; and $\Psi(T \circ g) = \Psi(T) \circ g$ for every admissible T and every $g \in \mathcal{N}$.

Proof. An element of $\mathcal{N}$ carries $\mathcal{P}$ to itself, so Theorem 4.9(i) holds for it and clause (v) supplies the map of classes; the assignment is a homomorphism because $(gh)^{\times N} = g^{\times N}h^{\times N}$. The last identity is the definition of Ψ applied to the composite. $\square$

Proposition 5.3 (The induced action is not faithful; Trap 2). [A] Let $N \geq 2$.

- (i) The kernel of the action of N on X is $\langle \rho \rangle$, of order two. Every other element of N moves some class.
- (ii) ρ is central in N , so $K := N/\langle \rho \rangle$ is a group, isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and it acts faithfully on X .
- (iii) Ψ is two-to-one; its fibres are the cosets of $\langle \rho \rangle$ in the torsor of admissible transformations; and its image, the set of induced bijections $X \rightarrow X'$, has four elements and is a torsor under K .
- (iv) The subgroup $G/\langle \rho \rangle = \{1, \kappa\}$ of K is canonical, κ being the common image of the two elements τ and $\tau\rho$. The other coset $\{\lambda, \kappa\lambda\}$ is canonical; neither of its elements is, by Remark 4.3.

At $N = 1$ the statement is false: the classes are then the two classes of $\mathcal{P}$, the kernel is G , and Ψ is four-to-one onto a two-element image.

Proof. (i) That ρ acts trivially is Lemma 4.4: $\rho^{\times N} = \text{flip}$ lies in Fr , and Fr preserves each of its own classes. For the converse we exhibit, for each $g \in N \setminus \langle \rho \rangle$, a word whose class is moved. If g exchanges the two classes of $\mathcal{P}$, take w constant with letter a ; then $\text{Fr} \cdot w$ consists of the constant words with letter in the class of a , as computed in the proof of Theorem 4.9, while $g^{\times N}(w)$ is constant with letter $g(a)$, which lies in the other class. If $g \in G \setminus \langle \rho \rangle$, so that $g \in \{\tau, \tau\rho\}$, take $w = (b, a, a, \dots, a)$ with a fixed by g and b moved by g , which exists because each such g fixes one class pointwise and exchanges the other. Then $g^{\times N}(w) = (g(b), a, \dots, a)$, while the four elements of $\text{Fr} \cdot w$ are w itself; $\text{flip}(w)$, whose last letter $\rho(a) \neq a$; $\text{rev}(w) = (\tau a, \dots, \tau a, \tau b)$, whose first letter is $\tau a \in \{a, \rho a\}$ and whose last letter differs from a because $\tau b \neq a$ as $b \notin \{a, \tau a\}$; and $\text{fliprev}(w)$, whose last letter is $\rho \tau b \neq a$ likewise. Since $N \geq 2$ the word $g^{\times N}(w)$ has last letter a and first letter $g(b) \neq b$, so it equals none of the four. (ii) Centrality of ρ is Lemma 2.5; the quotient of a group of order eight with centre containing ρ by that central subgroup has order four, and is elementary abelian because N/G and $G/\langle \rho \rangle$ both have order two and every element of N squares into $\langle \rho \rangle$, as is checked on the four class-exchanging elements directly. Faithfulness is (i). (iii) By Lemma 5.2, $\Psi(T \circ g) = \Psi(T)$ if and only if g acts trivially on X , which by (i) holds exactly for $g \in \langle \rho \rangle$; the fibres are therefore the $\langle \rho \rangle$ -cosets, of which there are four, and the image is a K -torsor by (ii). (iv) G is normal in N of index two and contains ρ , so its image is a subgroup of order two, canonical because G is; the complement is the other coset. For the assertion at $N = 1$, the classes of Fr on Σ are the orbits of $G = \text{Fr}$, which are the classes of $\mathcal{P}$ by [Sch26d, Thm. 4.9], and G preserves each. $\square$

Corollary 5.4 (Four, not eight). [D] Under H_{bij} , H_{sec} and $N \geq 2$, the eight admissible transformations induce exactly four distinct bijections of class sets. No function of the induced bijection alone separates more than four of them, and any selection principle formulated on classes selects among four.

5.2 The action in registry coordinates

Proposition 5.3 is stated abstractly. Under the identification of Theorem 3.3, which represents a class by its registry class $\bar{D}$, the action of K becomes explicit, and the two non-trivial cosets behave differently in a way that Section 6 will use.

Proposition 5.5 (K on registry sequences). [D] Assume H_{mk} , H_{stack} , H_{frm} and the offset convention of Definition 2.9, and identify X with the set of registry classes. Then:

- (i) κ acts by $D_{2i-1} \mapsto D_{2i-1}$ and $D_{2i} \mapsto D_{2i-1} + D_{2i} + D_{2i+1}$, fixing every odd-indexed entry; it fixes the central entry D_N on every reversal-symmetric class, and on every class when N is odd, but not in general when N is even;
- (ii) a class-exchanging element acts by $D \mapsto D + 1$, and therefore inverts D_N on every class;
- (iii) both actions are well defined modulo reversal;
- (iv) consequently λ acts freely on X for every N , and $\{1, \kappa\}$ preserves the partition of the size-two classes by stabiliser type while the other coset exchanges the two families.

Proof. (i) The letterwise map τ exchanges the two markers of each vessel, so $m'_{2i-1} = m_{2i}$ and $m'_{2i} = m_{2i-1}$. Hence $D'_{2i-1} = m_{2i} + m_{2i-1} = D_{2i-1}$ and $D'_{2i} = m_{2i-1} + m_{2i+2} = (m_{2i-1} + m_{2i}) + (m_{2i} + m_{2i+1}) + (m_{2i+1} + m_{2i+2})$, which is the stated sum. If N is odd the central index is odd and D_N is fixed. If N is even the central entry is carried to $D_{N-1} + D_N + D_{N+1}$, which equals D_N precisely when $D_{N-1} = D_{N+1}$; that holds on every reversal-symmetric class, reversal carrying the index $N-1$ to the index $N+1$, and fails in general, as the enumeration of Remark 5.7 records at $2N = 8$. (ii) A class-exchanging letterwise map alters b_1 in every vessel and therefore flips exactly one of the two markers of each vessel; taking the map $(b_1, b_2) \mapsto (b_1 + 1, b_2)$, which fixes $b_2 = \text{mk}(s) + f$ and hence fixes the second marker of each vessel, every marker of odd index is flipped and every marker of even index is fixed, so every consecutive pair contains exactly one flipped marker and $D_n \mapsto D_n + 1$ for all n . Any other class-exchanging element differs from this one by an element of G , and the composite acts by (i) followed by the increment. (iii) The increment commutes with reversal. For κ , reversal carries

the index n to $2N - n$, which preserves parity because $2N$ is even, and carries the triple $(D_{2i-1}, D_{2i}, D_{2i+1})$ to a triple of the same form; the displayed formula is symmetric in its two outer entries, so it commutes with reversal. In particular the central entry, being fixed by reversal, is a well-defined function on classes, and (i) fixes it. (iv) If λ fixed the class of D then $D + 1$ would equal D or its reversal. The first is impossible in $\mathbb{Z}/2$. For the second, compare the entries of index N : the registry sequence has odd length $2N - 1$, so reversal fixes that entry, giving $D_N + 1 = D_N$, which is false. The argument uses only the oddness of the length and holds for every N . For the last assertion, a size-two class is reversal-symmetric by Proposition 3.6(iii), so (i) applies to it and κ fixes its value of D_N , while (ii) inverts that value; by Proposition 3.6(iv) the value is the stabiliser type. $\square$

Remark 5.6 (Consistency with Theorem 4.9). Proposition 5.5(iv) is the registry-coordinate form of Remark 4.13. The stabiliser type of a size-two class is the value of D_N , by Proposition 3.6(iv); the automorphism α of Corollary 4.12 exchanges the two stabiliser subgroups; and the transformations inducing α are exactly the class-exchanging ones, which by Proposition 5.5(ii) invert D_N . The three statements are the same fact in three notations, and their agreement is a check on all three.

Remark 5.7 (Numerical verification). At $2N = 6$ the four maps of K were computed on all twenty classes. The enumeration confirms that all four are well defined modulo reversal; that κ , λ and $\kappa\lambda$ are pairwise distinct and distinct from the identity, so that K acts faithfully as Proposition 5.3(ii) asserts; that κ fixes eight classes while λ and $\kappa\lambda$ fix none; and that κ preserves the central entry on all twenty classes, $N = 3$ being odd, while the other two invert it. The same enumeration at $2N = 4, 8$ and 10 gives the class counts 6, 72 and 272; the fixed-class counts of λ are 0 throughout, as Proposition 5.5(iv) asserts; and at the even lengths $2N = 4$ and $2N = 8$ the map κ does not preserve D_N on every class, the counts of classes fixed by $\kappa\lambda$ being 2 and 8 rather than 0, which is the failure recorded in Proposition 5.5(i). The eight classes fixed by κ at $2N = 6$ are tabulated in Appendix A.

5.3 What the two-element indeterminacy is

Proposition 5.3 leaves a fibre of two. The following identifies that fibre with a datum of a kind the formalism already knows, and states precisely which hypothesis would resolve it and which would not.

Theorem 5.8 (Identification of the residual indeterminacy). [D] Assume H_{bij} , H_{sec} , H_{frm} and $N \geq 2$, and let T be admissible.

- (i) $T \circ \rho = c' \circ T$, and the fibre of Ψ through T is $\{T, c' \circ T\} = \{T, T \circ \rho\}$.
- (ii) Precomposition with ρ is the change $f \mapsto f + 1$ of the projection reference of the frame, which by [Sch26d, Def. 6.2] is the coordinate Φ_{typ} of the frame-setter and by [Sch26d, Prop. 4.4] is the change of the zero designated in $\Sigma_{\mathbf{v}}$.
- (iii) The two members of the fibre agree at no letter, c' being fixed-point-free, and they are separated by the designation of a single element of $\Sigma'_{\mathbf{v}}$: the image of the zero that the frame designates in $\Sigma_{\mathbf{v}}$. On the source side that designation is supplied by H_{frm} , by [Sch26d, Prop. 4.4] with [Sch26d, Cor. 6.9] and [Sch26d, Cor. A.8]. On the target side H_{sec} supplies the partition of $\Sigma'_{\mathbf{v}}$, which is its canonically pointed datum, and supplies no designated element; no other hypothesis of this paper supplies one.
- (iv) Consequently the four induced bijections of class sets determine the admissible transformation up to the designation, in $\Sigma'_{\mathbf{v}}$, of the image of the source zero, and no quantity defined on classes determines it further.

Proof. (i) is Theorem 4.9(iii) rearranged, together with Proposition 5.3(iii). (ii) is [Sch26d, Prop. 6.7(i)], which identifies the letterwise action of $f \mapsto f + 1$ with ρ , together with [Sch26d, Prop. 4.4], by which the frame enters $\Sigma_{\mathbf{v}}$ exactly through the coordinate b_2 and designates the zero; H_{frm} is what supplies f . (iii) The element c' is the unique fixed-point-free element of G' , by Remark 2.24; hence T and $c' \circ T$ take different values at every letter, and prescribing the value at one letter separates them. That the frame designates a zero of $\Sigma_{\mathbf{v}}$, that the zero is aligned, and that the assignment of a zero to a frame-setter is two-to-one onto the aligned class are [Sch26d, Prop. 4.4], [Sch26d, Cor. 6.9] and [Sch26d, Cor. A.8]. That H_{sec} supplies the partition and nothing else is Hypothesis 2.23; no torsor structure on $\Sigma'_{\mathbf{v}}$ is used or available, the partition being the whole of the assumed structure. (iv) follows from (i) and (iii). $\square$

Remark 5.9 (What Theorem 5.8 does and does not resolve). The two members of a fibre are related by a global relabelling, and the relabelling may be read on either side: as the change of the source frame's projection reference, or as the change of the designated image in $\Sigma'_{\mathbf{v}}$ of the source zero. For an admissible T the two readings coincide, by clause (i). It follows that H_{frm} , which designates the zero of $\Sigma_{\mathbf{v}}$ and of nothing else, does not by itself select one member of the fibre: both members are maps out of the same coordinatised $\Sigma_{\mathbf{v}}$, and they differ by a relabelling of $\Sigma'_{\mathbf{v}}$. What clause (iv) states is the more useful thing: the residual datum is of exactly the kind that H_{frm} supplies on

the source side, namely a designated zero, and supplying it for the secondary realization is the second coordinate of [Sch26d, Target 10.5], which asks whether Φ_{typ} corresponds to the type of the secondary structure. Until that target is discharged the indeterminacy stands, and this paper asserts nothing about which member of the fibre is realized.

Remark 5.10 (The indeterminacy is not an indeterminacy of the structure). The two members of a fibre agree on every quantity invariant under global marker reversal and differ on quantities that are not. They differ at word level: by [Sch26d, Prop. 4.21] the word operator is injective on Σ_v^N , so a word and its image under flip have distinct word operators, and the two transformations produce distinct words from the same stack. The two-element ambiguity of Proposition 5.3 is therefore an ambiguity of frame-invariant quantities and not of the structure; what Theorem 3.3 forbids is not the distinction but its use in a quantity attributed to the assembly.

Remark 5.11 (The two coordinates of the frame-setter are not on the same footing). By [Sch26d, Prop. 8.7] the family $\{T_\varphi\}_{\varphi \in \Phi}$ is a torsor under G , the four frame-setters giving four transformations that differ by elements of G . Of the two coordinates, Φ_{typ} contributes the factor ρ and Φ_{dir} the factor τ , by [Sch26d, Prop. 6.7(i), (ii)] applied to the letterwise effect of each coordinate change. By Proposition 5.3 the first is invisible on classes and the second is not, κ being the image of τ and acting non-trivially by Proposition 5.5. A class-level quantity therefore distinguishes the two values of Φ_{dir} and cannot distinguish the two values of Φ_{typ} . Nothing in this paper asserts that either distinction is realized by a secondary structure; that is [Sch26d, Target 10.5] in both of its coordinates.

6 A Variational Criterion and its Well-Posedness

Section 5 left four candidate maps of class sets. This section asks what could distinguish them. Section 6.1 shows that nothing transported from the source can; Section 6.2 explains why the criterion is formulated on classes and what a word-level criterion would cost; Section 6.3 constructs the functional; Section 6.4 determines exactly when its maximiser fails to be unique; and Section 6.5 exhibits the four values in closed form and a tie.

6.1 Trap 3: relabelling-invariant quantities select nothing

Proposition 6.1 (The invariance trap). [D] Assume H_{bij} , H_{sec} and H'_{ens} . Let Y be any function defined on the probability measures on X' with the property that $Y(\beta_* \nu) = Y(\nu)$ for every bijection $\beta : X' \rightarrow X'$, and put $F(T) := Y(\Psi(T)_* \mu)$. Then F is constant on the admissible transformations. The same holds with X' replaced by the word space $(\Sigma'_v)^N$ and $\Psi(T)$ by $T^{\times N}$.

Proof. For admissible T and S the composite $\beta := \Psi(S) \circ \Psi(T)^{-1}$ is a bijection of X' , and $\Psi(S)_* \mu = \beta_*(\Psi(T)_* \mu)$; apply the invariance of Y . For the word space, $T^{\times N}$ and $S^{\times N}$ are bijections and the same argument applies without any appeal to admissibility, so the conclusion there holds on all twenty-four bijections. $\square$

Corollary 6.2 (Entropy, and every quantity like it). [D] The Shannon entropy of the pushforward ensemble, its Rényi entropies, the cardinality of its support, the multiset of its values, and every other isomorphism invariant of the pushforward measure take the same value at all eight admissible transformations. In particular no such quantity selects among them.

Proof. Each listed quantity is invariant under bijections of the underlying set; apply Proposition 6.1. $\square$

Remark 6.3 (Trap 3, and what it forces). The eight admissible transformations are relabellings, and Proposition 6.1 says that anything blind to labels is blind to them. Hence

invariance under relabelling of the target $\not\Rightarrow$ any selection,

and a criterion must be built from a datum attached to the labels of the target and not transported from the source. That datum is the weight w' of H_{fit} , and it is the only place where the secondary realization enters the construction. Nothing is computed here about entropies: Corollary 6.2 is a consequence of a one-line invariance and requires no evaluation, and evaluating one would not strengthen it.

6.2 Why the criterion is formulated on classes

Proposition 6.4 (The word-level alternative and its cost). [D] Assume H_{bij} and H_{sec} , let ν be a probability measure on the source word space Σ_v^N , let v be a real function on $(\Sigma'_v)^N$, and put $\Lambda(T) := \sum_w \nu(w) v(T^{\times N} w)$.

(i) If ν satisfies the invariance clause of H_{fit} , that is if ν

is Fr' -invariant, then $\Lambda(T) = \Phi(T)$ for the functional of Definition 6.6 computed with the pushforward of v to X ; in particular Λ takes at most four values, and the word-level formulation separates nothing that the class-level formulation does not.

(ii) Without that clause,

$$\Lambda(T) - \Lambda(T \circ \rho) = \langle v - \text{flip}_* v, v \circ T^{\times N} \rangle = \langle T_*^{\times N} v, v - v \circ \text{flip}' \rangle. \quad (3)$$

Hence Λ separates the two members of a fibre of Ψ only if v is not flip -invariant and v is not flip' -invariant.

Proof. (i) If v is Fr' -invariant it is a function w' on X' , and $v(T^{\times N} w) = w'(\Psi(T)(\bar{D}_w))$ where $\bar{D}_w$ is the class of w , by Theorem 4.9(v); summing first over each class gives the stated identity with μ the pushforward of v . The bound of four values is Corollary 5.4. (ii) Substituting $u = \text{flip}(w)$ in the sum defining $\Lambda(T \circ \rho) = \sum_w v(w) v(T^{\times N} \text{flip}(w))$ gives $\langle \text{flip}_* v, v \circ T^{\times N} \rangle$, which is the first equality; the second follows from $T^{\times N} \text{flip} = \text{flip}' T^{\times N}$, which is Theorem 4.9(iv). Each factor of a vanishing pairing being zero is sufficient for the difference to vanish, so both must be nonzero for it not to. $\square$

Remark 6.5 (The cost, stated plainly). By Proposition 6.4(ii) a word-level criterion separates all eight admissible transformations only if the ensemble distinguishes a stack from its image under global marker reversal. By [Sch26d, Prop. 4.3] the registry is invariant under that reversal, by [Sch26d, Prop. 6.7(i)] the reversal is the action of the frame's projection reference, and by Theorem 3.3 no quantity attributable to the assembly distinguishes the two. An ensemble that does distinguish them is therefore a datum of the reading and not of the assembly, and a criterion built on it selects a transformation relative to a frame-setter rather than relative to the stack. That is the cost, and it is the reason H'_{ens} is imposed on X and the invariance clause is imposed on w' . The same conclusion is reached from the other side by Theorem 5.8: the residual pair is a relabelling, and a quantity that separates it is a quantity that has fixed a zero.

6.3 The functional

Definition 6.6 (The ensemble functional). Assume H_{bij} , H_{sec} , H'_{ens} and H_{fit} . For an admissible transformation T set

$$\Phi(T) := \sum_{O \in X} \mu(O) w'(\Psi(T)(O)), \quad (4)$$

$\Psi(T)$ being the induced bijection of Definition 5.1.

Proposition 6.7 (Well-posedness). [D] Under the hypotheses of Definition 6.6, Φ is well defined; it is constant on the fibres of Ψ , hence a function of the four induced bijections alone; and it takes at most four values. For $N \geq 2$ the four fibres are the $\langle \rho \rangle$ -cosets of Proposition 5.3(iii).

Proof. Well-definedness is Theorem 4.9(v), which supplies $\Psi(T)$ as a map of class sets, together with the invariance clause of H_{fit} , which makes w' a function on X' ; the rest of (4) is a finite sum. That Φ depends on T only through $\Psi(T)$ is immediate from the formula, and the count of values is Corollary 5.4. $\square$

Remark 6.8. Proposition 6.7 is a corollary of Theorem 4.9(v) and is recorded separately only because the corresponding statement fails for the word-level functional of Proposition 6.4(ii), which is well defined but not constant on fibres.

6.4 Uniqueness of the maximiser

Fix an admissible T_0 and write $\Psi_0 := \Psi(T_0)$ and $\widehat{w} := w' \circ \Psi_0$, a real function on X . By Proposition 5.3(iii) every induced bijection is $\Psi_0 \circ k$ for exactly one $k \in K$, so that

$$\Phi_k := \sum_{O \in X} \mu(O) \widehat{w}(kO) = \langle k_* \mu, \widehat{w} \rangle, \quad k \in K, \quad (5)$$

the second equality by the substitution $O \mapsto kO$ and $k^{-1} = k$.

Definition 6.9 (Parameter space and degeneracy locus). The parameter space is $\mathcal{P} := \Delta(X) \times \mathbb{R}^{X'}$, where $\Delta(X)$ is the simplex of probability measures on X , carrying the reference measure obtained as the product of normalised Lebesgue measure on $\Delta(X)$, of dimension $|X| - 1$, with Lebesgue measure on $\mathbb{R}^{X'}$. The degeneracy locus is

$$Z := \bigcup_{\{k,l\} \subset K, k \neq l} \{(\mu, w') \in \mathcal{P} : \langle k_* \mu - l_* \mu, \widehat{w} \rangle = 0\}, \quad (6)$$

a union of six sets, one for each unordered pair of elements of K .

Lemma 6.10 (Independence of the base point). [A] The set Z does not depend on the choice of T_0 .

Proof. Replacing T_0 by $T_0 \circ g$, $g \in \mathcal{N}$, replaces Ψ_0 by $\Psi_0 \circ h$ with h the image of g in K , hence $\widehat{w}$ by $\widehat{w} \circ h$ and the index k by hk . The quantity Φ_k is unchanged as a function of the transformation it labels, and the six differences are permuted among themselves by translation in the abelian group K ; their common zero set is therefore unchanged. $\square$

Proposition 6.11 (The locus is proper and closed). [A] Let $N \geq 2$. Each of the six conditions in (6) is a nonzero bilinear condition on (μ, w') . Consequently Z is a closed subset of $\mathcal{P}$ with empty interior and reference measure zero, and the same is true for every measure on $\mathcal{P}$ that is absolutely continuous with respect to the reference measure.

Proof. Fix $k \neq l$ in K and put $m := kl \neq 1$. By Proposition 5.3(ii) the action of K on X is faithful, so there is a class O with $mO \neq O$, whence $kO \neq lO$: equality would give $(lk)O = O$, that is $mO = O$. Take $\mu := \delta_O$, the point mass, which lies in $\Delta(X)$, and take w' to be the indicator of $\Psi_0(kO)$, so that $\widehat{w}$ is the indicator of kO . Then $k_*\mu = \delta_{kO}$ and $l_*\mu = \delta_{lO}$, and the pairing is $1 - 0 = 1 \neq 0$. The condition is therefore a nonzero bilinear form, its zero set is closed and is a proper algebraic subset of $\mathcal{P}$, and a finite union of six such sets is closed with empty interior and null for the reference measure; absolute continuity transfers nullity. $\square$

Theorem 6.12 (Uniqueness of the maximiser). [D] Assume $H_{\text{bij}}, H_{\text{sec}}, H'_{\text{ens}}, H_{\text{fit}}, H_{\text{gen}}$ and $N \geq 2$. Then Φ attains its maximum at exactly one of the four induced bijections of class sets, equivalently at exactly two of the eight admissible transformations, and those two form a fibre of Ψ , that is a pair $\{T, c' \circ T\}$ in the sense of Theorem 5.8(i).

Proof. By Proposition 6.7 the functional takes the four values (5). By H_{gen} the pair (μ, w') lies outside Z , so by (6) no two of the four values coincide; a finite set of pairwise distinct reals has a unique maximum. The fibre statement is Proposition 5.3(iii) with Theorem 5.8(i). $\square$

Proposition 6.13 (Universal ties). [A] Let $k \in K, k \neq 1$.

- (i) If $k_*\mu = \mu$ then $\Phi_g = \Phi_{gk}$ for every $g \in K$ and every w' .
- (ii) If $\widehat{w} \circ k = \widehat{w}$ then $\Phi_g = \Phi_{gk}$ for every $g \in K$ and every μ .
- (iii) A constant w' ties all four values.

These are strata of Z and do not exhaust it: membership of Z is the vanishing of a pairing, not a symmetry of either factor, and Example 6.16 exhibits a point of Z at which neither μ nor $\widehat{w}$ is invariant under any $k \neq 1$.

Proof. (i) $\Phi_{gk} = \langle (gk)_*\mu, \widehat{w} \rangle = \langle g_*(k_*\mu), \widehat{w} \rangle = \langle g_*\mu, \widehat{w} \rangle$. (ii) $\Phi_{gk} = \sum_O \mu(O) \widehat{w}(gkO) = \sum_O \mu(O) \widehat{w}(kgO) = \sum_O \mu(O) \widehat{w}(gO)$, using that K is abelian. (iii) is (ii) with k arbitrary. That the strata do not exhaust Z is Example 6.16. $\square$

6.5 Closed-form values, and a tie

Let $2N = 6$, identify X with the twenty registry classes, and write a class by the lexicographically smaller of the two sequences it contains. Fix a base point T_0 and define w' on X' by prescribing $\widehat{w} = w' \circ \Psi_0$ on X , which is legitimate since Ψ_0 is a bijection.

Example 6.14 (Four distinct values). [D] Take $\mu = \frac{1}{2}\delta_{O_1} + \frac{1}{2}\delta_{O_2}$ with O_1 the class 00000 and O_2 the class 00101, the first of size two and the second of size four; and take $\widehat{w}$ to vanish except for

$$\widehat{w}(00101) = 1, \quad \widehat{w}(01101) = 2, \quad \widehat{w}(01011) = 4, \quad \widehat{w}(01001) = 8.$$

The images of the two classes under K , computed by Proposition 5.5, are as follows.

k	kO_1	kO_2	Φ_k
1	00000	00101	1/2
κ	00000	01101	1
λ	11111	01011	2
$\kappa\lambda$	11111	01001	4

The four values are distinct, so $(\mu, w') \notin Z$, and the maximiser is the bijection $\Psi_0 \circ \kappa\lambda$, attained at exactly two of the eight admissible transformations, in accordance with Theorem 6.12. The measure μ is invariant under no nontrivial element of K , since λ moves both O_1 and O_2 and κ moves O_2 .

Example 6.15 (A tie, constructed). [D] Retain μ and replace $\widehat{w}$ by the indicator of the pair $\{00101, 01011\}$, which is an orbit of $\langle \lambda \rangle$ by Proposition 5.5, so that $\widehat{w} \circ \lambda = \widehat{w}$. Then

$$\Phi_1 = \Phi_\lambda = \frac{1}{2}, \quad \Phi_\kappa = \Phi_{\kappa\lambda} = 0,$$

so the maximum is attained at two of the four induced bijections and at four of the eight admissible transformations. The pair (μ, w') lies in Z , on the stratum of Proposition 6.13(ii).

Example 6.16 (A point of the locus with no symmetry). [D] Retain μ and take $\widehat{w}$ to vanish except for

$$\widehat{w}(00101) = \widehat{w}(01101) = 1, \quad \widehat{w}(01011) = 5, \quad \widehat{w}(00001) = 3.$$

Then $\Phi_1 = \Phi_\kappa = \frac{1}{2}, \Phi_\lambda = \frac{5}{2}$ and $\Phi_{\kappa\lambda} = 0$. The pair (μ, w') lies in Z , since two of the four values coincide; μ is invariant under no $k \neq 1$, as in Example 6.14, and $\widehat{w}$ is invariant under none either, as κ, λ and $\kappa\lambda$ each move a class on which $\widehat{w}$ takes a value the image does not. Nevertheless the maximum is attained at exactly one of the four.

Remark 6.17 (The locus is larger than the failure it guards against). Example 6.16 shows two things at once. First, the symmetry conditions of Proposition 6.13 are strata of Z and not a description of it. Second, H_{gen} is sufficient for Theorem 6.12 and is not necessary: Z is the locus on which *some* pair of values coincides, whereas uniqueness of the maximiser fails only on the smaller locus

$$Z_{\text{max}} := \{(\mu, w') : \text{the maximum of } \Phi \text{ is attained twice}\} \subseteq Z,$$

the inclusion being strict by the example. The definition (6) is retained because Z is a finite union of bilinear conditions and Z_{max} is not: the latter is a union of pieces cut out by inequalities as well, and is not an algebraic set. Nothing is lost by the strengthening, Z being null by Proposition 6.11 and Z_{max} smaller still.

Remark 6.18 (Why such weights are abundant). By Proposition 5.5(iv) the group $\langle \lambda \rangle$ acts freely on X , so at $2N = 6$ it has ten orbits and the λ -invariant weights form a ten-dimensional subspace of $\mathbb{R}^{X'}$. Example 6.15 is therefore not an isolated accident but a point of a stratum of codimension ten, and Theorem 6.12 is not vacuously true for want of counterexamples: the degeneracy locus has points, it has the structure Proposition 6.13 describes, and it is nevertheless null by Proposition 6.11.

Remark 6.19 (Verification policy). The values of Examples 6.14 and 6.15 were recomputed by direct enumeration over the twenty classes, as were the four images of O_1 and O_2 , the λ -invariance of the weight of Example 6.15, the absence of any invariance in Example 6.16, and the freeness of $\langle \lambda \rangle$. No random sampling is reported. A statement that a maximiser was unique in some number of randomly drawn trials would be evidence of nothing, the degeneracy locus being null by Proposition 6.11 and therefore missed by random draws with probability one; what requires exhibition is a point of the locus, and that is Example 6.15.

6.6 The variational reading

Definition 6.20 (The Φ -maximal transformation). Assume H_{bij} , H_{sec} , H'_{ens} , H_{fit} , H_{gen} , $N \geq 2$ and H_{var} . The Φ -maximal transformation of a process with ensemble μ and secondary weight w' is the induced bijection of class sets at which Φ attains its maximum, which exists and is unique by Theorem 6.12, together with the pair of admissible transformations lying over it.

Remark 6.21 (What Definition 6.20 does not say). Definition 6.20 is the only statement of this paper in which H_{var} occurs, apart from Target 7.4, and H_{var} is used in no proof, as Remark 2.33 records. Three things are therefore not asserted. That any process maximises Φ is H_{var} and is a hypothesis, not a result: Theorem 6.12 says that a functional has a unique maximiser, which is a statement about the functional and not about a process, and the non-implication is Proposition 7.1(v). That maximising Φ maximises persistence under persistent disturbance is not established and is Target 7.4. And that w' is supplied by the secondary realization is the provenance stipulation of Remark 2.30, which is used in no proof; what the theorems use is its invariance clause, and a w' supplied by any other means would serve them equally.

Remark 6.22 (Physical reading). [E] In the intended reading μ records how often a process presents each structural class of the stack and w' records what the secondary realization makes of each of its own classes; the functional is then the expected weight of the transformed structure. This remark enters no proof, and nothing in it is asserted of any physically realized system.

7 Non-Implications and Targets

The formalism's governing restraint is that restoring a foundational object never licenses restoring an unproved implication of that object [Sch26b, §3.1], of which the standing non-implications [Sch26b, Eqs. (2)–(3)] are the recorded instances. This section states, in that spirit, what the constructions above do not establish, and collects the open targets in one place. Each non-implication is refuted by a witness named in its proof, so the section adds no new argument; its purpose is to make the list checkable.

7.1 Non-implications

Proposition 7.1 (Non-implications). [D] Assume H_{bij} and H_{sec} where the statement refers to the secondary alphabet. None of the following implications holds in general; each is refuted by the witness named in its proof.

- (i) $|X| = |X'|$, with equal size profiles $\Rightarrow$ any correspondence between X and X' .
- (ii) $|X| = |\text{Mult}_N|$ at $N = 3 \Rightarrow$ any map between the classes and the multisets, or equality at any other length.
- (iii) $|\Sigma_v^3| = 64$ and $|X| = 20 \Rightarrow$ any correspondence with the genetic code.
- (iv) Bijectivity of $T \Rightarrow$ descent of T to the class sets.

- (v) *Descent of $T \Rightarrow$ preservation of stabiliser type.*
- (vi) *Intertwining of the two frame-group actions $\Rightarrow$ literal equivariance for a fixed labelling; and conversely the two are not equivalent.*
- (vii) *Invariance of a functional under relabelling of the target $\Rightarrow$ any selection among the admissible transformations.*
- (viii) *Uniqueness of the maximiser of $\Phi \Rightarrow$ selection of that maximiser by any process.*
- (ix) *Φ -maximality $\Rightarrow$ survival in the sense of [Sch26a], or any property of a transfer realization.*
- (x) *Determination of the induced bijection of class sets $\Rightarrow$ determination of the letter transformation; and the residual pair is not resolved by H_{frm} .*
- (xi) *The count $|X| = 2^{2N-2} + 2^{N-1} \Rightarrow$ that many classes are occupied, under an arbitrary constraint set of A_{12} .*
- (xii) *Injectivity of the word operator $\Rightarrow$ that the word is attributable to the assembly.*

Proof. (i) Proposition 4.6: the target class set has the stated cardinality and profile for every secondary alphabet satisfying H_{sec} , whether or not any transformation is given, so the equality is a fact about the two alphabets separately. Example 4.15 exhibits a bijection of word spaces for which the two class sets have equal counts and equal profiles and no map between them is induced. (ii) Proposition 3.15(i): the two counts are 72 and 35 at $N = 4$, and no map is constructed at $N = 3$. (iii) Proposition 3.15(ii): the fibre-size profiles of the two surjections differ, and the multiset of fibre sizes is an invariant of a surjection up to bijections of source and target. (iv) Example 4.15, where T is a bijection and the image of a class of size four meets four distinct classes; this is the class-level form of [Sch26d, Prop. 10.1(vii)]. (v) Remark 4.13 with Proposition 5.5(ii): a class-exchanging admissible transformation inverts the central entry, hence exchanges the two families of size-two classes, while preserving every class size. (vi) Proposition 4.11, whose witness is the transformation $T = (0\ 2)(1\ 3)$ of a structured alphabet with itself, admissible and satisfying $T\tau T^{-1} = \tau'\rho'$. (vii) Proposition 6.1 and Corollary 6.2. (viii) H_{var} is a hypothesis, registered at Hypothesis 2.32 and used in no proof, as Remark 2.33 records; Theorem 6.12 is a statement about a functional on a four-element set and mentions no process. (ix) No transfer realization occurs anywhere in this paper; by Proposition 2.7 no datum of A_{10} is used, and by [Sch26d, Rem. 9.12] the encoded stack supplies neither clause (N4) nor clause (N5) of [Sch26a, Def. 10.4], without which the transformation geometry of that paper is undefined and its dissipation is identically zero. This is Target 7.4 and not a result. (x) Theorem 5.8: the fibre of Ψ is $\{T, c' \circ T\}$, whose two members are maps out of the same frame-coordinatised Σ_0 and differ by a relabelling

of Σ'_0 ; H_{frm} designates a zero of Σ_0 and of nothing else, as Remark 5.9 records. (xi) Remark 3.12: Proposition 3.11 holds for the constraint set of [Sch26d, Def. 9.2], which [Sch26d, Rem. 9.4] licenses replacing; a constraint functional coupling the markers of distinct sites would cut Ω_{adm} inside the classes. (xii) [Sch26d, Prop. A.7]: the four frame-setters read four distinct words from one and the same stack, while by [Sch26d, Prop. 4.21] the word operator is injective. The two statements are compatible and neither bears on the other; this is [Sch26d, Prop. 9.13] read as a restriction. $\square$

Remark 7.2 (Status of the individual items). Items (ii), (iii), (v) and (vi) are refuted by witnesses using no hypothesis and could carry the label [A] separately; the proposition is labelled [D] because items (i), (iv), (vii), (viii) and (x) refer to the secondary alphabet and therefore use H_{bij} and H_{sec} .

Remark 7.3 (The three that matter most for reading this paper). Items (i), (iii) and (viii) are the ones on which a reader is most likely to over-read. Item (i) states that the agreement of two counts is a fact about two alphabets and not evidence about a map between them; the content of Section 4 is descent, and descent fails for two thirds of the bijections. Item (iii) states that the classification is not a code: the surjection of sixty-four words onto twenty classes has all fibres of size two or four, and the genetic code does not, so no pair of bijections carries one to the other, whatever else the companion paper may construct. Item (viii) states that a functional with a unique maximiser is not thereby a selection principle: a process that maximises Φ is H_{var} , which is registered and used nowhere. A reader who grants all three retains exactly what this paper proves: a classification, a criterion for transporting it, and a functional whose maximiser is generically unique.

7.2 Targets

Target 7.4 (Φ as a survival quantity). [T] *Show that w' is a survival weight in the sense of [Sch26a], so that maximising Φ maximises persistence under persistent disturbance. Hypotheses required, in order: clause (N4) of [Sch26a, Def. 10.4], which is [Sch26d, Target 10.6], so that the forms carry comparable transfer realizations and the unitary-orbit distance δ_u of [Sch26a, Def. 5.1] is defined; clause (N5), which is [Sch26d, Target 10.7], so that the finiteness criterion [Sch26a, Thm. 4.17] is not vacuous; the transport conditions of Theorem 4.9, so that registry is preserved by the transformation; H_{var} , so that maximisation is the operative principle; and finally both halves of the survival conjunction, which by [Sch26a, Prop. 8.15] cannot be obtained from a single bound on the worst-case norm. Five steps, of*

which this paper takes none. By Proposition 7.1(ix) nothing here bears on the question, and by [Sch26d, Rem. 9.16] no survival property is asserted of the encoding.

Target 7.5 (Provenance of the weight). [T] *Construct w' from the structure of the secondary realization, thereby discharging the provenance stipulation of Remark 2.30, which is at present recorded and used in no proof. Hypotheses required: a description of the secondary realization beyond the partition of Hypothesis 2.23, which neither this paper nor [Sch26d] supplies. Until this is discharged the physical reading of Section 6.6 has no referent, and Φ is a functional of two given data rather than of the process.*

Target 7.6 (Provenance of the ensemble). [T] *Determine whether A_{12} supplies the measure μ of H'_{ens} , for instance through the constraint functionals of [Sch26d, Def. 9.2] together with an admissibility statement on the stacking, which is the hypothesis [Sch26d, Prop. 3.8] identifies and does not supply. This target and Target 3.13 are two halves of one question: the first asks which classes are occupied, the second with what weight.*

Target 7.7 (A zero for the secondary alphabet). [T] *Supply for Σ'_0 a designated element, namely the image of the zero that the frame designates in Σ_0 — a datum of the kind H_{frm} supplies on the source side through [Sch26d, Prop. 4.4] and [Sch26d, Cor. 6.9] — and thereby resolve the residual pair of Theorem 5.8. This is the second coordinate of [Sch26d, Target 10.5], which asks whether Φ_{typ} corresponds to the type of the secondary structure; by Proposition 7.1(x) no class-level quantity resolves it, and by Remark 5.10 the two members of the pair are distinguished at word level, so the target is not vacuous.*

Target 7.8 (The even-length anomaly). [T] *Determine the fixed-class counts of κ and $\kappa\lambda$ on X for general N , and in particular explain the dependence on the parity of N recorded at Proposition 5.5(i) and verified at Remark 5.7: for odd N the element $\kappa\lambda$ acts freely, and for even N it does not. The stabiliser refinement of Proposition 3.6 is unaffected, since it concerns reversal-symmetric classes only, on which κ fixes the central entry at every length; what is open is the orbit structure of K on the whole of X .*

Remark 7.9 (Inventory of Targets). This paper introduces eight targets: Targets 3.13 and 3.14 in Section 3, Target 4.17 in Section 4, and Targets 7.4, 7.5, 7.6, 7.7 and 7.8 here.

Seven of them ask for a datum or a discharge within the formalism; Target 7.8 is of a different kind, being a question about the combinatorics of the classification itself. Their number is not offered as evidence of anything. Two are load-bearing in the sense that the paper's connection to its own physical reading is incomplete without them: Target 7.5, without which w' is a given datum and not a datum of the secondary realization, and Target 7.4, which is the claim the functional was constructed to make possible and which this paper does not make. Target 7.7 is the first obligation of the companion paper constructing the transformation, and is the same obligation as the second coordinate of [Sch26d, Target 10.5].

Remark 7.10 (Status of the deferred program of the predecessors). No deferred target of [Sch26c] is discharged here, and none of the nine targets of [Sch26d] is discharged. Two are touched. [Sch26d, Target 10.5], the construction of the transformation, acquires in Theorem 4.9 a criterion that any such construction must satisfy, and in Target 7.7 a statement of what it must supply beyond that criterion; neither is a construction. [Sch26d, Target 10.8], the candidate survival invariants, acquires in Theorem 3.3 an enumeration of the objects those invariants are functions of; that is a classification of the candidates and not a proof that any of them is preserved.

8 Conclusion

The question this paper set out to answer was whether the maximal frame invariant of the encoded stack has elementary objects, whether a transformation of the stack acts on them, and whether anything the process supplies distinguishes one such action from another. All three answers are affirmative, and in each case the content lies in what had to be excluded before the affirmative statement could be made.

The classification. The frame group has $2^{2N-2} + 2^{N-1}$ orbits on the marker assignments of a stack of $2N$ sites, twenty for $2N = 6$, and the orbit map is the registry sequence modulo reversal (Lemma 3.1, Theorem 3.3). Every orbit has two elements or four, never one, and its stabiliser is a subgroup of the frame group determined by the class; the two families of size-two classes are separated by the central entry of the registry sequence, which is a class function because the sequence has odd length (Proposition 3.6). By [Sch26d, Prop. 9.13] these twenty classes are the objects through which every candidate assembly invariant factors, so they are the elementary objects of the survival question as that paper poses it. Whether all of them are occupied depends on a datum of A_{12} that the predecessor licenses replacing, and the count is exact for the constraint set given there and an upper

bound otherwise (Proposition 3.11, Remark 3.12).

Descent, not counting. The secondary alphabet has a classification of its own, with the same cardinality and the same split into eight and twelve, whatever transformation is given and whether or not any is (Proposition 4.6). Agreement of counts is therefore evidence of nothing. What is substantive is descent, and descent is equivalent to four conditions already present in the corpus and to one that is not: a letterwise transformation transports the registry invariant, conjugates the structural group, conjugates complementation, intertwines the two frame-group actions, and carries classes onto classes, or it does none of these (Theorem 4.9). Exactly eight of the twenty-four bijections descend, and a transformation that does not can carry a single class of size four onto a set meeting four distinct classes, two of size two and two of size four (Proposition 4.14, Example 4.15). Three refinements are worth separating from the theorem. The intertwining condition is the correct form of equivariance: the literal generator-matching condition is strictly stronger and fails for half of the admissible transformations (Proposition 4.11). Class size is transported and stabiliser type is not, the transformations inducing the nontrivial automorphism of the frame group being exactly those that exchange the two families of size-two classes (Remark 4.13, Proposition 5.5). And the hypothesis on the secondary alphabet can be reduced to a single clause, a distinguished partition into two classes, from which the group, the invariant and the complementation follow (Remark 2.24).

Four maps, and the datum that is missing. The eight admissible transformations induce only four bijections of class sets: the action of $\mathcal{N}$ on the classes has kernel $\langle \rho \rangle$, and the four induced maps form a torsor under $\mathcal{N}/\langle \rho \rangle$ (Proposition 5.3). Any criterion formulated on classes therefore selects among four, and the residual pair is not a defect of the criterion. It is a relabelling, and it is exactly the datum that the frame supplies on the source side and that nothing supplies on the target side: a designated zero (Theorem 5.8). The pair is distinguished at word level, so the indeterminacy is one of frame-invariant quantities and not of the structure (Remark 5.10). One asymmetry between the two coordinates of the frame-setter follows, and appears to be new: a class-level quantity distinguishes the two reading directions and provably cannot distinguish the two projection references (Remark 5.11).

The criterion and its exact genericity. Nothing invariant under relabelling of the target selects anything, so a criterion must be built from a weight attached to the target's own classes (Proposition 6.1). Given such a weight and an ensemble on the source classes, the functional (4) is well posed, is constant on the fibres of the induced map, and takes at most four values (Proposition 6.7). Its maximiser is unique off an explicitly described locus: the union of six bilinear conditions $\langle k_*\mu - l_*\mu, \widehat{w} \rangle = 0$, which is closed, proper and null, which contains

as strata, but is not exhausted by, the pairs in which either the ensemble or the weight has a symmetry matching the action, and which is itself larger than the locus on which uniqueness actually fails (Definition 6.9, Proposition 6.11, Theorem 6.12, Proposition 6.13, Remark 6.17). The four values are computed in closed form for a stated pair, and a tie is constructed rather than reported as absent, the weights invariant under a free element of the acting group forming a ten-dimensional space at $2N = 6$ (Examples 6.14 and 6.15).

What is not done. No axiom is introduced (Remark 2.2) and Axiom A_7 is used nowhere, so nothing here is a statement about sequence dependence (Proposition 2.7, Remark 2.8). No target of [Sch26c] or of the predecessor is discharged (Remark 7.10). No transformation is constructed. Twelve non-implications are recorded (Proposition 7.1) and eight targets opened (Remark 7.9), of which Targets 7.5 and 7.4 are load-bearing: until the first is discharged the weight is a given datum and not a datum of the secondary realization, and the second is the claim the functional was constructed to make possible and which this paper does not make. That any process maximises the functional is a hypothesis, registered and used in no proof (Hypothesis 2.32, Remark 2.33). And the two numerical coincidences that the classification invites — with the twenty three-letter multisets, and with the sixty-four codons and twenty amino acids of the genetic code — are refuted and not used, the first at $N = 4$ and the second at $N = 3$ itself, by fibre-size profile (Proposition 3.15). What remains is what the paper proves: a classification of twenty objects, a criterion for transporting it, a four-element indeterminacy with its missing datum named, and a functional whose maximiser is generically unique.

A The Twenty Classes of a Six-Site Stack

Table 2 lists the twenty frame classes of a stack of $2N = 6$ sites, in the notation fixed in Section 2. Everything in it is determined by Theorem 3.3 and Propositions 3.6 and 5.5; the table is a computation and not a construction, and is included so that every statement of Sections 3–6 may be checked against a finite object rather than against a formula.

Reading the table. The representative is the lexicographically least marker assignment of the class, written in the convention $F = 0$, $B = 1$ of [Sch26d, Def. 3.12]; the class is displayed as $\overline{D}$, the lexicographically smaller of the registry sequence and its reversal; the size is the cardinality of the Fr-orbit; the stabiliser type is 1 for the trivial stabiliser, r for {id, rev} and fr for {id, fliprev}, in the sense of Proposition 3.6(i); D_N is the central entry, which by Proposition 3.6(iv) equals 0 exactly on the classes of type r and 1 exactly on those of type fr; and the last two columns give the images of the class under the two

	Rep.	$\overline{D}$	Size	Stab	D_N	κ	λ
1	FFFFF	00000	2	r	0	00000*	11111
2	FFFFB	00001	4	1	0	00011	01111
3	FFFFBB	00010	4	1	0	00010*	10111
4	FFFFBF	00011	4	1	0	00001	00111
5	FFBFF	00100	2	fr	1	01110	11011
6	FFBFFB	00101	4	1	1	01101	01011
7	FFBFF	00110	4	1	1	00110*	10011
8	FFBFFB	00111	4	1	1	01111	00011
9	FFBFFB	01001	4	1	0	01011	01101
10	FFBFF	01010	2	r	0	01010*	10101
11	FFBFFB	01011	4	1	0	01001	00101
12	FFBFFB	01101	4	1	1	00101	01001
13	FFBFFB	01110	2	fr	1	00100	10001
14	FFBFFB	01111	4	1	1	00111	00001
15	FBFFB	10001	2	r	0	11011	01110
16	FBFFB	10011	4	1	0	10011*	00110
17	FBFFB	10101	2	fr	1	10101*	01010
18	FBFFB	10111	4	1	1	10111*	00010
19	FBFFB	11011	2	r	0	10001	00100
20	FBFFB	11111	2	fr	1	11111*	00000

Table 2: The twenty frame classes at $2N = 6$. Asterisks mark the eight classes fixed by κ .

generators of K , computed by Proposition 5.5. A class fixed by κ is marked with an asterisk.

What the table verifies. Eight classes have size two and twelve have size four, as Corollary 3.7 asserts; of the eight, four are of type r (rows 1, 10, 15, 19) and four of type fr (rows 5, 13, 17, 20), and in every case D_N agrees with the type, which is Proposition 3.6(iv) checked on every class rather than on a representative. No class has size one. The map κ fixes eight classes and preserves D_N throughout, $N = 3$ being odd; it carries size-two classes to size-two classes of the same type, exchanging rows 5 and 13 and rows 15 and 19. The map λ fixes no class, pairs the twenty into ten pairs, and inverts D_N , exchanging the four classes of type r with the four of type fr: rows 1 and 20, 5 and 19, 10 and 17, 13 and 15. This is Remark 4.13 and Proposition 5.5(iv) in a form that can be read off.

Cross-references. Row 6 is the witness of [Sch26d, App. A], whose marker assignment is (F, F, F, B, B, F) and whose registry sequence is 00101; it is a class of size four, and it is the class used in Example 4.15. Rows 1 and 6 are the classes O_1 and O_2 of Examples 6.14 and 6.15, and the columns κ and λ of those rows supply the four images tabulated there. Rows 1 and 20 are the two classes of constant and alternating registry, namely the stacks whose consecutive markers all agree and all disagree; they are exchanged by λ , which is the statement that a class-inverting transformation carries the aligned extreme to the unaligned one.

Scope. The table is specific to $2N = 6$. The counts for other even lengths are given by Lemma 3.1, and the behaviour

of K at even N differs from the behaviour displayed here, by Proposition 5.5(i) and Target 7.8; nothing in this appendix should be read as a statement about lengths other than six.

B Concordance, Chemistry Item, and Defect Ledger

B.1 Scope and method

This appendix records, for every prior statement of the Chrysene Formalism corpus that this paper touches, whether the statement is *preserved* (used unchanged), *extended* (used with a further hypothesis or conclusion, without alteration of its content), *restated* (reformulated here in a manner the reviewer must check is equivalent), *corrected* (found to be false or type-incorrect as written, with the repair given), *not applicable* (outside the scope of the objects considered here), or *noted* (recorded for information without change of status). The classification is made against the criterion C of [Sch26b, Def. 2]: a statement of this paper is concordant when it neither contradicts a prior statement of the formalism nor silently weakens a structure that a prior statement relies upon.

B.2 Concordance with the encoded-stack paper

B.3 Concordance with the twelve-axiom paper and the transfer-geometry paper

B.4 Chemistry and domain item

Remark B.1 (The single item for the chemistry review). [E] This paper contains no molecular construction and no new identification. Two items are placed before the chemistry review.

First, and the only substantive one, Proposition 3.15(ii) and (iii) assert that no correspondence with the genetic code, and no correspondence with any twenty-element biological set, is claimed. Two questions were put to the domain review and both are disposed. The quoted profile — fibres of sizes 1, 2, 3, 4 and 6 with multiplicities 2, 9, 1, 5 and 3, sixty-one coding codons and three stop codons — was recomputed from the standard codon table rather than read from the citation, and is confirmed. And the manuscript was read for assertions about biological systems that the refutation would then be obliged to defend: it contains none, the only statements in that register being Remark 3.16, which records a property of a published table, and the physical readings of Sections 3.3 and 6.6, which are marked [E] and enter no proof. The scope of the refutation was narrowed at the same pass, Remark 3.16 now recording that clause (ii) refutes the standard table and not every conceivable assignment.

Second, the marker of [Sch26d, Def. 3.3] is used here

unchanged. Its molecular reading, and the perimeter-orientation criterion that replaced the bay designation, were ruled on at [Sch26d, Rem. B.1] and are not reopened; this paper adds no molecular datum to them.

Remark B.2 (The locator audit). Every citation into [Sch26a, Sch26b, Sch26c, Sch26d] was checked individually against the source at the concordance pass, and not against the draft's own earlier use of it. Two features of the manuscript's history made the check necessary rather than formal. First, the results of the encoded-stack paper were at one stage keyed to the twelve-axiom paper, under which key they appear in [Sch26d] itself; the ninety affected locators were reassigned and then re-derived from the source, so that the reassignment is not itself the evidence. Second, five locators into [Sch26d] carry a clause number, and each clause was read rather than the statement it belongs to. Four discrepancies were found and are recorded as items 16 to 19 of Table 5; none of the four is a misattributed theorem, and all four are of the kind that a check against the draft rather than against the source would have passed.

B.5 Bibliographic items

Remark B.3 (Pagination). Three items are recorded for correction; none affects any result. The start page 675 of this article collides with no range assigned in the corpus as recorded in [Sch26b, Sch26a, Sch26d]; it leaves the range 628–674 unassigned, and the gap is noted so that it is not filled twice. The overlap already on record at [Sch26a, Rem. B.9], between the self-reported pagination of [Sch26c] and the range assigned to it by [Sch26b], remains outstanding. A second overlap is recorded here: [Sch26b] gives its own range as 476–488 and gives that of [Sch26c] as 476–547, so that two articles of Volume 1, Number 2 are assigned the same start page. This is distinct from the item at [Sch26a, Rem. B.9] and is not resolved by it.

B.6 Self-identified defect ledger

The following records, in the crit/rev/ver form of [Sch26b, Def. 3], the criticisms raised against this manuscript, their disposition, and whether the repair has been independently examined. Items 1 to 6 were raised before drafting began, against the plan of the paper rather than against a draft, and are recorded as such rather than presented as design decisions; items 7 to 12 arose during drafting or in the pre-review passes.

B.7 Targeted verification and meta-review

Remark B.4 (Targeted verification). Targeted verification does not repeat a disciplinary review. It determines, for the specified list of defects of Table 5, whether each has been repaired without introducing a new inconsistency elsewhere. Two operations were performed against the final manuscript. Each of the twenty-two entries was located in the text at the place its *ver* column names, and the repair confirmed present. The manuscript was then scanned for text left standing by a repair: the torsor language on Σ'_0 withdrawn at item 16, the designated-zero language it carried, the superseded locator of item 8, the restraint locator of item 17, and the forward commitment withdrawn at item 22. No occurrence survives outside Table 5 itself, where each is quoted in the description of the defect it records. The enumerations of Remarks 3.18, 4.18, 5.7 and 6.19 were re-run against the final statements, forty-five checks, with no discrepancy; the locator audit of Remark B.2 covers one hundred and eight citations; the domain review of Remark B.1 covers one item.

Remark B.5 (Meta-review). The meta-review treats the final manuscript, and not any earlier reviewer judgment, as the controlling object, and classifies each material criticism in the manner of [Sch26b, Def. 3] and [Sch26b, §4.3]. Of the twenty-two entries of Table 5, twenty-one are *resolved* and one, item 9, is *partially resolved*: it is a question about [Sch26d] rather than about this paper, it is recorded and referred rather than repaired, and Remark 3.12 is written so that this paper inherits nothing from it. None is *unresolved* and none is *no longer applicable*.

Three observations belong here rather than in the ledger. First, of the ten criticisms raised after drafting began, seven are over-claims in the manuscript's own direction — a hypothesis asserted to do more than it does, a scope asserted wider than the argument, a datum cited rather than computed — which is the asymmetry to expect when one system drafts and then reviews, and is the reason the stages are assigned disjoint questions. Second, the two criticisms with the largest consequence, items 16 and 21, were raised at the concordance and domain stages respectively and would not have been produced by the mathematical stage, neither being a defect of proof; this is the differentiation of correction type by review discipline that [Sch26b, Rem. 1] reports, reproduced here on a second manuscript. Third, no criticism at any stage required a result to be withdrawn: the repairs narrow claims, supply witnesses, correct locators and remove a forward commitment, and the statements of Sections 3 to 6 stand as first drafted in every case but item 13, where a hypothesis was deleted because the proof does not use it.

What the review does not certify is stated with the same

plainness. The eight Targets are open, and two of them are load-bearing by Remark 7.9. Three hypotheses — H'_{ens} , H_{fit} in its provenance clause, and H_{var} — are registered and undischarged, and the last is used in no proof. And the limitation of Remark B.6 applies to every entry of Table 5 marked *ver explicit*.

Remark B.6 (The limitation of the verification recorded here). Targeted verification for this manuscript re-runs the mathematical stage rather than employing an independent reviewing system. By [Sch26b, §9.4], independence of pass is not independence of reviewing system, and the majority of the Volume 1, Number 1 corpus is subject to exactly that limitation; every entry marked *ver explicit* in Table 5 inherits it and should be discounted accordingly. The concordance and chemistry stages are not affected, being assigned distinct questions in the sense of [Sch26b, §4.3]. The numerical verifications reported in Remarks 3.18, 4.18, 5.7 and 6.19 are of a different kind: each is an exhaustive enumeration over a finite set, and each is reproducible from the statement of the object enumerated without reference to any reviewing system. They are collected in a supplementary script accompanying this article, which rebuilds the alphabet, the frame group, the registry sequence and the transformations from their definitions and reports forty-five checks, among them a row-by-row comparison with Table 2; the script imports nothing from the manuscript, so that agreement is evidence and not a restatement.

C Established Mathematical Literature

As in the literature appendices of [Sch26c, Sch26b, Sch26d], none of the subjects below is claimed to be novel in isolation. The purpose is to identify the established machinery each part of this paper draws upon, and to keep that machinery visibly distinct from the specific organisation the classification proposes. The mathematics used here is elementary, and the list is correspondingly short; that is itself worth recording, since the paper's claims rest on finite computations that any reader can repeat rather than on machinery that must be taken on trust.

C.1 Group actions and orbit counting

The orbit-counting lemma used in Proof 1 of Lemma 3.1 is classical and is stated in [Bur11, §119]; a modern treatment, together with the orbit–stabiliser theorem used throughout Section 3.3 and the fact that stabilisers of points in one orbit are conjugate, is [Rot95, Ch. 3]. The attribution of the lemma to Burnside is conventional and inaccurate, the result being due to Cauchy and Frobenius; the name is retained here because the corpus will be searched under it.

C.2 Torsors

The notion of a torsor under a finite abelian group, used for the family of admissible transformations in Proposition 4.14, for the induced maps in Proposition 5.3, and inherited from [Sch26d, Prop. 8.7], is standard; see [Bou70, Ch. I, §5] and, for the finite case, [Ser79, Ch. I]. The property used here is that a torsor has no canonical base point, so that a statement about its elements must be invariant under the acting group or must name the choice that breaks the symmetry. That discipline is applied at Remarks 4.3 and 2.37.

C.3 Finite symmetric and dihedral groups

The subgroup structure of $\text{Sym}(4)$ used in Lemma 2.5 and Proposition 4.14 — the Klein four-group of permutations preserving each class of a partition into two pairs, its normaliser of order eight, and the centrality of the fixed-point-free involution in that normaliser — is elementary and may be read off from the tables in [Rot95, Ch. 3]. The identification of the normaliser with the dihedral group of order eight is recorded at Remark 4.2 and used nowhere.

C.4 Elementary enumeration

The count of multisets used in Proposition 3.15(i) is the standard stars-and-bars formula; see [Sta12, Ch. 1]. The invariance of the multiset of fibre sizes of a surjection under bijections of source and target, on which Proposition 3.15(ii) turns, is immediate from the definition and is recorded here only because the proposition rests on it.

C.5 The genetic-code datum

The fibre profile of the standard genetic code quoted in Proposition 3.15(ii) is taken from [Cri68]. It is an external datum, cited and not derived, and it is used only to refute a coincidence; no statement of this paper depends on it, and no assertion about a biological system is made. See Remark 3.16.

C.6 What is drawn from the Chrysene Formalism itself

Axioms A_1 – A_{12} , the status vocabulary, the non-vacuity theorem and the acyclicity of the axiom ladder are [Sch26c]. The encoded stack, the alphabet Σ_0 , the structural group, the registry invariant, the frame and its group, the maximal-frame-invariant theorem, the transport conditions and the Target inventory this paper extends are [Sch26d]. The review architecture, the editorial classification and the non-implication conventions are [Sch26b]. The survival programme against which Target 7.4 is stated, and the instance schema whose clauses (N4) and (N5) it requires, are [Sch26a]. None of these is external literature in the sense of this appendix; they are listed here so that the boundary between

established machinery and the formalism’s own claims is visible in one place.

References

- [Bou70] Nicolas Bourbaki. *Algebra I: Chapters 1–3*. Hermann / Addison-Wesley, 1970.
- [Bur11] William Burnside. *Theory of Groups of Finite Order*. Cambridge University Press, Cambridge, 2nd edition, 1911.
- [Cri68] Francis H. C. Crick. The origin of the genetic code. *Journal of Molecular Biology*, 38(3):367–379, 1968.
- [Rot95] Joseph J. Rotman. *An Introduction to the Theory of Groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 4th edition, 1995.
- [Sch26a] Charles D. Schaper. Stable Transfer Geometry of the Substituted Chrysene Network: H^∞ Transformation Metrics, Dissipative Well-Posedness, and Survival Under Persistent Disturbance. *Annals of the Chrysene Formalism*, 1(2):559–600, 2026.
- [Sch26b] Charles D. Schaper. The Chrysene Formalism as Candidate Mathematics and Candidate Biogenesis: A Quantitative and Categorical Survey of Volume 1, Number 1. *Annals of the Chrysene Formalism*, 1(2):476–488, 2026.
- [Sch26c] Charles D. Schaper. The Chrysene Formalism Revisited: A Twelve-Axiom Foundation for Molecular Characterization, Tensorial Non-commutative Geometry, and Operator Response. *Annals of the Chrysene Formalism*, 1(2), 2026. Pagination inconsistent between the article and [Sch26b]; see Remark B.3.
- [Sch26d] Charles D. Schaper. The Encoded Stack: A Frame-Indexed Base-Four Structure on the $\mathbb{Z}^3$ -Fibrated Chrysene Network, its Sequence Algebra, and the Set-Up for Survival. *Annals of the Chrysene Formalism*, 1(2):601–627, 2026.
- [Ser79] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.
- [Sta12] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2nd edition, 2012.

Prior statement	Disposition
Thm. 6.14, maximal frame invariant	Preserved and extended; Thm. 3.3 adds the count and Prop. 3.6 the stabiliser refinement
Prop. 9.13, candidate invariants factor through the registry class	Preserved; the objects it quantifies over are enumerated here
Props. 4.7, 4.8, 4.10, the structural group	Preserved; used as the definition of a structured alphabet
Lem. 8.4, G is the stabiliser of the registry partition	Preserved; it is what makes Def. 4.1 equivalent to the data of H_{sec}
Thm. 8.5, transport conditions (i)–(iii)	Preserved and extended; Thm. 4.9 adds (iv), (v) and the size refinement, and weakens the hypothesis
Hyp. 2.17, structured secondary alphabet	Restated in the weaker form of Hyp. 2.23; the equivalence of the four data is Rem. 2.24 and is to be checked
Hyp. 2.13, ensemble on the word space	Restated as H'_{ens} on the class set; the two are related by pushforward, and Prop. 6.4 records what the change costs
Hyp. 2.11 and Prop. 3.19, vessel independence	Preserved; used in Prop. 3.11, and only through the four constant words in Thm. 4.9
Prop. 3.8, the marker is information bearing	Preserved as a Target; not discharged, see Rem. 3.12
Def. 9.2 and Rem. 9.4, the frame-consistency functional	Preserved; its replaceability is what makes the count of Prop. 3.11 conditional
Prop. 6.7, decomposition of the frame action	Preserved; used in Lem. 2.15 and Thm. 5.8
Prop. 4.21, injectivity of the word operator	Preserved; used only in Rem. 5.10
Prop. 8.7, the family of transformations is a torsor under G	Preserved and extended; Prop. 5.3 exhibits the eight admissible maps as an N -torsor and the induced maps as a K -torsor
Thm. 4.17 and Prop. 4.18, sequence dependence from A_7	Not applicable; no statement here uses A_7 (Prop. 2.7, Rem. 2.8)
Rem. 4.19 and Rem. A.15, the twenty multisets of three letters	Noted; the coincidence with $ X $ is refuted at Prop. 3.15(i)
Target 10.5, construction of the transformation	Not discharged; a criterion it must satisfy is Thm. 4.9, and a datum it must supply is Target 7.7
Target 10.8, candidate survival invariants	Not discharged; the candidates are enumerated, none is shown preserved
Defect ledger, item 4	Corrected: the transport theorem is Thm. 8.5 of that paper, Lem. 8.4 being the stabiliser lemma; the ledger entry cites it as Thm. 8.4

Table 3: Concordance with [Sch26d]. One entry is disposed *corrected*, and it concerns a citation in that paper’s own ledger and no result.

Prior statement	Disposition
$A_1, A_2, A_6, A_8, A_{12}$	Preserved; no axiom added (Rem. 2.2)
Defs. 1, 3, 5, 6, status vocabulary	Preserved; used as defined there
Thm. 61, non-vacuity	Preserved, not re-established; no witness is constructed here
Prop. 39, acyclicity of the axiom ladder	Preserved, not re-established; no axiom is added and the imports of Prop. 2.7 are confined to $A_1, A_2, A_6, A_8, A_{12}$. Monotonicity of the provenance assignment for the ladder is not claimed here, as it is at [Sch26d, Prop. 4.26]
Thm. 41, the $GF(4)$ target	Not applicable; this paper concerns Σ_v and not $\Sigma_{\mathfrak{C}}$
[Sch26b, §3.1], the governing restraint, and [Sch26b, Eqs. (2)–(3)], its recorded instances	Preserved; Prop. 7.1 is written in their form
[Sch26a, Def. 2.10], clauses (N4) and (N5)	Not supplied, as in [Sch26d, Rem. 9.12]; recorded at Target 7.4
[Sch26a, Prop. 8.15], the survival conjunction	Preserved; cited only to state what Target 7.4 would require

Table 4: Concordance with [Sch26c] and [Sch26a]. No entry is disposed *corrected*.

Critique	Review and Verification
1 Literal generator-matching equivariance was stated as a member of the transport equivalence; it is strictly stronger and fails for four of the eight admissible maps	Fully accepted; clause (iv) restated as intertwining and the literal form separated. Ver explicit: Prop. 4.11
2 Trap 1 was stated as a property of every bijection T ; it is a property of the target alphabet alone	Fully accepted. Ver explicit: Prop. 4.6, Rem. 4.7
3 The genericity condition was stated as a symmetry of the weight; that is the condition for a tie at every ensemble, not at a given one	Fully accepted; the exact locus is given and the symmetry conditions demoted to strata. Ver explicit: Def. 6.9, Prop. 6.13
4 The word <i>kernel</i> was applied to a map defined on a torsor	Fully accepted. Ver explicit: Prop. 5.3
5 No non-implication guarded the number twenty	Fully accepted. Ver explicit: Prop. 3.15
6 Verification was to be reported as a count of random trials, which a null degeneracy locus makes vacuous	Fully accepted; closed-form values and a constructed tie replace it. Ver explicit: Ex. 6.14, Ex. 6.15, Rem. 6.19
7 The coincidence of the sixty-four words and twenty classes with the genetic code was not identified in the planning pass	Fully accepted; refuted by fibre profile, which is decisive at $N = 3$ itself. Ver explicit: Prop. 3.15(ii)
8 Drafts before Pass 1 cited Prop. 9.10 of [Sch26d] for the factorisation of candidate invariants; the statement is Prop. 9.13	Fully accepted. Ver explicit: Rem. 3.5
9 Prop. 3.19 of [Sch26d] asserts that each vessel factor is all of Σ_v by definition, while its Prop. 3.8 leaves the range of admissible stacks open	Recorded, not repaired: it is a question about that paper and not about this one, and Rem. 3.12 is written so that nothing is inherited. Ver none; referred to the concordance stage
10 The planned proof of (v) $\Rightarrow$ (ii) required $N \geq 2$ and the whole of H_{ind}	Superseded during drafting: the constant-word argument requires neither. Ver explicit: Thm. 4.9, Rem. 4.10
11 The planned theorem asserted that the frame datum determines the transformation outright; both members of a fibre are maps out of the same coordinatised Σ_v	Fully accepted; the conclusion is replaced by the identification of the residual datum as a zero of Σ'_v . Ver explicit: Thm. 5.8(iv), Rem. 5.9
12 The claim that κ preserves the central entry was stated without a parity hypothesis; it is false for even N	Fully accepted; the claim is restricted to reversal-symmetric classes and to odd N , which is where it is used. Ver explicit: Prop. 5.5(i), Rem. 5.7, Target 7.8
13 Raised at mathematical review: Thm. 4.9 carried a hypothesis on the source word space which its proof does not use, the abstract statement being over Σ_v^N in full	Fully accepted; the clause is deleted and the scope of the constant-word argument stated instead. Ver explicit: Thm. 4.9, Rem. 4.10
14 Raised at mathematical review: Prop. 6.13 asserted that the symmetry strata do not exhaust Z without exhibiting a point outside them	Fully accepted; a witness is supplied, and with it the observation that H_{gen} is sufficient and not necessary for Thm. 6.12. Ver explicit: Ex. 6.16, Rem. 6.17
15 Raised at mathematical review: the growth comparison in Prop. 3.15(i) asserted a ratio exceeding three without the step that gives it	Fully accepted; the estimate $a \geq 8b$ is supplied. Ver explicit: proof of Prop. 3.15
16 Raised at concordance review: Thm. 5.8(iii) described the residual datum as a zero of a torsor Σ'_v , and cited [Sch26d, Prop. 4.4] as applying to it; that proposition is about Σ_v , and the weakened H_{sec} supplies a partition and no torsor	Fully accepted; the clause is restated in terms of the designated image in Σ'_v of the source zero, with the source-side designation cited to [Sch26d, Prop. 4.4, Cor. 6.9, Cor. A.8]. Ver explicit: Thm. 5.8(iii)–(iv), Target 7.7
17 Raised at concordance review: the governing restraint was cited to [Sch26b, Eq. (2)], which displays two instances of it; the corpus cites the restraint itself both that way and as [Sch26b, §3.1]	Fully accepted; the restraint is cited to §3.1 and its instances separately. The divergence between [Sch26d] and [Sch26a] on this locator is recorded and not resolved here. Ver explicit: opening of Section 7, Table 4
18 Raised at concordance review: Table 4 claimed that the provenance assignment of Prop. 2.7 is monotone for the axiom ladder, which this paper does not prove	Fully accepted; the claim is withdrawn and replaced by what is shown, the imports being confined to five axioms. Ver explicit: Table 4
19 Raised at concordance review: Rem. 5.11 attributed the factors ρ and τ to the two coordinates of the frame-setter without a locator	Fully accepted; [Sch26d, Prop. 6.7(i), (ii)] supplied. Ver explicit: Rem. 5.11
20 Raised at domain review: the degeneracy profile of the standard code was cited and not recomputed	Fully accepted; recomputed from the standard codon table and confirmed, with [Cri68] retained for the code itself. Ver explicit: Rem. 3.16
21 Raised at domain review: clause (ii) of Prop. 3.15 was open to being read as excluding every surjection $64 \rightarrow 20$ of the shape in question, whereas a surjection with the profile of q is arithmetically possible	Fully accepted; the scope is stated, and clause (iii) is identified as what covers targets other than the standard table. Ver explicit: Rem. 3.16
22 Raised at domain review: Rem. 3.17 asserted the intended reading of a companion paper, an assertion this paper is not in a position to make and did not mark	Fully accepted; rephrased to assert nothing about the companion, with the sources of the expectation cited instead. Ver explicit: Rem. 3.17

Table 5: Self-identified defect ledger. Twenty-one items are disposed fully accepted and repaired; item 9 is recorded and referred, and is the only entry whose verification is none. Items 13 to 15 were raised at mathematical review, items 16 to 19 at concordance review and items 20 to 22 at domain review, in every case against the completed draft.