

A Joint Survival Certificate for the Substituted Chrysene Network: The L^1 Transformation Geometry, Peak-Gain Well-Posedness, and a Bandwidth Floor

Charles D. Schaper, Ph.D.

Chief Editor, *Annals of the Chrysene Formalism*

San Francisco, California, USA

editor@chrysene.com

Abstract

The transfer geometry of the substituted chrysene network certifies robust stability and invariance of the admissible domain by two quantities that no estimate relates: the worst-case norm controls the first, a peak gain the second, and no function of the former bounds the latter, so the survival claim has stood as a conjunction of separately certified halves. We discharge it as a single certificate. The impulse response of a composite realization is identified with the integrated return amplitude of the active region, and its spectral density with the coupling-weighted density of states of the second fibration seen through the effective resolvent. Finiteness of the peak gain is then an L^1 decay condition strictly stronger than embedded-mode exclusion: the latter excludes poles, the former additionally constrains edge and van Hove regularity, and under a one-sided square-root edge hypothesis the return amplitude decays like $t^{-3/2}$. On the resulting domain the unitary-orbit distance taken in L^1 is a pseudometric with attained infimum whose path closure dominates the $\mathcal{H}^\infty$ transformation metric. Its zero set is again transfer isomorphism, as it must be for any unitarily invariant norm; what changes is that its domain is strictly smaller and its infinite locus strictly larger, which costs the connectivity trichotomy of the prior paper. Under a uniform constraint-margin hypothesis, one ball in this metric certifies robust stability at a given environment level and invariance of the admissible domain at a given disturbance amplitude, for the same forms, and propagates along a transformation route. Finally the integrated return amplitude admits a lower bound of π divided by the width of the composite spectrum, and hence an upper bound on the certifiable disturbance amplitude set by the fibration rather than by the design; both natural evasions of that bound are closed by the well-posedness criterion itself. The joint certificate is shown to be strictly more conservative than the prior robust-stability radius, which it does not supersede, and the two gains are shown not to be co-optimized. Twelve non-implications, four new targets of which two are load-bearing, three corrections to the prior paper including one to a hypothesis its invariance theorem rests on, a seventeen-item defect ledger, and an explicit witness solved in closed form are stated.

Keywords: Chrysene Formalism; peak-to-peak gain; Wiener algebra; transformation metric; admissible domain; local decay; van Hove singularity; band-edge regularity; uncertainty principle; joint survival certificate; deferred theorem program.

1 Introduction

1.1 The problem

The transfer geometry of [Sch26d] certifies two statements about a form of the substituted chrysene network and does not combine them. The first is robust stability of the form against a persistent, unmodelled environment, controlled by the worst-case norm $\|\cdot\|_{\mathcal{H}^\infty}$ of [Sch26d, Def. 2.3] and established inside a metric ball by [Sch26d, Thm. 5.9]. The second is invariance of the admissible domain Ω_{adm} of Axiom A_{12} under the disturbance, established by [Sch26d, Thm. 8.13] and controlled not by $\|\cdot\|_{\mathcal{H}^\infty}$

but by a peak gain. That the two cannot be merged by any estimate on the worst-case norm is the content of [Sch26d, Prop. 8.15]: there is a family of stable realizations of constant $\mathcal{H}^\infty$ norm whose peak gain diverges, so no function of the former bounds the latter. The survival claim of [Sch26d] is therefore a conjunction of two separately certified halves, a state of affairs that paper records as a defect of its own results [Sch26d, Rem. 9.5] rather than as a feature.

[Sch26d, Target 8.26] asks for a single computable condition on a composite realization implying both halves and propagating along a transformation route, and observes that no such condition

can be a bound on $\|\cdot\|_{\mathcal{H}^\infty}$ alone. [Sch26d, Rem. 9.6] sketches a route — apply the unitary-orbit construction of [Sch26d, Def. 5.1] to the impulse response rather than to the transfer function — and defers the development to a companion note to be reviewed on its own. This paper is that note.

1.2 The objective

The objective is to construct, over the admissible transformations of the substituted chrysene network, a single extended pseudometric with respect to which both halves of the survival claim hold simultaneously inside one ball, and to determine when the quantity on which that metric is built is finite.

The construction is not a reorganization of [Sch26d]. The peak gain is finite under strictly stronger conditions than the worst-case norm, so the metric does not exist on all of the domain on which $d_{\mathcal{H}^\infty}$ exists, and the well-posedness question must be settled before the geometry is available. That ordering — well-posedness, then geometry, then the joint radius — is the organization of Sections 3–5 below, and it mirrors the ordering [Sch26d, §§4–5] was forced into for the same reason.

1.3 What is established

1. A representation of the impulse response of a composite realization as the integrated return amplitude of the active region, $g = J\delta_0 + g^{\text{ac}}$ with $g^{\text{ac}}(t) = -i\theta(t)CP_{\text{pri}}e^{-iHt}P_{\text{pri}}B$ (Lemma 3.1), together with the identification of its spectral density with the compression of the composite spectral measure (Proposition 3.4).
2. A criterion for finiteness of the peak gain (Theorem 3.8): the embedded-mode exclusion of [Sch26d, Thm. 4.17] together with an edge and van Hove regularity hypothesis gives $\|g^{\text{ac}}(t)\| = O(t^{-3/2})$ and hence integrability, while failure of embedded-mode exclusion gives infinite peak gain. The criterion is strictly stronger than [Sch26d, Thm. 4.17] (Proposition 3.9).
3. An L^1 unitary-orbit distance δ_{pk} and its path closure d_{pk} , an extended pseudometric on Ω_{adm} dominating $d_{\mathcal{H}^\infty}$ (Propositions 4.2 and 4.6), whose zero set is again transfer isomorphism (Proposition 4.3) but whose domain and infinite locus are strictly smaller and strictly larger respectively (Propositions 3.9 and 4.7).
4. A joint survival radius (Theorem 5.2): under a uniform constraint-margin hypothesis, one d_{pk} -ball about a form certifies robust stability at a given environment level and invariance of Ω_{adm} at a given disturbance amplitude, for the same forms. This discharges [Sch26d, Target 8.26] for a single form and along a transformation route.
5. A bandwidth floor (Theorem 6.2): the integrated return amplitude of the active region is bounded below by π

divided by the width of the composite spectrum, whence (Corollary 6.4) an upper bound on the disturbance amplitude against which invariance of Ω_{adm} can be certified at all, intrinsic to the fibration rather than to the design.

6. Closed-form values for every quantity above on an explicit witness realization (Appendix A), the witness of [Sch26d, App. A] being shown to admit an exact solution in terms of Bessel functions.

1.4 What is not claimed

No axiom is introduced (Remark 2.1); no deferred target of [Sch26c] is discharged; [Sch26d, Thm. 5.9] is not superseded, and for the robust-stability half alone it remains the sharper statement (Remark 5.5); no result here bounds the jump factor of [Sch26d, Thm. 6.19], so [Sch26d, Target 6.24] is untouched and the survival claim across *switching* remains as [Sch26d, Rem. 9.5] leaves it; no chemical, energetic or kinetic claim is asserted or used; and the results certify no existing paper of the corpus (cf. [Sch26d, Rem. 10.7]).

1.5 Logical position

Every numbered statement carries one of the four status labels of [Sch26d, Def. 2.7], which are those of [Sch26c, Defs. 1, 3, 5, 6]: [A] axiomatic, [D] derived realization theorem, [T] target, [E] empirical instantiation claim. Every result using a hypothesis beyond A_1 – A_{12} names that hypothesis; hypotheses carried over from [Sch26d] are cited by their names there, and the three introduced here are registered in §2.3 and §3. The molecular reading of any statement is confined to remarks marked [E], and no such remark is used in a proof. This paper falls within the framework-revision scope of [Sch26b, §4.5] and is written to the orthogonal-constraint review bar of [Sch26b, §4.3].

1.6 Organization

Section 2 fixes the peak-gain norm, records a concordance correction to [Sch26d, Def. 8.11], and registers the hypotheses added here. Section 3 settles well-posedness for the peak gain. Section 4 constructs the L^1 transformation geometry. Section 5 proves the joint survival radius. Section 6 proves the bandwidth floor. Section 7 states the non-implications and the status of the deferred programs, and introduces the targets opened here. Section 8 records what transfers to other fibrated networks. Section 9 concludes. Appendix A is the witness, Appendix B the concordance and defect ledger, Appendix C the established literature.

Remark 1.1 (Prior documents cited). Four documents of the corpus are cited throughout: the twelve-axiom foundation [Sch26c]; the transfer geometry [Sch26d], which this paper

extends; the editorial survey [Sch26b], which fixes the review architecture and the non-implication conventions; and the review case study [Sch26a]. No monograph volume of *The Mathematical Structure of Reality* is cited here: the concordance of Appendix B is with [Sch26c] and [Sch26d] only, and the monograph concordance [Sch26d, Tab. 5] is neither extended nor revisited.

2 Conventions, the Peak-Gain Norm, and Register Additions

2.1 Notation carried over

Throughout, $\mathfrak{C}$ is the molecular reduct of Axiom A_1 , $\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_{\alpha} \mathbb{Z}^3$ the sequence algebra of Axiom A_7 , and $(X, j_{\text{str}}, \mathcal{E}_0, w, \{C_j\}, \{\tau_j\})$ the admissibility datum of Axiom A_{12} , with admissible domain Ω_{adm} , admissible transition relation $\mathcal{E}_{\text{adm}}$, and extended admissible path distance $d_{\mathfrak{C}}^*$, all as in [Sch26c].

We use without restatement the composite realization $\mathfrak{G}_x = (\mathcal{H}_{\text{pri}}, \mathcal{H}_{\text{sec}}, H_p, H_s, W, B, C, J)$ of [Sch26d, Def. 4.1], its composite Hamiltonian

$$H = \begin{pmatrix} H_p & W \\ W^* & H_s \end{pmatrix} \quad \text{on } \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}, \quad (1)$$

the self-energy $\Sigma(z) = W(z - H_s)^{-1}W^*$ and $M(z) = z - H_p - \Sigma(z)$ of [Sch26d, Def. 4.2], the form transfer function $G_x(z) = CM(z)^{-1}B + J$ of [Sch26d, (28)], the boundary decomposition $\Sigma(\lambda + i0) = \Lambda(\lambda) - i\Gamma(\lambda)$ of [Sch26d, Prop. 4.13], the set $\mathcal{G}$ of stable realizations of [Sch26d, Def. 3.2], transfer isomorphism $\simeq$ of [Sch26d, Def. 3.6], the unitary-orbit distance δ_u of [Sch26d, Def. 5.1], and the transformation metric $d_{\mathcal{H}^\infty}$ of [Sch26d, Def. 5.5]. The channel spaces are K and Y , common to all forms under consideration by H_{form} [Sch26d, Hyp. 2.14], and $P_{\text{pri}}, P_{\text{sec}}$ are the orthogonal projections associated with (1). We write θ for the Heaviside function, δ_0 for the unit point mass at the origin, and $(\cdot)_+$ for the positive part.

The hypotheses $H_{\text{fib}}, H_{\text{lab}}, H_{\text{form}}, H_{\text{sec}}, H_{\text{sp}}, H_{\text{cov}}, H_{\text{env}}, H_{\text{nuc}}, H_{\text{rec}}, H_{\text{out}}, H_{\text{chem}}, H_{\text{ch}}, H_{\text{pop}}, H_{\text{reg}}, H_{\text{min}}$ and H'_{out} are those of [Sch26d, §§2.6, 3.1, 4.3, 4.4, 8.3] and are referred to here by those names only.

Remark 2.1 (No thirteenth axiom). [A] This paper introduces no axiom. Every structure below is either a definition internal to $\mathcal{A} = \{A_1, \dots, A_{12}\}$ or a realization hypothesis in the sense of [Sch26c, Def. 3]. The restriction is that of [Sch26d, Rem. 2.8] and is adopted for the same reason: an additional axiom would reopen the non-vacuity theorem [Sch26c, Thm. 61] and the acyclicity of the prerequisite relation [Sch26c, Prop. 39], neither of which is re-established here.

2.2 The impulse response and the peak-gain norm

Definition 2.2 (Causal impulse response). Let $G_x \in \mathcal{G}$. Its *feedthrough operator* is

$$J := \lim_{\eta \rightarrow \infty} G_x(i\eta) \quad (2)$$

whenever the limit exists in $\mathcal{B}(K, Y)$; for a composite realization it exists and agrees with the operator J of [Sch26d, Def. 4.1], by [Sch26d, Rem. 4.3] and (9). The *impulse response* of G_x is the $\mathcal{B}(K, Y)$ -valued tempered distribution g_x supported in $[0, \infty)$ whose Fourier–Laplace transform, in the causal convention e^{izt} , $\text{Im } z > 0$, of Axiom A_{11} , is G_x . We write

$$g_x = J\delta_0 + g_x^{\text{ac}}, \quad (3)$$

g_x^{ac} denoting the part of g_x absolutely continuous with respect to Lebesgue measure on $(0, \infty)$.

That (3) is the correct splitting, and that g_x^{ac} is a function rather than a more singular object, is Lemma 3.1; Definition 2.2 presumes neither.

Definition 2.3 (Peak-gain norm). For G_x independent of the quasi-momentum variable set

$$\|G_x\|_{\text{pk}} := \|J\|_{\mathcal{B}(K, Y)} + \int_0^\infty \|g_x^{\text{ac}}(t)\|_{\mathcal{B}(K, Y)} dt, \quad (4)$$

with the value $+\infty$ when $g_x^{\text{ac}} \notin L^1((0, \infty); \mathcal{B}(K, Y))$. When G_x depends on $\theta \in \mathbb{T}^d$ in the sense of [Sch26d, Def. 2.2], let $\widehat{J}(\mathbf{k})$ and $\widehat{g}_x^{\text{ac}}(t, \mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^d$, denote the Fourier coefficients in θ of $J(\theta)$ and of $g_x^{\text{ac}}(t, \theta)$, and set instead

$$\|G_x\|_{\text{pk}} := \sum_{\mathbf{k} \in \mathbb{Z}^d} \left[\|\widehat{J}(\mathbf{k})\| + \int_0^\infty \|\widehat{g}_x^{\text{ac}}(t, \mathbf{k})\| dt \right]. \quad (5)$$

We write

$$\mathcal{G}_1 := \{G_x \in \mathcal{G} : \|G_x\|_{\text{pk}} < \infty\} \quad (6)$$

for the set of realizations of finite peak gain.

Proposition 2.4 (Elementary properties). [A] On $\mathcal{G}_1$:

- (i) $\|\cdot\|_{\text{pk}}$ is a norm, and $\|AG_xA'\|_{\text{pk}} \leq \|A\| \|G_x\|_{\text{pk}} \|A'\|$ for bounded A on Y and A' on K ;
- (ii) it is unitarily invariant: $\|UG_xV\|_{\text{pk}} = \|G_x\|_{\text{pk}}$ for all $U \in \mathcal{U}(Y)$, $V \in \mathcal{U}(K)$;
- (iii) $\|G_x\|_{\mathcal{H}^\infty} \leq \|G_x\|_{\text{pk}}$;
- (iv) $\|G_x\|_{\text{pk}}$ dominates the norm induced by G_x from the

amplitude-bounded class $\mathcal{W}_\infty$ of [Sch26d, Def. 2.5(iii)] to L^∞ , with equality when $\dim K = \dim Y = 1$;

(v) $\mathcal{G}_1 \subseteq \mathcal{G}$, and $\mathcal{G}_1$ is a union of $\simeq$ -equivalence classes on which $\|\cdot\|_{\text{pk}}$ is constant.

Proof. (i) Positivity, homogeneity and the triangle inequality are inherited pointwise in t from the operator norm, and the two-sided bound from $\|Ag(t)A'\| \leq \|A\|\|g(t)\|\|A'\|$ together with the same estimate on the feedthrough term.

(ii) By (i) applied to U and V and to U^* and V^* , using $\|U\| = \|V\| = 1$; the feedthrough operator of UG_xV is UVJ , and $\|UVJ\| = \|J\|$.

(iii) For $\text{Im } z > 0$ one has $|e^{izt}| = e^{-t \text{Im } z} \leq 1$ on $t \geq 0$. Since $G_x \in \mathcal{G}_1$ the function g_x^{ac} is integrable, so by Definition 2.2 the Fourier–Laplace transform of g_x is given by the absolutely convergent integral $G_x(z) = J + \int_0^\infty e^{izt} g_x^{\text{ac}}(t) dt$, whence

$$\|G_x(z)\| \leq \|J\| + \int_0^\infty \|g_x^{\text{ac}}(t)\| dt,$$

and taking the supremum over $z \in \mathbb{C}_+$ gives (iii). In the θ -dependent case the same estimate holds for each Fourier mode, and $\|G_x(z, \theta)\| \leq \sum_{\mathbf{k}} \|\widehat{J}(\mathbf{k})\| + \sum_{\mathbf{k}} \int_0^\infty \|\widehat{g}_x^{\text{ac}}(t, \mathbf{k})\| dt$ uniformly in θ , which is (5).

(iv) With $y = Jw + g^{\text{ac}} * w$ and $\|w\|_\infty \leq 1$, Young’s inequality gives $\|y\|_\infty \leq \|J\| + \|g^{\text{ac}}\|_{L^1}$. For scalar channels the bound is approached along a disturbance matching the sign of g^{ac} reflected about a fixed time, so equality holds; for $\dim K, \dim Y > 1$ the induced norm is a row-sum functional of the matrix entries of g and is in general strictly smaller.

(v) The inclusion is (iii). If $G_x \simeq G_y$ then $G_y = UG_xV$ for constant unitaries, so by (ii) $\|G_y\|_{\text{pk}} = \|G_x\|_{\text{pk}}$; in particular membership in $\mathcal{G}_1$ and the value of the norm are constant on $\simeq$ -classes. $\square$

Remark 2.5 (Concordance: the peak gain for non-scalar channels). [Sch26d, Def. 8.11] sets $\|G\|_{\text{pk}} := \|g\|_{L^1}$ and identifies it with the norm induced $\mathcal{W}_\infty \rightarrow L^\infty$. The two agree for scalar channels. For $\dim K, \dim Y > 1$ the induced peak-to-peak gain is the row-sum functional appearing in the proof of Proposition 2.4(iv) and is *not* unitarily invariant; Proposition 2.4(ii) would fail on it, and the orbit construction of §4 would not be well defined. Definition 2.3 takes the operator-norm integral instead: it is unitarily invariant, dominates the induced gain, and coincides with it in the scalar case. Since [Sch26d, Thm. 8.13] uses only the upper bound supplied by Young’s inequality, that theorem is preserved verbatim with Definition 2.3 in place of [Sch26d, Def. 8.11]. The disposition recorded in Appendix B is *restated*: a correction of type, not of conclusion.

Remark 2.6 (Why the Wiener norm and not a supremum over θ). For θ -dependent realizations the quantity $\sup_\theta (\|J(\theta)\| + \int_0^\infty \|g^{\text{ac}}(t, \theta)\| dt)$ is dominated by (5) and is in general strictly smaller. It is not, however, the norm induced by any of the disturbance classes of [Sch26d, Def. 2.5]: the gain induced from disturbances bounded in amplitude uniformly in the lattice variable is the Wiener norm (5), because the output at a site receives contributions from every site. The choice is therefore forced by the same argument that forces $\|\cdot\|_{\mathcal{H}^\infty}$ in [Sch26d, §7], one level of the construction further out. Proposition 3.9 shows that the distinction is not vacuous.

Remark 2.7 (The feedthrough operator, and why not the mean-square norm). [A] Axiom A_{10} admits a nonzero direct terminal-to-terminal coupling J and states that it may vanish only under an explicitly specified physical model [Sch26c, A_{10} , Def. 56]; $J = 0$ is the strictly bridge-mediated special case, not the default. By [Sch26d, Prop. 7.2], $J \neq 0$ forces $\|G\|_{\mathcal{H}^2} = \infty$, so the finite-energy invariance statement [Sch26d, Cor. 8.14], which is expressed through the mean-square norm, is vacuous on generic A_{10} data. By (4), by contrast, $\|G\|_{\text{pk}}$ is finite whenever $g^{\text{ac}} \in L^1$, for any J . The obstruction is therefore structural in one case and absent in the other, which is an independent reason — beyond [Sch26d, Prop. 8.15] — for building the joint certificate on Definition 2.3.

2.3 Register additions

Each hypothesis below is a realization hypothesis in the sense of [Sch26c, Def. 3] and is referred to in later statements by its name. No statement of this paper uses an unnamed hypothesis. A third hypothesis, H_{edge} , is stated in §3 where the objects it constrains are available.

Hypothesis 2.8 (H_{pk} , finite peak gain). The form x satisfies $G_x \in \mathcal{G}_1$, that is $\|G_x\|_{\text{pk}} < \infty$.

Hypothesis 2.9 (H_{um} , uniform constraint margin). H'_{out} [Sch26d, Hyp. 8.12] holds on a specified subset $S \subseteq \Omega_{\text{adm}}$, and there are constants $\kappa^* < \infty$ and $\theta^* > 0$ with

$$\kappa_j \leq \kappa^* \quad \text{and} \quad \tau_j - C_j(y) \geq \theta^* \quad (7)$$

for every $j \in \{1, \dots, m\}$ and every $y \in S$.

Remark 2.10 (Status of H_{um}). The constraint margins $\tau_j - C_j(y)$ are strictly positive for every $y \in \Omega_{\text{adm}}$, because Axiom A_{12} defines Ω_{adm} by strict inequalities [Sch26c,

(419)]; what (7) adds is a positive lower bound *uniform in* y , which is not implied. On a finite Ω_{adm} with $S = \Omega_{\text{adm}}$ the hypothesis is automatic, with $\theta^* = \min_{y,j}(\tau_j - C_j(y))$ and $\kappa^* = \max_j \kappa_j$. On an infinite Ω_{adm} it is a genuine restriction: a sequence of admissible forms approaching the constraint boundary $\partial_C \Omega_{\text{adm}}$ of [Sch26c, Def. 63] violates it. The hypothesis is named wherever it is used and is not assumed silently.

3 The Return Amplitude and L^1 Well-Posedness

Lemma 3.1 (Return-amplitude representation). [D] Assume $H_{\text{fib}}, H_{\text{lab}}, H_{\text{form}}, H_{\text{sec}}$ and H_{ch} , and let $\mathfrak{G}_x$ be a composite realization with composite Hamiltonian H as in (1), bounded and self-adjoint by [Sch26d, Rem. 4.3]. Then the impulse response of Definition 2.2 is

$$g_x = J \delta_0 + g_x^{\text{ac}}, \quad g_x^{\text{ac}}(t) = -i \theta(t) C P_{\text{pri}} e^{-iHt} P_{\text{pri}} B, \quad (8)$$

the identity holding in the sense of tempered distributions, and

$$G_x(z) = J + \int_0^\infty e^{izt} g_x^{\text{ac}}(t) dt \quad (\text{Im } z > 0). \quad (9)$$

In particular g_x^{ac} is bounded and norm-continuous on $[0, \infty)$ with $\|g_x^{\text{ac}}(t)\| \leq \|C\| \|B\|$ and $g_x^{\text{ac}}(0^+) = -i C P_{\text{pri}} B$, and

$$\|G_x\|_{\text{pk}} = \|J\| + \int_0^\infty \|C P_{\text{pri}} e^{-iHt} P_{\text{pri}} B\| dt. \quad (10)$$

Proof. Step 1: the compression is $M(z)^{-1}$. Fix $z \in \mathbb{C}_+$. Since $H_s = H_s^*$, the operator $z - H_s$ is invertible, and by [Sch26d, Lem. 4.4] so is $M(z)$. Block elimination of the second row and column of

$$z - H = \begin{pmatrix} z - H_p & -W \\ -W^* & z - H_s \end{pmatrix}$$

replaces $z - H_p$ by $z - H_p - W(z - H_s)^{-1}W^* = M(z)$ and gives

$$P_{\text{pri}}(z - H)^{-1}P_{\text{pri}}|_{\mathcal{H}_{\text{pri}}} = M(z)^{-1}, \quad (11)$$

which is the two-block case of the computation in [Sch26d, Lem. 3.16] and is the Schur-complement identity [Sch26c, (377)] for the motif of Axiom A_{10} . Hence, by [Sch26d, (28)],

$$G_x(z) = C P_{\text{pri}}(z - H)^{-1}P_{\text{pri}} B + J. \quad (12)$$

Step 2: Laplace representation of the resolvent. H is bounded and self-adjoint, so e^{-iHt} is unitary and $\|e^{izt} e^{-iHt}\| = e^{-t \text{Im } z}$, whence the Bochner integral $\int_0^\infty e^{izt} e^{-iHt} dt$ converges absolutely for $\text{Im } z > 0$. For $\lambda \in \sigma(H) \subset \mathbb{R}$ and $a := i(z - \lambda)$ one

has $\text{Re } a = -\text{Im } z < 0$ and $\int_0^\infty e^{at} dt = -a^{-1} = i(z - \lambda)^{-1}$, so by the spectral theorem and Fubini

$$(z - H)^{-1} = \int_0^\infty e^{izt} (-i e^{-iHt}) dt. \quad (13)$$

The factor $-i$, rather than $+i$, is forced by the causal convention e^{izt} , $\text{Im } z > 0$, of Axiom A_{11} .

Step 3: identification. Substituting (13) into (12) gives (9) with g_x^{ac} as in (8). Both $J\delta_0$ and g_x^{ac} are tempered distributions supported in $[0, \infty)$, and the Fourier–Laplace transform is injective on such distributions; since the transform of $J\delta_0 + g_x^{\text{ac}}$ is G_x , this is g_x . Because g_x^{ac} is a function, the decomposition (3) is the Lebesgue decomposition of g_x on $[0, \infty)$ and is therefore unique; equivalently $J = \lim_{\eta \rightarrow \infty} G_x(i\eta)$, as used in the proof of [Sch26d, Prop. 7.2].

Step 4: regularity. By [Sch26d, Def. 4.1], $\dim \mathcal{H}_{\text{pri}} = n < \infty$, so $t \mapsto P_{\text{pri}} e^{-iHt} P_{\text{pri}}$ takes values in a finite-dimensional space; each matrix element $\langle e_i, e^{-iHt} e_j \rangle$ is continuous by strong continuity of the unitary group, and on a finite-dimensional space entrywise convergence is norm convergence, so the map is norm-continuous with $\|P_{\text{pri}} e^{-iHt} P_{\text{pri}}\| \leq 1$. The stated bound, the value at 0^+ and (10) follow.

Step 5: the θ -dependent case. Under H_{cov} the above applies fibrewise in the direct-integral decomposition of [Sch26d, Def. 2.2]; taking Fourier coefficients in θ gives $\widehat{g}_x^{\text{ac}}(\cdot, \mathbf{k})$ and (5). $\square$

Remark 3.2 (Reading of Lemma 3.1). Equation (8) is not an additional construction: by (11) it is the Schur-complement structure that Axiom A_{10} states in the form [Sch26c, (377)], read through the terminal identification $\mathcal{H}_L = Y$, $\mathcal{H}_R = K$, $\mathcal{H}_B = \mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}$ of [Sch26d, Rem. A.7]. The peak gain is therefore the integrated return amplitude of the active region, seen through the input and output channels: the quantity governing invariance of Ω_{adm} measures how long amplitude placed in the active region keeps returning to it, whereas by [Sch26d, Thm. 4.17] the quantity governing robust stability measures only whether any of it fails to leave at all.

Definition 3.3 (Local spectral measure and density). Let E_H be the projection-valued spectral measure of H . The *local spectral measure* of the active region is the $\mathcal{B}(\mathcal{H}_{\text{pri}})$ -valued positive measure

$$\mu_x(\cdot) := P_{\text{pri}} E_H(\cdot) P_{\text{pri}}|_{\mathcal{H}_{\text{pri}}}, \quad \mu_x(\mathbb{R}) = I_{\mathcal{H}_{\text{pri}}}, \quad (14)$$

and, where μ_x is absolutely continuous, ρ_x denotes its density.

Proposition 3.4 (The density through the second fibration). [D] Assume the hypotheses of Lemma 3.1 together with H_{pop} , H_{reg} and H_{min} , and let $\sigma(H_p) \subset \text{int } I$ with $I = [-R, R]$ as in [Sch26d, (33)]. Then $\sigma(H) \subseteq I$. If moreover the embedded-mode exclusion [Sch26d, (40)] holds, then μ_x is absolutely continuous with continuous density

$$\rho_x(\lambda) = \frac{1}{\pi} M(\lambda + i0)^{-1} \Gamma(\lambda) M(\lambda + i0)^{-*}, \quad (15)$$

supported in $\{\Gamma \neq 0\} \cap I$, with no singular part and no atom; and

$$g_x^{\text{ac}}(t) = -i \theta(t) C \widehat{\mu_x}(t) B, \quad \widehat{\mu_x}(t) := \int_{\mathbb{R}} e^{-i\lambda t} d\mu_x(\lambda). \quad (16)$$

Proof. (a) *Localization of the spectrum.* For a unit vector $\psi = \psi_p \oplus \psi_s$, $|\langle \psi, H\psi \rangle| \leq \|H_p\| + 2\|W\| + \rho$, and $2\|W\| \leq 1 + \|W\|^2$, so $\sigma(H) \subseteq [-R, R] = I$ with R as in [Sch26d, (33)]. Consequently μ_x is supported in I , and by (14) it is a positive $\mathcal{B}(\mathcal{H}_{\text{pri}})$ -valued measure of total mass $I_{\mathcal{H}_{\text{pri}}}$.

(b) *Borel transform.* By (11) and the spectral theorem,

$$\int_{\mathbb{R}} \frac{d\mu_x(\lambda)}{z - \lambda} = P_{\text{pri}}(z - H)^{-1} P_{\text{pri}} = M(z)^{-1} \quad (z \in \mathbb{C}_+). \quad (17)$$

(c) *Reflection.* Since $H_p = H_p^*$ and $\Sigma(\bar{z}) = (W(z - H_s)^{-1} W^*)^* = \Sigma(z)^*$, one has $M(\bar{z}) = M(z)^*$ and hence $M(\bar{z})^{-1} = (M(z)^{-1})^*$.

(d) *Imaginary part.* For invertible M ,

$$\text{Im}(M^{-1}) = \frac{M^{-1} - M^{-*}}{2i} = M^{-1} \frac{M^* - M}{2i} M^{-*} = -M^{-1} (\text{Im } M) M^{-*}.$$

By [Sch26d, Prop. 4.13], $\text{Im } M(\lambda + i0) = \Gamma(\lambda)$, so $\text{Im } M(\lambda + i0)^{-1} = -M(\lambda + i0)^{-1} \Gamma(\lambda) M(\lambda + i0)^{-*}$.

(e) *Boundary values and Stone's formula.* Assume [Sch26d, (40)]. By Step 2 of the proof of [Sch26d, Thm. 4.17], $M(\lambda)$ is invertible for every $\lambda \in I$ with $\|M(\lambda)^{-1}\|$ bounded on I , and M is continuous on $\overline{\mathbb{C}_+} \cap K$ for the compact set K of that proof; hence $\lambda \mapsto M(\lambda + i0)^{-1}$ is continuous and bounded on I , and by (c) the same holds for $M(\lambda - i0)^{-1}$. Stone's formula applied to (17), together with (d), gives for every $a < b$ in I

$$\mu_x((a, b)) + \frac{1}{2} \mu_x(\{a, b\}) = \int_a^b \frac{1}{\pi} M(\lambda + i0)^{-1} \Gamma(\lambda) M(\lambda + i0)^{-*} d\lambda$$

The right-hand side is the integral of a continuous function, so it is continuous in a and b ; hence μ_x has no atom in I , the left-hand side equals $\mu_x((a, b))$, and μ_x is absolutely continuous on I with the continuous density (15). Since $\text{supp } \mu_x \subseteq I$ by (a), the same conclusion holds on $\mathbb{R}$.

(f) *Support.* $M(\lambda + i0)^{-1}$ is invertible for $\lambda \in I$, so $\rho_x(\lambda) = 0$ if and only if $\Gamma(\lambda) = 0$.

(g) *Transform.* Equation (16) is (8) together with the spectral theorem and (14). $\square$

Remark 3.5 (What (15) localizes). By [Sch26d, Prop. 4.13], Γ is the coupling-weighted density of states of the second fibration. Equation (15) states that the local spectral density of the active region is that same quantity, seen through the resolvent of the effective Hamiltonian. Finiteness of the peak gain is therefore a statement about the *regularity* of the density of states of the second fibration, whereas by [Sch26d, Thm. 4.17] finiteness of the worst-case norm is a statement about the *zero set* of M on the real axis. This is the precise sense in which the two criteria are not of the same kind, and it is why Theorem 3.8 requires a hypothesis that [Sch26d, Thm. 4.17] does not.

Hypothesis 3.6 (H_{edge} , edge and van Hove regularity). Assume H_{pop} , H_{reg} and [Sch26d, (40)], and let ρ_x be the density of Proposition 3.4, extended by zero to $\mathbb{R}$. There are a finite set $V \subset I$, containing every band edge and every critical value of every non-flat band function, and $\delta > 0$, such that

- (i) ρ_x is of class C^2 on every compact subset of $\mathbb{R} \setminus V$;
- (ii) for each $\lambda_* \in V$ there are r_* , h_*^+ , $h_*^- \in C^2((\lambda_* - \delta, \lambda_* + \delta); \mathcal{B}(\mathcal{H}_{\text{pri}}))$ with

$$\rho_x(\lambda) = r_*(\lambda) + (\lambda - \lambda_*)_+^{1/2} h_*^+(\lambda) + (\lambda_* - \lambda)_+^{1/2} h_*^-(\lambda) \quad (18)$$

for $|\lambda - \lambda_*| < \delta$.

Remark 3.7 (Scope of H_{edge}). Form (18) is the square-root vanishing characteristic of a band edge of a translation-covariant operator of finite range, and the van Hove form of a nondegenerate critical point of a band function in the interior, where a nonzero regular part r_* is present and the square-root correction is one-sided. It is stated as a hypothesis rather than derived: H_{reg} supplies continuity of Λ and Γ on I and nothing about the rate, and [Sch26d, Rem. 4.14] already records that the continuity clause of H_{reg} is itself hypothesized rather than derived from finite range. Whether H_{edge} follows from H_{pop} and H_{reg} at rank 3 is Target 3.11. The exponent $\frac{1}{2}$ and the regularity C^2 of r_* , $h_*^\pm$ are exactly what the proof of Theorem 3.8(ii) consumes; the weaker requirement $h_*^\pm \in C^1$ does not suffice, since the second integration by parts in that proof is then unavailable. H_{edge} is verified in closed form on the witness of Appendix A, where $V = \{\pm 2\}$, $r_\pm \equiv 0$, and $h_\pm^\mp$ is real-analytic.

Theorem 3.8 (L^1 criterion for the peak gain). [D] Assume [Sch26d, Hyps. 2.12–2.15], H_{ch} , H_{pop} , H_{reg} , H_{min} , and

$\sigma(H_p) \subset \text{int } I$. Then:

(i) if the embedded-mode exclusion [Sch26d, (40)] fails, $\|G_x\|_{\text{pk}} = \infty$;

(ii) if [Sch26d, (40)] holds and in addition H_{edge} holds, then

$$\|g_x^{\text{ac}}(t)\| = O(t^{-3/2}) \quad (t \rightarrow \infty), \quad (19)$$

and consequently $\|G_x\|_{\text{pk}} < \infty$, that is H_{pk} holds.

Proof. (i) If [Sch26d, (40)] fails then, by Step 3 of the proof of [Sch26d, Thm. 4.17], $\|G_x\|_{\mathcal{H}^\infty} = \infty$; by Proposition 2.4(iii), $\|G_x\|_{\text{pk}} = \infty$.

(ii) By Proposition 3.4 and (16),

$$\|g_x^{\text{ac}}(t)\| \leq \|C\| \|B\| \|\widehat{\rho_x}(t)\|, \quad \widehat{\rho_x}(t) = \int_{\mathbb{R}} e^{-i\lambda t} \rho_x(\lambda) d\lambda,$$

so it suffices to prove $\|\widehat{\rho_x}(t)\| = O(t^{-3/2})$. Since $\dim \mathcal{H}_{\text{pri}} < \infty$, all norms on $\mathcal{B}(\mathcal{H}_{\text{pri}})$ are equivalent and it suffices to treat each matrix entry, so we may assume ρ_x scalar.

Claim. Let $\psi \in C_c^2(\mathbb{R})$ and $\lambda_* \in \mathbb{R}$. Then there is $C(\psi) < \infty$ with

$$\left| \int_{\mathbb{R}} (\lambda - \lambda_*)_+^{1/2} \psi(\lambda) e^{-i\lambda t} d\lambda \right| \leq C(\psi) t^{-3/2} \quad (t \geq 1). \quad (20)$$

Proof of claim. Multiplying by the unimodular factor $e^{-i\lambda_* t}$ we may take $\lambda_* = 0$ and write $I(t) = \int_0^\infty u^{1/2} \psi(u) e^{-iut} du$. Put $a := 1/t$ and split at a . On $(0, a)$, $|\int_0^a| \leq \|\psi\|_\infty \frac{2}{3} a^{3/2} = \frac{2}{3} \|\psi\|_\infty t^{-3/2}$. On (a, ∞) set $F(u) := u^{1/2} \psi(u)$, which is C^2 there with

$$|F| \leq \|\psi\|_\infty u^{1/2}, \quad |F'| \leq u^{-1/2}, \quad |F''| \leq u^{-3/2},$$

the implied constants depending on $\|\psi\|_{C^2}$ and $\text{supp } \psi$. Two integrations by parts, the boundary terms at $+\infty$ vanishing by compact support, give

$$\int_a^\infty F e^{-iut} du = \frac{F(a) e^{-iat}}{it} + \frac{F'(a) e^{-iat}}{(it)^2} + \frac{1}{(it)^2} \int_a^\infty F'' e^{-iut} du.$$

The three terms are bounded respectively by $\|\psi\|_\infty a^{1/2}/t = \|\psi\|_\infty t^{-3/2}$; by $C a^{-1/2}/t^2 = C t^{-3/2}$; and, since $\int_a^\infty |F''| \lesssim a^{-1/2} + 1 \lesssim t^{1/2}$, by $C t^{1/2}/t^2 = C t^{-3/2}$. This proves (20); the statement for $(\lambda_* - \lambda)_+^{1/2}$ follows by reflection. $\diamond$

Now let V and δ be as in H_{edge} and choose a C^∞ partition of unity $1 = \chi_0 + \sum_{\lambda_* \in V} \chi_*$ on a neighbourhood of $\text{supp } \rho_x$, with $\text{supp } \chi_* \subset (\lambda_* - \delta, \lambda_* + \delta)$ and $\text{supp } \chi_0 \cap V = \emptyset$. Then $\widehat{\rho_x} = \chi_0 \widehat{\rho_x} + \sum_{\lambda_* \in V} \widehat{\chi_* \rho_x}$. By H_{edge} (i), $\chi_0 \rho_x \in C_c^2(\mathbb{R})$, so two integrations by parts give $|\widehat{\chi_0 \rho_x}(t)| \leq C t^{-2}$.

For each $\lambda_* \in V$, substitute (18): $\chi_* \rho_x = \chi_* r_* + (\lambda - \lambda_*)_+^{1/2} \chi_* h_*^+ + (\lambda_* - \lambda)_+^{1/2} \chi_* h_*^-$. The first summand lies in

C_c^2 , contributing $O(t^{-2})$; the second and third are of the form treated in the Claim with $\psi = \chi_* h_*^\pm \in C_c^2$, contributing $O(t^{-3/2})$ each.

Summing the finitely many contributions gives $\|\widehat{\rho_x}(t)\| = O(t^{-3/2})$, which is (19). Finally g_x^{ac} is bounded and continuous on $[0, \infty)$ by Lemma 3.1, so $\int_0^1 \|g_x^{\text{ac}}\| < \infty$, while (19) gives $\int_1^\infty \|g_x^{\text{ac}}\| \lesssim \int_1^\infty t^{-3/2} dt < \infty$. Hence $\|G_x\|_{\text{pk}} < \infty$. $\square$

Proposition 3.9 (The L^1 criterion is strictly stronger). [D]

(i) For every realization, $\|G_x\|_{\text{pk}} < \infty \Rightarrow \|G_x\|_{\mathcal{H}^\infty} < \infty$.

(ii) The converse fails on the class of admissible fibrated transfer networks of [Sch26d, Def. 2.10]: there is such a network of rank 1 and a form x of it with $\|G_x\|_{\mathcal{H}^\infty} < \infty$ and $\|G_x\|_{\text{pk}} = \infty$. Hence $\mathcal{G}_1 \subsetneq \mathcal{G}$.

Proof. (i) is Proposition 2.4(iii).

(ii) Take $d = 1$, $\Lambda = \mathbb{Z}$, $\Sigma_{\text{sp}} = \{\mathfrak{p}\}$ with the constant labelling, $\mathcal{S} = \{x\}$, $\mathcal{T} = \emptyset$, $\mathcal{K} = \mathcal{Y} = \mathbb{C}$, $\mathcal{H}_x = \mathbb{C}$, and for the dissipation datum (N5) the decomposition $\mathcal{H}_x = \mathbb{C} \oplus \{0\}$ with zero coupling; the network is admissible with the single constraint functional $C_1 \equiv 0$ and threshold $\tau_1 = 1$, so that $\Omega_{\text{adm}} = \{x\}$. For the transfer realization (N4) take

$$G_x(z, \theta) := J(\theta) \quad (z \in \mathbb{C}_+, \theta \in \mathbb{T}),$$

independent of z , where $J \in C(\mathbb{T})$ is chosen with $\sum_{k \in \mathbb{Z}} |\widehat{J}(k)| = \infty$. Such a J exists because the Wiener algebra $A(\mathbb{T})$ is a proper subset of $C(\mathbb{T})$ [Kat04, Ch. I]. The map is analytic in z , being constant, and continuous in θ , so (N4) holds and [Sch26d, Def. 2.3] reduces to a supremum over the spatial torus alone. Then

$$\|G_x\|_{\mathcal{H}^\infty} = \sup_{\theta} |J(\theta)| < \infty, \quad \|G_x\|_{\text{pk}} = \sum_k |\widehat{J}(k)| = \infty$$

by (5), the impulse response having no absolutely continuous part. Hence $G_x \in \mathcal{G} \setminus \mathcal{G}_1$. $\square$

Remark 3.10 (What Proposition 3.9 does and does not show). The witness of Proposition 3.9(ii) separates $\mathcal{G}_1$ from $\mathcal{G}$ in the quasi-momentum variable and uses no energy dependence at all. It therefore establishes that $\mathcal{G}_1$ is a proper subset, and with it that Definition 4.1 below has a smaller domain than [Sch26d, Def. 5.1]; and it shows that the Wiener norm of Remark 2.6 is not equivalent to a supremum over θ . It does *not* show that H_{edge} cannot be dropped from Theorem 3.8(ii), because the witness satisfies neither H_{pop} nor the hypotheses under which Proposition 3.4 produces a density at all. Whether a composite realization satisfying H_{pop} and H_{reg} can have finite worst-case norm

and infinite peak gain is open and is stated as Target 3.11. An earlier draft of this paper asserted the stronger reading; it is withdrawn, and the correction is recorded in the defect ledger of Appendix B.

Target 3.11 (Is $\mathcal{G}_1 = \mathcal{G}$ under H_{pop} and H_{reg} ?). [T] Target: determine whether, for a second fibration of rank 3 satisfying H_{pop} and H_{reg} , the embedded-mode exclusion [Sch26d, (40)] alone implies $\|G_x\|_{\text{pk}} < \infty$; equivalently, whether H_{edge} is automatic under those hypotheses. Finite range makes each band function a real-analytic trigonometric polynomial on $\mathbb{T}^3$, so its critical values are finite in number and its level sets are semianalytic; the question is whether the resulting van Hove exponents always produce an integrable transform. A negative answer requires a symbol satisfying H_{pop} for which $\widehat{\rho}_x \notin L^1$, and would also settle the question left open in [Sch26d, Rem. 4.14]. Proposition 3.9(ii) does not decide this, for the reason given in Remark 3.10.

Remark 3.12 (Position in the logical ladder). Both criteria are conditions on A_{10} and A_{12} data together with the second fibration of H_{sec} ; neither uses the molecular content of $A_1 - A_3$; and neither is an axiom. Under the prerequisite relation [Sch26c, Def. 80], with $A_4 \prec A_{10}$ and $A_9 \prec A_{12}$ among its generators, they sit at the same level of the ladder as [Sch26d, Thm. 4.17], and Theorem 3.8 refines that theorem rather than displacing it.

Remark 3.13 (Chemical reading). [E] Under H_{chem} , condition (19) is the assertion that amplitude placed on the rearranged unit of [Sch26d, Def. 2.28] not only leaves the active region but leaves it fast enough to be integrable. That an actual substituted chrysene assembly satisfies this is an empirical instantiation claim; it is used in no proof, and Theorem 3.8 is a statement about $\mathfrak{G}_x$ alone.

4 The L^1 Transformation Geometry

Definition 4.1 (L^1 unitary-orbit distance). Assume H_{ch} . For $G_x, G_y \in \mathcal{G}_1$ set

$$\delta_{\text{pk}}(G_x, G_y) := \inf_{\substack{U \in \mathcal{U}(\mathcal{Y}) \\ V \in \mathcal{U}(\mathcal{K})}} \|UG_xV - G_y\|_{\text{pk}}. \quad (21)$$

Proposition 4.2 (δ_{pk} is a pseudometric with attained infimum). [D] Assume H_{ch} . Then on $\mathcal{G}_1$:

(i) δ_{pk} is finite and the infimum in (21) is attained;

(ii) $\delta_{\text{pk}} \geq 0$, $\delta_{\text{pk}}(G_x, G_x) = 0$, and δ_{pk} is symmetric;

(iii) δ_{pk} satisfies the triangle inequality;

(iv) $\delta_u(G_x, G_y) \leq \delta_{\text{pk}}(G_x, G_y)$.

Proof. (i) Finiteness: $UG_xV - G_y$ has feedthrough $UJ_xV - J_y$ and absolutely continuous part $Ug_x^{\text{ac}}V - g_y^{\text{ac}}$, so by Proposition 2.4(i)–(ii) its peak-gain norm is at most $\|G_x\|_{\text{pk}} + \|G_y\|_{\text{pk}} < \infty$. For attainment, note that by Proposition 2.4(i),

$$\|UG_xV - G_y\|_{\text{pk}} - \|U'G_xV' - G_y\|_{\text{pk}} \leq \|UG_xV - U'G_xV'\|_{\text{pk}},$$

and

$$\|UG_xV - U'G_xV'\|_{\text{pk}} \leq \|G_x\|_{\text{pk}} (\|U - U'\| + \|V - V'\|),$$

by writing $UG_xV - U'G_xV' = (U - U')G_xV + U'G_x(V - V')$. Thus $(U, V) \mapsto \|UG_xV - G_y\|_{\text{pk}}$ is continuous on $\mathcal{U}(\mathcal{Y}) \times \mathcal{U}(\mathcal{K})$, which is compact under H_{ch} ; the infimum is attained.

(ii) Nonnegativity and vanishing on the diagonal are clear. Symmetry follows from Proposition 2.4(ii): $\|UG_xV - G_y\|_{\text{pk}} = \|G_x - U^*G_yV^*\|_{\text{pk}}$, and $(U, V) \mapsto (U^*, V^*)$ is a bijection of the index set.

(iii) Let (U_1, V_1) attain $\delta_{\text{pk}}(G_x, G_y)$ and (U_2, V_2) attain $\delta_{\text{pk}}(G_y, G_z)$. Then

$$\begin{aligned} \delta_{\text{pk}}(G_x, G_z) &\leq \|U_2U_1G_xV_1V_2 - G_z\|_{\text{pk}} \\ &\leq \|U_2(U_1G_xV_1 - G_y)V_2\|_{\text{pk}} + \|U_2G_yV_2 - G_z\|_{\text{pk}} \\ &= \delta_{\text{pk}}(G_x, G_y) + \delta_{\text{pk}}(G_y, G_z), \end{aligned}$$

the last equality by Proposition 2.4(ii).

(iv) For each admissible (U, V) , $\|UG_xV - G_y\|_{\mathcal{H}^\infty} \leq \|UG_xV - G_y\|_{\text{pk}}$ by Proposition 2.4(iii); take the infimum over (U, V) on both sides. $\square$

Proposition 4.3 (Zero set). [D] For $G_x, G_y \in \mathcal{G}_1$,

$$\delta_{\text{pk}}(G_x, G_y) = 0 \iff G_x \simeq G_y$$

in the sense of [Sch26d, Def. 3.6]. Hence the zero set of δ_{pk} is the restriction to $\mathcal{G}_1$ of the zero set of δ_u , and the two induced quotients of $\mathcal{G}_1$ coincide.

Proof. If $G_x \simeq G_y$, the witnessing pair (U, V) of [Sch26d, (15)] gives $UG_xV - G_y = 0$ identically, hence $\delta_{\text{pk}} = 0$. Conversely, suppose $\delta_{\text{pk}}(G_x, G_y) = 0$. By Proposition 4.2(i) the infimum is attained at some (U, V) , so $\|UG_xV - G_y\|_{\text{pk}} = 0$. By (4) this forces $UJ_xV = J_y$ and $Ug_x^{\text{ac}}V = g_y^{\text{ac}}$ almost everywhere on $(0, \infty)$; both are continuous by Lemma 3.1, so the second identity holds everywhere. Hence the impulse responses of UG_xV and G_y coincide as tempered distributions, and by injectivity of the Fourier–Laplace transform $UG_xV =$

G_y on $\mathbb{C}_+$, which is [Sch26d, (15)]. The last assertion follows because δ_u vanishes exactly on $\approx$ by [Sch26d, Prop. 5.2(iv)]. $\square$

Remark 4.4 (Where the two geometries actually differ). Proposition 4.3 refutes the natural expectation that the finer norm separates more forms. The reason is Proposition 2.4(ii): $\approx$ acts by constant unitaries, and every unitarily invariant norm has the same orbit zero set, so no choice of norm in (21) can change it. The differences between d_{pk} and $d_{\mathcal{H}^\infty}$ are therefore located elsewhere — in the domain, by Proposition 3.9, and in the infinite locus, by Proposition 4.7 — and any claim that the L^1 geometry is finer because it distinguishes more forms would be false. Compare [Sch26d, Prop. 5.7].

Definition 4.5 (Edge cost and the L^1 transformation metric). For $(x, y) \in \mathcal{E}_{\text{adm}}$ set $c_{\text{pk}}(x, y) := \delta_{\text{pk}}(G_x, G_y)$ when both G_x and G_y lie in $\mathcal{G}_1$, and $c_{\text{pk}}(x, y) := +\infty$ otherwise. For a transformation route $\gamma = (x = x_0, \dots, x_N = y) \in P_{\text{adm}}(x, y)$ set $L_{c_{\text{pk}}}(\gamma) := \sum_{i=1}^N c_{\text{pk}}(x_{i-1}, x_i)$, and define

$$d_{\text{pk}}(x, y) := \begin{cases} \inf_{\gamma \in P_{\text{adm}}(x, y)} L_{c_{\text{pk}}}(\gamma), & P_{\text{adm}}(x, y) \neq \emptyset, \\ +\infty, & P_{\text{adm}}(x, y) = \emptyset. \end{cases} \quad (22)$$

Proposition 4.6 (Basic properties and domination). [D] d_{pk} is an extended pseudometric on Ω_{adm} , and for all $x, y \in \Omega_{\text{adm}}$

$$\delta_{\text{pk}}(G_x, G_y) \leq d_{\text{pk}}(x, y), \quad d_{\mathcal{H}^\infty}(x, y) \leq d_{\text{pk}}(x, y). \quad (23)$$

Proof. Nonnegativity is clear and $d_{\text{pk}}(x, x) = 0$ via the empty route. Symmetry holds because $\mathcal{E}_{\text{adm}}$ is symmetric by Axiom A_{12} [Sch26c, (420)] and c_{pk} is symmetric by Proposition 4.2(ii), so route reversal is a bijection $P_{\text{adm}}(x, y) \rightarrow P_{\text{adm}}(y, x)$ preserving $L_{c_{\text{pk}}}$. The triangle inequality follows by concatenation of routes with the convention $r + \infty = \infty$. The first inequality in (23) follows by iterating Proposition 4.2(iii) along a route and taking the infimum, trivially when no route exists. For the second, compare edge costs: if both endpoints lie in $\mathcal{G}_1$ then $c(x, y) = \delta_u \leq \delta_{\text{pk}} = c_{\text{pk}}(x, y)$ by Proposition 4.2(iv); if some endpoint lies outside $\mathcal{G}_1$ then $c_{\text{pk}}(x, y) = +\infty \geq c(x, y)$. Since $c \leq c_{\text{pk}}$ edgewise, $L_c \leq L_{c_{\text{pk}}}$ on every route and the infima are ordered. $\square$

Proposition 4.7 (The infinite locus is strictly larger). [D] By [Sch26d, Prop. 5.6], $d_{\mathcal{H}^\infty}(x, y) = +\infty$ exactly when x and y lie in distinct connected components of $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$.

The corresponding statement for d_{pk} is false: there is an admissible fibrated transfer network and a pair x, y in the same connected component of $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$ with $d_{\mathcal{H}^\infty}(x, y) < \infty$ and $d_{\text{pk}}(x, y) = +\infty$. Consequently the equivalence $d_{\mathcal{H}^\infty} = +\infty \Leftrightarrow d_{\mathcal{C}}^* = +\infty$ of [Sch26d, Rem. 5.23] does not extend to d_{pk} , and the connectivity trichotomy of [Sch26d, Prop. 8.18] acquires a fourth condition strictly stronger than the other three.

Proof. Extend the network of Proposition 3.9(ii) to three forms $\mathcal{S} = \{x, m, y\}$ with $\mathcal{T} = \{(x, m), (m, x), (m, y), (y, m)\}$, constraint functional $C_1 \equiv 0 < \tau_1 = 1$ so that $\Omega_{\text{adm}} = \mathcal{S}$ and $\mathcal{E}_{\text{adm}} = \mathcal{T}$, and transfer realizations $G_x = G_y := 0$ and $G_m := J(\theta)$ with J as in Proposition 3.9(ii). Then x and y lie in the same connected component of $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$, and every route from x to y passes through m . All three realizations lie in $\mathcal{G}$, so $d_{\mathcal{H}^\infty}(x, y) \leq \delta_u(G_x, G_m) + \delta_u(G_m, G_y) = 2 \sup_\theta |J(\theta)| < \infty$; whereas $G_m \notin \mathcal{G}_1$, so $c_{\text{pk}}(x, m) = c_{\text{pk}}(m, y) = +\infty$ by Definition 4.5 and $d_{\text{pk}}(x, y) = +\infty$ by (22). The final assertions follow because [Sch26d, Rem. 5.23] and [Sch26d, Prop. 8.18] both rest on the characterization of the infinite locus of $d_{\mathcal{H}^\infty}$ as disconnection, which by the above fails for d_{pk} . $\square$

Proposition 4.8 (The peak gain is 1-Lipschitz). [D] For all $x, y \in \Omega_{\text{adm}}$ with $G_x, G_y \in \mathcal{G}_1$,

$$|\|G_x\|_{\text{pk}} - \|G_y\|_{\text{pk}}| \leq \delta_{\text{pk}}(G_x, G_y) \leq d_{\text{pk}}(x, y). \quad (24)$$

Proof. For any admissible (U, V) , $\|G_y\|_{\text{pk}} \leq \|UG_x V\|_{\text{pk}} + \|UG_x V - G_y\|_{\text{pk}} = \|G_x\|_{\text{pk}} + \|UG_x V - G_y\|_{\text{pk}}$ by Proposition 2.4(ii). Taking the infimum over (U, V) gives $\|G_y\|_{\text{pk}} - \|G_x\|_{\text{pk}} \leq \delta_{\text{pk}}(G_x, G_y)$; the symmetric argument and Proposition 4.2(ii) give the first inequality, and the second is (23). $\square$

Proposition 4.9 (No converse comparison). [D] There is no nondecreasing finite $\varphi : (0, \infty) \rightarrow (0, \infty)$ with $\|G\|_{\text{pk}} \leq \varphi(\|G\|_{\mathcal{H}^\infty})$ for all stable realizations, and no nondecreasing finite ψ with $d_{\text{pk}} \leq \psi(d_{\mathcal{H}^\infty})$. The two transformation metrics are therefore not equivalent and (23) cannot be reversed.

Proof. The first assertion is [Sch26d, Prop. 8.15]: the all-pass family $\widehat{G}_n(s) = ((1-s)/(1+s))^n$ satisfies $\|G_n\|_{\mathcal{H}^\infty} = 1$ for every n while $\|G_n\|_{\text{pk}} \rightarrow \infty$. For the second, let $\mathcal{S} := \{y\} \cup \{x_n\}_{n \geq 1}$ with the edges $(x_n, y) \in \mathcal{E}_{\text{adm}}$ for every n , constraint functional $C_1 \equiv 0 < \tau_1 = 1$ so that $\Omega_{\text{adm}} = \mathcal{S}$, and realizations $G_{x_n} := G_n$ and $G_y := 0$. Each G_n is rational with poles in the open left half-plane, so $G_n \in \mathcal{G}_1$, and $0 \in \mathcal{G}_1$ trivially. Then within this single network $d_{\mathcal{H}^\infty}(x_n, y) = \delta_u(G_n, 0) = \|G_n\|_{\mathcal{H}^\infty} = 1$ for every n , while $d_{\text{pk}}(x_n, y) = \delta_{\text{pk}}(G_n, 0) = \|G_n\|_{\text{pk}} \rightarrow \infty$, so no ψ as stated can exist. $\square$

Definition 4.10 (Bottleneck closure). With c_{pk} as in Definition 4.5, set $L_{c_{pk}}^\vee(\gamma) := \max_{1 \leq i \leq N} c_{pk}(x_{i-1}, x_i)$, with $L_{c_{pk}}^\vee = 0$ for the empty route, and $d_{pk}^\vee(x, y) := \inf_\gamma L_{c_{pk}}^\vee(\gamma)$, with value $+\infty$ when no route exists.

Proposition 4.11 (Ultrametric property and comparison). [D] $d_{pk}^\vee$ is an extended pseudo-ultrametric on Ω_{adm} ,

$$d_{pk}^\vee(x, z) \leq \max\{d_{pk}^\vee(x, y), d_{pk}^\vee(y, z)\},$$

and $d_{pk}^\vee \leq d_{pk}$ pointwise.

Proof. Symmetry and vanishing on the diagonal are as in Proposition 4.6. For the ultrametric inequality, the concatenation γ of a route γ_1 from x to y with a route γ_2 from y to z satisfies $L_{c_{pk}}^\vee(\gamma) = \max\{L_{c_{pk}}^\vee(\gamma_1), L_{c_{pk}}^\vee(\gamma_2)\}$; taking infima over γ_1 and γ_2 independently gives the inequality, which implies the triangle inequality. Finally $L_{c_{pk}}^\vee \leq L_{c_{pk}}$ on every route, whence $d_{pk}^\vee \leq d_{pk}$. This is the argument of [Sch26d, Prop. 5.14] applied to c_{pk} . $\square$

5 The Joint Survival Radius

Definition 5.1 (Joint tolerance). Assume H_{um} on $S \subseteq \Omega_{adm}$ with constants κ^*, θ^* . For an environment level $\delta > 0$ and a disturbance amplitude $\alpha > 0$ set

$$\beta(\delta, \alpha) := \min\left\{\frac{1}{\delta}, \frac{\theta^*}{\kappa^* \alpha}\right\}. \quad (25)$$

Theorem 5.2 (Joint survival radius). [D] Assume H_{ch} , H_{env} , H'_{out} and H_{um} on $S \subseteq \Omega_{adm}$. Let $x \in S$ with $\gamma_x^{pk} := \|G_x\|_{pk} < \infty$, and let $\delta, \alpha > 0$. Then for every $y \in S$ satisfying

$$d_{pk}(x, y) < \beta(\delta, \alpha) - \gamma_x^{pk} \quad (26)$$

the following hold simultaneously:

- (i) for every causal environment operator Δ with $\|\Delta\|_{\mathcal{H}^\infty} \leq \delta$, the feedback interconnection of G_y with Δ is well posed and internally stable;
- (ii) for every disturbance $w \in \mathcal{W}_\infty$ with $\|w\|_\infty \leq \alpha$, the trajectory issuing from y satisfies $C_j(t) < \tau_j$ for every j and every $t \geq 0$; that is, it remains in Ω_{adm} .

Proof. Write $\gamma_y^{pk} := \|G_y\|_{pk}$ and let η denote the output signal of G_y .

Step 1: y lies in $\mathcal{G}_1$, with peak gain bounded on the ball. By (26), $d_{pk}(x, y) < \infty$, so by (22) there is a route $\gamma \in P_{adm}(x, y)$

with $L_{c_{pk}}(\gamma) < \infty$. Every edge of γ therefore has finite cost, and by Definition 4.5 finite cost of an edge requires both its endpoints to lie in $\mathcal{G}_1$; in particular $G_y \in \mathcal{G}_1$. Proposition 4.8 now applies and, with (26), gives

$$\gamma_y^{pk} \leq \gamma_x^{pk} + \delta_{pk}(G_x, G_y) \leq \gamma_x^{pk} + d_{pk}(x, y) < \beta(\delta, \alpha). \quad (27)$$

Both conclusions below are consequences of this single bound. *Step 2: robust stability.* By Proposition 2.4(iii), (27) and (25),

$$\|G_y\|_{\mathcal{H}^\infty} \leq \gamma_y^{pk} < \beta(\delta, \alpha) \leq \frac{1}{\delta},$$

so $\|G_y\|_{\mathcal{H}^\infty} \|\Delta\|_{\mathcal{H}^\infty} < 1$. The realization G_y is stable, and Δ is causal and stable by H_{env} , so the small-gain theorem [ZDG96, Thm. 9.1] yields well-posedness and internal stability of the interconnection, exactly as in the proof of [Sch26d, Thm. 5.9]. This is (i).

Step 3: invariance of Ω_{adm} . Let $w \in \mathcal{W}_\infty$ with $\|w\|_\infty \leq \alpha$, and let the trajectory issue from y with zero initial state, so that by Lemma 3.1 the output is $\eta = Jw + g_y^{ac} * w$. Young's inequality gives, for every $t \geq 0$,

$$\|\eta(t)\| \leq \left(\|J\| + \int_0^\infty \|g_y^{ac}\|\right) \|w\|_\infty = \gamma_y^{pk} \|w\|_\infty \leq \gamma_y^{pk} \alpha. \quad (28)$$

Estimate (28) is the upper bound of Proposition 2.4(iv), and it is the only way Definition 2.3 enters [Sch26d, Thm. 8.13]; this is the verification promised in Remark 2.5 that replacing [Sch26d, Def. 8.11] by Definition 2.3 leaves that theorem intact.

By H'_{out} [Sch26d, Hyp. 8.12] and Hypothesis 2.9, for every j and every $t \geq 0$,

$$C_j(t) \leq C_j(y) + \kappa_j \|\eta(t)\| \leq C_j(y) + \kappa^* \gamma_y^{pk} \alpha.$$

By (27) and (25),

$$\kappa^* \gamma_y^{pk} \alpha < \kappa^* \beta(\delta, \alpha) \alpha \leq \kappa^* \cdot \frac{\theta^*}{\kappa^* \alpha} \cdot \alpha = \theta^*,$$

while (7) gives $C_j(y) + \theta^* \leq \tau_j$. Hence $C_j(t) < \tau_j$ for every j and every $t \geq 0$, which is (ii). $\square$

Corollary 5.3 (Joint balls). [D] Under the hypotheses of Theorem 5.2, the set of forms simultaneously robustly stable against every environment perturbation of size at most δ and Ω_{adm} -invariant under every disturbance of amplitude at most α contains the open d_{pk} -ball about x of radius $\beta(\delta, \alpha) - \gamma_x^{pk}$, which is nonempty precisely when $\gamma_x^{pk} < \beta(\delta, \alpha)$.

Proof. The inclusion is Theorem 5.2, whose hypothesis (26) is exactly membership in the stated ball. The ball is nonempty if and only if its radius is positive, that is $\gamma_x^{pk} < \beta(\delta, \alpha)$; when it is, x itself belongs to it, since $d_{pk}(x, x) = 0$ by

Proposition 4.6. $\square$

Remark 5.4 (Discharge of [Sch26d, Target 8.26], and its scope). Theorem 5.2 is the single computable condition asked for in [Sch26d, Target 8.26]: one inequality, in one metric, implying both halves of the survival claim for the same forms, and propagating along a transformation route because d_{pk} is a path closure. It is a bound on $\|\cdot\|_{\text{pk}}$ and not on $\|\cdot\|_{\mathcal{H}^\infty}$, consistently with the impossibility recorded in [Sch26d, Prop. 8.15]. Three limits are stated here and repeated in §7. First, the discharge is for a single form and along a route; nothing here bounds the jump factor μ of [Sch26d, (80)], so [Sch26d, Target 6.24] is untouched and the survival claim across *switching* remains a conjunction, as [Sch26d, Rem. 9.5] records. Second, the conclusion presupposes H_{um} , which by Remark 2.10 is a genuine restriction on an infinite Ω_{adm} . Third, the witness of Appendix A does not satisfy H_{ss} , so [Sch26d, Target 6.26] is inherited unchanged.

Remark 5.5 (The joint radius is strictly more conservative). By Proposition 2.4(iii) and (23), $\gamma_x^{\text{pk}} \geq \|G_x\|_{\mathcal{H}^\infty}$ and $d_{\text{pk}} \geq d_{\mathcal{H}^\infty}$, so the ball of Corollary 5.3 is contained in that of [Sch26d, Cor. 5.10] whenever both are defined. For the robust-stability half alone [Sch26d, Thm. 5.9] therefore remains the sharper statement and is not superseded by Theorem 5.2; the price of the joint certificate is a strictly smaller ball. The loss is quantified on the witness in Appendix A.

Proposition 5.6 (No common optimum). [D] *On the witness family of Appendix A, parametrized by the coupling strength $v^2 \in (0, 2)$, the map $v^2 \mapsto \|G\|_{\mathcal{H}^\infty}$ attains its minimum, of value 1, at $v^2 = 1$, whereas $v^2 \mapsto \|G\|_{\text{pk}}$ attains its minimum, of value 1.5067..., at $v^2 = 0.8027...$, where $\|G\|_{\mathcal{H}^\infty} = 1.2458...$. No coupling strength minimizes both.*

Proof. By [Sch26d, Prop. A.8], $\|G\|_{\mathcal{H}^\infty} = v^{-2}$ for $0 < v^2 \leq 1$ and $(2 - v^2)^{-1}$ for $1 \leq v^2 < 2$; this function is continuous, strictly decreasing on $(0, 1]$ and strictly increasing on $[1, 2)$, so its unique minimizer is $v^2 = 1$, with value 1. The second assertion is Proposition A.8 — the closed-form and numerical evaluation of $\|G\|_{\text{pk}}$ carried out in Appendix A, where the minimizer is located at $v^2 = 0.80269...$ with value 1.50674...; the corresponding worst-case norm is $1/0.80269... = 1.24581...$. Since $0.80269... \neq 1$ the two minimizers are distinct. $\square$

Remark 5.7 (Reading of Proposition 5.6). [Sch26d, Prop. 8.15] shows that the two quantities are not comparable as functionals. Proposition 5.6 shows that they are

not co-optimized even along a one-parameter family inside the formalism's own witness, so the incomparability has a consequence at the level of design and not only at the level of certificates: a form tuned to be maximally robust in the sense of [Sch26d, Thm. 5.9] is not the form that tolerates the largest disturbance amplitude in the sense of [Sch26d, Thm. 8.13].

6 The Bandwidth Floor

Definition 6.1 (Spectral width). For a composite realization with composite Hamiltonian H as in (1), bounded and self-adjoint, set

$$L_x := \text{diam conv } \sigma(H) < \infty. \quad (29)$$

Theorem 6.2 (Return floor). [D] *For every composite realization with bounded self-adjoint composite Hamiltonian H ,*

$$\int_0^\infty \|P_{\text{pri}} e^{-iHt} P_{\text{pri}}\| dt \geq \frac{\pi}{L_x}, \quad (30)$$

the left-hand side being $+\infty$ when the integrand is not integrable. The bound uses no hypothesis beyond boundedness of H .

Proof. Set $A(t) := P_{\text{pri}} e^{-iHt} P_{\text{pri}}|_{\mathcal{H}_{\text{pri}}} \in \mathcal{B}(\mathcal{H}_{\text{pri}})$. If $\int_0^\infty \|A(t)\| dt = \infty$ there is nothing to prove, so assume the integral finite. If $L_x = 0$ then $\sigma(H)$ is a single point λ_0 , so $H = \lambda_0 I$, $\|A(t)\| = 1$ for all t and the integral diverges; assume therefore $L_x > 0$.

(a) *Reflection.* Since $H = H^*$,

$$A(-t) = P_{\text{pri}} e^{iHt} P_{\text{pri}} = (P_{\text{pri}} e^{-iHt} P_{\text{pri}})^* = A(t)^*,$$

so $\|A(-t)\| = \|A(t)\|$, whence $A \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}_{\text{pri}}))$ with

$$\int_{\mathbb{R}} \|A(t)\| dt = 2 \int_0^\infty \|A(t)\| dt. \quad (31)$$

(b) *A is the transform of μ_x .* By the spectral theorem and Definition 3.3, $A(t) = \int_{\mathbb{R}} e^{-i\lambda t} d\mu_x(\lambda) = \widehat{\mu_x}(t)$, where μ_x is a positive $\mathcal{B}(\mathcal{H}_{\text{pri}})$ -valued measure of total mass $\mu_x(\mathbb{R}) = I_{\mathcal{H}_{\text{pri}}}$, supported in $\sigma(H)$.

(c) *Inversion.* Fix $u, v \in \mathcal{H}_{\text{pri}}$. The scalar measure $\langle u, \mu_x(\cdot)v \rangle$ is finite and its Fourier transform $\langle u, A(\cdot)v \rangle$ lies in $L^1(\mathbb{R})$; by the inversion theorem for finite measures with integrable transform [Kat04, Ch. VI], it is absolutely continuous with continuous density $\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \langle u, A(t)v \rangle dt$. Since $\dim \mathcal{H}_{\text{pri}} < \infty$, applying this to the pairs drawn from an orthonormal basis

shows that μ_x is absolutely continuous with continuous density

$$m(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} A(t) dt, \quad \sup_{\lambda} \|m(\lambda)\| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \|A(t)\| dt. \quad (32)$$

(d) *Mass against width.* By (b), $\text{supp } \mu_x \subseteq \sigma(H) \subseteq \text{conv } \sigma(H)$, an interval of length L_x . Hence, by the triangle inequality for Bochner integrals,

$$1 = \|I_{\mathcal{H}_{\text{pri}}}\| = \left\| \int_{\text{conv } \sigma(H)} m(\lambda) d\lambda \right\| \leq \int_{\text{conv } \sigma(H)} \|m(\lambda)\| d\lambda \leq L_x \sup_{\lambda} \|m(\lambda)\|$$

Combining with (32) and (31),

$$1 \leq \frac{L_x}{2\pi} \int_{\mathbb{R}} \|A(t)\| dt = \frac{L_x}{\pi} \int_0^\infty \|A(t)\| dt,$$

which is (30). $\square$

Corollary 6.3 (Bandwidth floor on the peak gain). [D] Suppose in addition that C is bounded below and B is surjective onto $\mathcal{H}_{\text{pri}}$, with $\sigma_{\min}(C), \sigma_{\min}(B) > 0$. Then

$$\|G_x\|_{\text{pk}} - \|J\| \geq \frac{\pi \sigma_{\min}(C) \sigma_{\min}(B)}{L_x}. \quad (33)$$

In particular, for scalar channels with $|C| = |B| = 1$, $\|G_x\|_{\text{pk}} \geq \|J\| + \pi/L_x$.

Proof. Since C is bounded below, $\|C\xi\| \geq \sigma_{\min}(C)\|\xi\|$ for every $\xi \in \mathcal{H}_{\text{pri}}$, so $\|CN\| \geq \sigma_{\min}(C)\|N\|$ for every N with range in $\mathcal{H}_{\text{pri}}$. Since B is surjective onto $\mathcal{H}_{\text{pri}}$, $BB^* \geq \sigma_{\min}(B)^2 I_{\mathcal{H}_{\text{pri}}}$, so for every N defined on $\mathcal{H}_{\text{pri}}$,

$$\|NB\|^2 = \|NBB^*N^*\| \geq \sigma_{\min}(B)^2 \|NN^*\| = \sigma_{\min}(B)^2 \|N\|^2,$$

using that $X \geq Y \geq 0$ implies $NXN^* \geq NYN^*$. Applying the second estimate with $N = CA(t)$ and then the first,

$$\|CA(t)B\| \geq \sigma_{\min}(B) \|CA(t)\| \geq \sigma_{\min}(B) \sigma_{\min}(C) \|A(t)\|.$$

By Lemma 3.1, $\|g_x^{\text{ac}}(t)\| = \|CA(t)B\|$ for $t > 0$, so integrating and applying Theorem 6.2,

$$\|G_x\|_{\text{pk}} - \|J\| = \int_0^\infty \|g_x^{\text{ac}}(t)\| dt \geq \sigma_{\min}(C) \sigma_{\min}(B) \int_0^\infty \|A(t)\| dt$$

which is (33). For scalar channels $\sigma_{\min}(C) = |C|$ and $\sigma_{\min}(B) = |B|$. $\square$

Corollary 6.4 (Maximum certifiable disturbance amplitude). [D] Assume H'_{out} and H_{um} , and let $\alpha_{\max}(x) := \theta^*/(\kappa^* \|G_x\|_{\text{pk}})$ be the largest amplitude at which Theo-

rem 5.2(ii) certifies invariance of Ω_{adm} at x . Then

$$\alpha_{\max}(x) \leq \frac{\theta^* L_x}{\pi \kappa^* \sigma_{\min}(C) \sigma_{\min}(B)}. \quad (34)$$

Proof. Applied at $y = x$, where $d_{\text{pk}}(x, x) = 0$, condition (26) of Theorem 5.2 reads $\gamma_x^{\text{pk}} < \beta(\delta, \alpha)$. For the invariance conclusion (ii) the binding term of (25) is $\theta^*/(\kappa^* \alpha)$, so the condition is $\alpha < \theta^*/(\kappa^* \gamma_x^{\text{pk}}) = \alpha_{\max}(x)$; thus $\alpha_{\max}(x)$ is indeed the supremum of amplitudes at which Theorem 5.2(ii) certifies invariance at x . By Corollary 6.3, $\gamma_x^{\text{pk}} \geq \|J\| + \pi \sigma_{\min}(C) \sigma_{\min}(B)/L_x \geq \pi \sigma_{\min}(C) \sigma_{\min}(B)/L_x$, and (34) follows on inverting. In particular a nonzero feedthrough operator only lowers $\alpha_{\max}$. $\square$

Remark 6.5 (Two caveats, stated plainly). The quantity L_x in (30) is the width of the *composite* spectrum and not of the second fibration alone; by the proof of Proposition 3.4(a), $\sigma(H) \subseteq [-R, R]$ with $R = \rho + 1 + \|H_p\| + \|W\|^2$. Two consequences must be stated rather than elided. First, a design that widens the active-region energy scale $\|H_p\|$ weakens the floor, so the correct claim is a floor uniform over realizations with a fixed second fibration and a bounded active-region energy scale, not an absolute one. Second, the cheapest way to widen $\sigma(H)$ is to add spectral mass far from the bands of the second fibration; but isolated spectral mass outside the bands is precisely a mode ψ with $\Gamma(\lambda)\psi = 0$, which by [Sch26d, Thm. 4.17] gives $\|G_x\|_{\mathcal{H}^\infty} = \infty$ and by Theorem 3.8(i) gives $\|G_x\|_{\text{pk}} = \infty$. Both escape routes from the floor are therefore closed by the well-posedness criterion itself, and it is this pairing, rather than (30) alone, that makes the bound a statement about the fibration rather than about a particular choice of parameters.

Remark 6.6 (Sharpness on the witness). For the witness of Appendix A one has $\sigma(H) = [-2, 2]$, hence $L_x = 4$ and a floor of $\pi/4 = 0.78539 \dots$, uniformly in the coupling. The minimum of the peak gain over the family is $1.50674 \dots$, attained at $v^2 = 0.80269 \dots$, so the floor is satisfied with a ratio of $1.9185 \dots$ and is not attained. Because the floor is constant along the family while the peak gain is not, the ratio is extremized where the gain is, and the figure $1.9185 \dots$ is therefore the best that this family achieves; it is an upper bound on the sharp constant of Target 6.7 and not evidence that the bound is close to tight in general.

Target 6.7 (The sharp constant). [T] Target: determine

$$c^* := \inf L_x \int_0^\infty \|P_{\text{pri}} e^{-iHt} P_{\text{pri}}\| dt, \quad (35)$$

the infimum taken over composite realizations with bounded self-adjoint composite Hamiltonian and finite left-hand side. Theorem 6.2 gives $c^* \geq \pi$ and Remark 6.6 gives $c^* \leq 4 \times 1.50674 \dots = 6.0270 \dots$. In the scalar case the question is the extremal Fourier problem of minimizing $\|\widehat{\rho}\|_{L^1(\mathbb{R})}$ over probability densities supported in an interval of unit length; whether the infimum is attained, and by what density, is not settled here.

Remark 6.8 (Chemical reading). [E] Under H_{chem} , Corollary 6.4 is the assertion that the bandwidth of the second fibration sets a ceiling on the disturbance amplitude any rearranged form of the network can be certified against, independently of how the active region is designed. That an actual substituted chrysene assembly realizes the stated composite Hamiltonian is an empirical instantiation claim, conditional in addition on the Identification Hypothesis Φ of [Sch26b], and it enters no proof.

7 Non-Implications, the Deferred Programs, and New Targets

The governing restraint of the formalism is that restoring a foundational object never licenses restoring an unproved implication of that object [Sch26b, §3.1]. This section states, in the style of [Sch26d, Prop. 9.1] and of the separations [Sch26c, (444)], what the constructions above do not establish. Each item is refuted, or delimited, by a witness or a hypothesis already exhibited; the section adds no new argument and exists to make the list checkable in one place.

7.1 Non-implications

Proposition 7.1 (Non-implications). [D] *None of the following holds in general; each is refuted, or delimited, by the item named in its proof.*

- (i) *For an admissible fibrated transfer network, $\|G_x\|_{\mathcal{H}^\infty} < \infty \Rightarrow \|G_x\|_{\text{pk}} < \infty$.*
- (ii) *δ_{pk} separates more forms than δ_u ; in particular $\delta_{\text{pk}}(G_x, G_y) = 0 \Rightarrow x = y$.*
- (iii) *$d_{\text{pk}}(x, y) = +\infty \Rightarrow x$ and y lie in distinct connected components of $(\Omega_{\text{adm}}, \mathcal{E}_{\text{adm}})$.*
- (iv) *$\lambda_2(L) > 0 \Rightarrow d_{\text{pk}}(x, y) < \infty$ for all x, y .*
- (v) *$d_{\mathcal{H}^\infty}(x, y)$ small $\Rightarrow d_{\text{pk}}(x, y)$ small.*
- (vi) *$\|G_x\|_{\text{pk}} < \infty \Rightarrow$ the trajectory from x remains in Ω_{adm} at some positive disturbance amplitude.*

- (vii) *Theorem 5.2 $\Rightarrow$ the network survives arbitrary admissible switching among the forms of the ball.*
- (viii) *a form minimizing $\|\cdot\|_{\mathcal{H}^\infty}$ minimizes $\|\cdot\|_{\text{pk}}$.*
- (ix) *Theorem 6.2 $\Rightarrow$ a limit on tolerable disturbance that no design can improve.*
- (x) *$\|G_x\|_{\text{pk}} = +\infty \Rightarrow$ autonomous physical scission, or any physical event whatever.*
- (xi) *$H_{\text{edge}} \Rightarrow$ the continuity clause of H_{reg} is derived rather than hypothesized.*
- (xii) *the discharge of [Sch26d, Target 8.26] $\Rightarrow$ the discharge of [Sch26d, Target 7.19].*

Proof. (i) Proposition 3.9(ii). Under H_{pop} and H_{reg} the implication is neither proved nor refuted here; that is Target 3.11, and Remark A.15 records that the witness does not decide it. (ii) Proposition 4.3: the zero sets of δ_u and δ_{pk} coincide on $\mathcal{G}_1$, because $\simeq$ acts by constant unitaries and every unitarily invariant norm has the same orbit zero set (Remark 4.4). The forms $x_1 \neq x_3$ of (38), with $\delta_{\text{pk}}(G_1, G_3) = 0$ by Proposition A.9, are an explicit witness.

(iii) and (iv) Proposition 4.7, whose witness places x and y in one connected component with $d_{\text{pk}}(x, y) = +\infty$; since $\lambda_2(L) > 0$ is equivalent to connectedness [Sch26d, Prop. 8.18], the same witness refutes (iv).

(v) Proposition 4.9, with the all-pass family: there $d_{\mathcal{H}^\infty} = 1$ for every n while $d_{\text{pk}} \rightarrow \infty$.

(vi) By H_{out} [Sch26d, Hyp. 2.22] and [Sch26d, Rem. 5.12], absent a constraint readout no gain bound of any kind constrains membership in Ω_{adm} ; and with a readout, the amplitude certified by Theorem 5.2(ii) is $\theta^*/(\kappa^*\|G_x\|_{\text{pk}})$, which tends to 0 as the uniform margin θ^* of Hypothesis 2.9 does. Finiteness of the peak gain is therefore necessary and not sufficient.

(vii) Theorem 5.2 certifies each y in the ball separately, so [Sch26d, Prop. 6.6] applies verbatim: there are forms, individually robustly stable, between which admissible switching is exponentially unstable. Nothing here bounds the jump factor μ of [Sch26d, (80)]; see Remark 5.4.

(viii) Proposition 5.6, with the values of Proposition A.8.

(ix) Remark 6.5(i): the width L_x of (29) is that of the composite spectrum, and by [Sch26d, (33)] it grows with $\|H_p\|$. The floor is uniform over realizations with a fixed second fibration and a bounded active-region energy scale, and is not absolute. Remark 6.5(ii) records why the two obvious ways of exploiting this are closed by the well-posedness criterion, but that is a limitation on the evasion, not an extension of the bound.

(x) The standing restraint [Sch26b, Eq. (3)], [Sch26c, Rem. 39] and [Sch26d, Rem. 4.21]; nothing in §§3–6 weakens it. Divergence of $\|G_x\|_{\text{pk}}$ is a statement about an operator norm, and by [Sch26d, Prop. 4.20] no norm is lost from $\mathcal{H}_{\text{pri}} \oplus \mathcal{H}_{\text{sec}}$

under the composite evolution.

(xi) Hypothesis 3.6 is stated *under* H_{reg} and assumes its continuity clause; the question raised in [Sch26d, Rem. 4.14] of deriving that clause from structural hypotheses on the symbol remains open, and Remark 3.7 records this.

(xii) [Sch26d, Target 7.19] asks for a quantity bounding simultaneously the unstructured stability radius $\|G_x\|_{\mathcal{H}^\infty}^{-1}$ and the self-adjoint embedded-mode margin $r_{\text{emb}}(x)$ of [Sch26d, Thm. 7.15]. The pair combined by Theorem 5.2 is a different one — robust stability and invariance of Ω_{adm} — and r_{emb} appears nowhere in this paper. $\square$

Remark 7.2 (The two items most likely to be over-read). Items (vii) and (ix). The first is the boundary of the discharge: this paper makes the survival claim a single theorem *for one form and along a transformation route*, and leaves it a conjunction across switching, exactly where [Sch26d, Rem. 9.5] left it. The second is the boundary of the floor: Theorem 6.2 is a statement about a fixed fibration and not a universal constant of the formalism.

Remark 7.3 (The feedthrough operator cuts both ways). Remark 2.7 records that a nonzero J leaves $\|G\|_{\text{pk}}$ finite where it forces $\|G\|_{\mathcal{H}^2} = \infty$, and the proof of Corollary 6.4 records that a nonzero J only lowers α_{max} . Both are correct and they are not in tension: the first says the certificate exists, the second says it is weaker. The mean-square route of [Sch26d, Cor. 8.14] does not trade margin for existence in this way; it simply has none to trade, being vacuous whenever $J \neq 0$, which by [Sch26c, A₁₀] is the default.

7.2 Status of the deferred program of [Sch26c]

Remark 7.4 (Reading of Table 1). [Sch26c, §11] states ten targets, Theorems 40–49. [Sch26b, §3.6] and [Sch26d, Tab. 2] compress these to eight entries, merging Theorems 42 and 43 (directional transfer and polarization anisotropy) and Theorems 45 and 46 (real spectral triple and spectral dimension). Table 1 lists all ten, so that a reviewer checking against [Sch26c, §11] finds a one-to-one correspondence. No target is discharged.

7.3 Status of the twelve targets of [Sch26d]

7.4 Targets introduced here

Two targets have been stated in place: Target 3.11, whether $\mathcal{G}_1 = \mathcal{G}$ under H_{pop} and H_{reg} , and Target 6.7, the sharp constant in the bandwidth floor. Two more are opened now.

Target of [Sch26c]	Relation to this paper
Thm. 40, $N + 2$ closure	not touched
Thm. 41, $GF(4)$ field law	not touched
Thm. 42, directional transfer	not touched
Thm. 43, polarization anisotropy	not touched
Thm. 44, spectral–path metric	not approached: d_{pk} is not a candidate, for the reason [Sch26d, Rem. 5.23] gives for $d_{\mathcal{H}^\infty}$
Thm. 45, real spectral triple	not touched
Thm. 46, spectral dimension	not touched
Thm. 47, emergent chirality	not touched
Thm. 48, noncocommutativity	not touched
Thm. 49, dynamical scission	not discharged; explicitly not licensed, Prop. 7.1(x)

Table 1: The deferred program of [Sch26c] against the present paper.

Target 7.5 (The peak gain under admissible switching). [T] Target: establish the analogue of [Sch26d, Thm. 6.13] and [Sch26d, Cor. 6.16] for the peak gain — a common storage functional, or a structural sufficient condition on the couplings $\{\Gamma_x\}$, under which arbitrary admissible switching among a family of forms preserves a uniform bound on the peak-to-peak gain, and hence, through Theorem 5.2(ii), preserves invariance of Ω_{adm} . The obstruction is that the bounded real lemma [ZDG96, Ch. 13] characterizes the $\mathcal{H}^\infty$ norm by a quadratic storage functional and has no peak-to-peak counterpart of the same form; the natural substitute is an invariant-set rather than a dissipativity argument. Until this is discharged, the survival claim across switching remains the conjunction that [Sch26d, Rem. 9.5] records, and Theorem 5.2 does not change that.

Target 7.6 (Weakening the uniform constraint margin). [T] Target: replace Hypothesis 2.9 by a local condition. Axiom A₁₂ supplies $\tau_j - C_j(y) > 0$ for every $y \in \Omega_{\text{adm}}$ but no uniform lower bound, and by Remark 2.10 the uniform version fails along any sequence approaching $\partial_C \Omega_{\text{adm}}$. Target: give a version of Theorem 5.2 in which θ^* is replaced by the margin at y itself, which requires bounding $\tau_j - C_j(y)$ below in terms of $\tau_j - C_j(x)$ and $d_{\text{pk}}(x, y)$ — that is, a Lipschitz property of the constraint margin with respect to the transformation metric, in analogy with Proposition 4.8 for the gain. No such property is established here, and δ_{pk} compares transfer data while C_j is a functional on structural states, so the two are not obviously related; compare the obstruction recorded in [Sch26d, Target 6.24].

Remark 7.7 (Inventory, and which are load-bearing). This paper opens four targets: 3.11, 6.7, 7.5 and 7.6. Their

Target of [Sch26d]	Relation to this paper	Proof.
4.24, mixed populations	not touched	Identical to [Sch26d, Prop. 10.1]; no datum of the present paper enters the verification of (N1)–(N5). $\square$
4.25, extended active regions	inherited, and now also required by Thm. 6.2, whose proof uses $\dim \mathcal{H}_{\text{pri}} < \infty$	Theorem 8.2 (Portability). [D] <i>Let $\mathfrak{R}$ be an admissible fibrated transfer network of rank d whose data satisfy the hypotheses named in the statement of a given result below. Then that result holds for $\mathfrak{R}$, with the same proof, $\mathbb{Z}^3$ replaced by $\mathbb{Z}^d$ throughout. This applies to: Proposition 2.4; Lemma 3.1; Proposition 3.4; Theorem 3.8; Proposition 3.9; all of §4; all of §5; and all of §6.</i>
4.26, scattering-theoretic form	not touched	
6.24, jump factor	not touched; load-bearing	
6.25, optimal policies	not touched	
6.26, non-Markovian forms	inherited unchanged; the witness still fails H_{ss}	
7.19, combined margin	not touched, Prop. 7.1(xii)	
7.20, margin along a route	not touched	
8.25, commutant as scaling class	not touched	
8.26, joint certificate	discharged for a single form and along a route, under H_{um} : Thm. 5.2, Rem. 5.4	
8.27, nucleation dynamics	not touched	
10.9, distinguished instance	not touched	<i>Proof.</i> Inspection of the proofs. Each uses only: the direct-sum decomposition (N5) of [Sch26d, Def. 2.10] and self-adjointness of the two diagonal blocks; the translation covariance and finite range of (N1)–(N2); the analyticity and continuity of the realizations (N4); the admissibility structure of (N3); and the named hypotheses of [Sch26d, §2.6] and §2.3. No proof invokes Axioms A_1 – A_3 , the comparison class $\mathcal{B}_4$, the locant structure, or any statement about the reference molecule. Replacing $\mathbb{Z}^3$ by $\mathbb{Z}^d$ affects only the dimension of the torus in (5) and in [Sch26d, Def. 2.2], and the Hausdorff measure in [Sch26d, (35)], which becomes $(d-1)$ -dimensional; Hypothesis 3.6 is stated on the density ρ_x and is dimension-free as written. $\square$

Table 2: The target program of [Sch26d] against the present paper. One entry is discharged, with the scope stated in Remark 5.4.

number is not offered as evidence of anything. Two are load-bearing in the sense of [Sch26d, Rem. 9.5], that the paper’s own results are incomplete without them: Target 7.5, because without it the discharge of [Sch26d, Target 8.26] does not extend across switching and the survival claim remains a conjunction in that regime; and Target 7.6, because Theorem 5.2 presupposes H_{um} , which on an infinite Ω_{adm} is a restriction and not a formality. Target 3.11 is not load-bearing: Theorem 3.8(ii) is complete with H_{edge} as a named hypothesis, which Proposition A.7 verifies on the witness, and Target 3.11 asks only whether the hypothesis is redundant. Target 6.7 is likewise a sharpening and not a gap.

8 Transfer to Other Fibrated Networks

The proofs of §§3–6 use the molecular content of Axioms A_1 – A_3 nowhere. This section states that precisely, in the form of [Sch26d, §10], and states equally precisely what an instance must exhibit beyond the schema given there.

Proposition 8.1 (Induced network). [D] *Let $x \in \Omega_{\text{adm}}$ carry a composite realization $\mathfrak{G}_x$ under the hypotheses of Lemma 3.1 together with H_{ch} and H_{pop} . Then the data (1) and (6) constitute an admissible fibrated transfer network of rank 3 in the sense of [Sch26d, Def. 2.10], exactly as in [Sch26d, Prop. 10.1].*

Remark 8.3 (Two clauses the instance schema must acquire). [Sch26d, Def. 10.4] lists what must be exhibited to assert that a system is an admissible fibrated transfer network. Two further items are required to invoke the results of this paper, and neither is implied by that list.

First, to invoke §5 one must exhibit the constraint readout of H'_{out} together with the uniform margin constants κ^*, θ^* of Hypothesis 2.9. [Sch26d, Def. 10.4(N3)] requires the constraint functionals to be stated; it does not require them to factor through the output channel, and without that factorization no gain bound constrains membership in Ω_{adm} at all [Sch26d, Hyp. 2.22]. Remark A.3 records that the factorization must be read incrementally.

Second, to invoke §6 one must exhibit $\sigma_{\min}(C) > 0$ and surjectivity of B onto $\mathcal{H}_{\text{pri}}$. Without them Theorem 6.2 still holds, being a statement about $P_{\text{pri}}e^{-iHt}P_{\text{pri}}$ alone, but Corollary 6.3 and with it the amplitude ceiling of Corollary 6.4 are unavailable, since $\|CAB\|$ then admits no lower bound in terms of $\|A\|$.

As in [Sch26d, Rem. 10.5], the verification burden is not nominal: both clauses fail silently rather than visibly.

Remark 8.4 (What is not portable). Three classes of statement do not transfer. First, everything of type [E]: Remarks 3.13 and 6.8 concern the reference molecule and have no analogue elsewhere. Second, the closed forms of Appendix A,

which are properties of the particular half-line fibration chosen there and not of fibrated transfer networks in general — in particular Propositions A.5–A.7. Third, the verification of H_{edge} itself: Target 3.11 is open even at rank 3 with finite range, so no instance acquires H_{edge} by being a fibrated transfer network.

Remark 8.5 (No retroactive certification). Theorem 8.2 makes results available to a network satisfying [Sch26d, Def. 10.4] together with Remark 8.3. It certifies no existing paper. In particular the papers of Volume 1, Number 1 applying H^∞ methods to clinical, cryptographic and financial settings [Sch26b, §6] were reviewed under the Level 1 protocol, were not evaluated against $F_{\text{adm}} = M \cap C \cap H$, and have not been checked here against Remark 8.3 or against the corrections recorded in Appendix B. This restates [Sch26d, Rem. 10.7] and does not weaken it; the correction to [Sch26d, Def. 8.11] recorded in Remark 2.5 and the reading of H_{out} recorded in Remark A.3 both bear on any such application.

9 Conclusion

The question this paper set out to answer was whether the two halves of the survival claim of [Sch26d] — robust stability, controlled by the worst-case norm, and invariance of the admissible domain, controlled by a peak gain — can be certified by a single condition, given that [Sch26d, Prop. 8.15] forbids any such condition from being a bound on the worst-case norm. The answer is affirmative and conditional, and the conditions are the substance.

The peak gain is the integrated return amplitude of the active region (Lemma 3.1), and its spectral density is the coupling-weighted density of states of the second fibration seen through the effective resolvent (Proposition 3.4). It is therefore finite under a strictly stronger condition than the worst-case norm: embedded-mode exclusion excludes poles, while integrability additionally constrains the edge and van Hove regularity of that density (Theorem 3.8). The two criteria are not of the same kind, and the second is not implied by the first for a general fibrated transfer network (Proposition 3.9), though whether it is implied under finite range remains open (Target 3.11).

Granted finiteness, the geometry behaves. The L^1 orbit distance is a pseudometric with attained infimum, its path closure dominates d_{H^∞} , and one ball in it certifies both halves at once (Theorem 5.2), which discharges [Sch26d, Target 8.26] for a single form and along a transformation route. What changes relative to d_{H^∞} is not the zero set, which is transfer isomorphism in both cases and cannot be otherwise for any unitarily invariant norm (Proposition 4.3), but the domain and the infinite locus (Propositions 3.9 and 4.7); the second of these costs [Sch26d, Prop. 8.18] its trichotomy.

Two findings run against the reading the corpus has so far taken. Survival certified jointly is strictly more expensive than survival certified in halves: on the witness the fraction of the available radius consumed by the same pair of forms rises from 0.67 to 0.99 (Remark A.12), and [Sch26d, Thm. 5.9] is not superseded for the robust-stability half alone. And the two gains are not co-optimized: the coupling strength minimizing the worst-case norm is not the one minimizing the peak gain (Proposition 5.6), so the incomparability of [Sch26d, Prop. 8.15] has a consequence for design and not only for certificates.

Finally, the integrated return amplitude admits a lower bound set by the width of the composite spectrum (Theorem 6.2), and with it an upper bound on the disturbance amplitude against which invariance of Ω_{adm} can be certified at all (Corollary 6.4). The bound is a statement about a fixed fibration and a bounded active-region energy scale, not a universal constant (Proposition 7.1(ix)); what makes it more than an inequality is that both natural ways of evading it — widening the active region, or adding distant spectral mass — are closed by the well-posedness criterion itself (Remark 6.5).

What the paper does not do is stated as plainly. No axiom is introduced. No deferred target of [Sch26c] is discharged (Table 1). Twelve non-implications are recorded (Proposition 7.1), four targets are opened, of which two are load-bearing (Remark 7.7), and the concordance appendix records three corrections to [Sch26d], one of them to a hypothesis on which [Sch26d, Thm. 8.13] rests. Fourteen defects found during drafting and in review, including two in the hypothesis of this paper’s own well-posedness criterion, one in the definition on which the whole construction rests, and a set of provisional numerical values wrong in the third significant figure, are recorded with their repairs in Table 5, together with three items accepted and not repaired. The survival claim across switching remains a conjunction, as [Sch26d, Rem. 9.5] left it; discharging Target 7.5 is what would change that.

A An Explicit Witness Realization

The witness $\mathfrak{R}_{\text{wit}}$ of [Sch26d, App. A] instantiates the A_{10} and A_{12} data of the explicit model $\mathfrak{R}_{\text{ex}}$ of [Sch26c, App. B], a replacement of downstream data that [Sch26c, Rem. 62] licenses while retaining membership in $\text{Ext}_{12}(\mathbb{C}^{\text{chr}})$. This appendix does three things: it augments that datum with a constraint readout, so that H'_{out} and H_{um} are instantiated and Theorem 5.2 has a worked instance; it solves the witness in closed form, so that the quantities of §§3–6 are exhibited rather than estimated; and it records, in §A.9, what the witness does not establish.

A.1 The augmented admissibility datum

We retain from [Sch26d, Def. A.1] the five-point set $X = \{x_1, \dots, x_5\}$ with mixing parameters $s_i = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, 1$, the spectral realization $J_{\text{str}}(x_i) = (1-s_i)\omega + s_i\varphi$, the constraint functional

$C_1(x_i) = s_i$ with threshold $\tau_1 = 1$, the edge set and the weights. By [Sch26d, Prop. A.2], $\Omega_{\text{adm}} = \{x_1, x_2, x_3, x_4\}$, $x_5 \in \partial_C \Omega_{\text{adm}}$, and $\mathcal{E}_{\text{adm}} = \{\{x_1, x_3\}, \{x_1, x_2\}, \{x_2, x_4\}\}$.

Definition A.1 (Constraint readout). Set $\kappa_1 := 1$ and declare the constraint functional evaluated along a trajectory issuing from $y \in \Omega_{\text{adm}}$ with output signal η to be

$$C_1(t) := C_1(y) + \kappa_1 |\eta(t)|. \quad (36)$$

Proposition A.2 (The witness instantiates H'_{out} and H_{um}). [D] With Definition A.1, H'_{out} [Sch26d, Hyp. 8.12] holds with $\kappa_1 = 1$, and H_{um} holds on $S := \Omega_{\text{adm}}$ with

$$\kappa^* = 1, \quad \theta^* = \min_{y \in \Omega_{\text{adm}}} (\tau_1 - C_1(y)) = 1 - \frac{3}{5} = \frac{2}{5}. \quad (37)$$

Proof. Equation (36) is [Sch26d, Hyp. 8.12] with equality and $\kappa_1 = 1$. The margins are $\tau_1 - C_1(x_i) = 1 - s_i$ for $i \leq 4$, that is $1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}$, all strictly positive with minimum $\frac{2}{5}$ attained at x_4 ; Ω_{adm} is finite, so Remark 2.10 applies. $\square$

Remark A.3 (Concordance: the literal form of [Sch26d, Hyp. 2.22]). H_{out} as stated in [Sch26d, Hyp. 2.22] requires $C_j(x) = c_j(y_x)$ for functions c_j of the output signal. Taken literally with a single c_j independent of the form, this cannot hold whenever C_j is non-constant on Ω_{adm} and the undisturbed output vanishes, as it does here: $c_j(0)$ would have to equal $C_j(x)$ for every x . The form actually used in [Sch26d, Thm. 8.13], and the one instantiated by (36), is the incremental version H'_{out} , in which the readout is per-form and only the increment factors through the output channel. The disposition recorded in Appendix B is *repeated*: no conclusion of [Sch26d, Thm. 8.13] depends on the stronger reading.

Remark A.4 (Environment interconnection). H_{env} is instantiated by taking Ω_{env} to be the unstructured causal class of [Sch26d, Thm. 7.7], with $E_e := \Delta$ and $\|\Delta\|_{\mathcal{H}^\infty} \leq \delta$; well-posedness of the interconnection for the pairs considered is Step 2 of the proof of Theorem 5.2. [Sch26d, Tab. 3] records H_{env} and H_{out} as not instantiated on $\mathfrak{R}_{\text{wit}}$; with Definition A.1 and the present remark both are instantiated, and the gap reported as item 7 of [Sch26d, Tab. 6] does not recur for §5. It does recur for H_{ss} ; see §A.9.

A.2 Closed form of the return amplitude

We use the forms of [Sch26d, Def. A.6]: $K = Y = \mathbb{C}$, $\mathcal{H}_{\text{pri}} = \mathbb{C}$, $H_p = \varepsilon = 0$, $C = 1$, $J = 0$, and

$$x_1 : (v^2, B) = (1, 1), \quad x_2 : (1, \rho), \quad x_3 : (1, e^{i\psi}), \quad x_4 : (\tfrac{1}{2}, 1), \quad (38)$$

with $\rho = \frac{1}{2}$, $\psi = \pi/3$, and the half-line second fibration of [Sch26d, Def. A.3]. Write $w := v^2$.

Proposition A.5 (Local spectral density and Chebyshev moments). [D] For $0 < w < 2$ and $\varepsilon = 0$ the local spectral measure μ_x of Definition 3.3 is absolutely continuous with density

$$\rho_w(\lambda) = \frac{w\sqrt{4-\lambda^2}}{2\pi(\lambda^2(1-w) + w^2)} \quad (|\lambda| < 2), \quad (39)$$

and $\rho_w = 0$ for $|\lambda| \geq 2$. Its Chebyshev moments $\mu_n := \int T_n(\lambda/2) \rho_w(\lambda) d\lambda$ are

$$\mu_0 = 1, \quad \mu_{2k} = \tfrac{1}{2}(2-w)(-1)^k(1-w)^{k-1} \quad (k \geq 1), \quad \mu_{\text{odd}} = 0. \quad (40)$$

Proof. Equation (39) is (15) with $\Gamma(\lambda) = w\sqrt{4-\lambda^2}/2$ and $|M(\lambda)|^2 = \lambda^2(1-w) + w^2$, both from [Sch26d, Prop. A.4, (112)]. The denominator is strictly positive on $[-2, 2]$ for $0 < w < 2$: for $w \leq 1$ its minimum is w^2 at $\lambda = 0$, and for $w \geq 1$ its minimum is $(2-w)^2$ at $\lambda^2 = 4$. For (40), put $z = \zeta + \zeta^{-1}$ with $|\zeta| < 1$, so that $g_s(z) = \zeta$ by [Sch26d, Prop. A.4] and

$$M(z) = \zeta + \zeta^{-1} - w\zeta = \zeta^{-1}(1 + (1-w)\zeta^2), \quad G(z) = \frac{\zeta}{1 + (1-w)\zeta^2}.$$

On the other hand, with $\lambda = 2 \cos \theta$, $(z - \lambda)^{-1} = \zeta(1 - 2\zeta \cos \theta + \zeta^2)^{-1}$ and $(1 - 2\zeta \cos \theta + \zeta^2)^{-1} = (1 - \zeta^2)^{-1}[1 + 2 \sum_{n \geq 1} \zeta^n \cos n\theta]$, so integrating against ρ_w and using $T_n(\cos \theta) = \cos n\theta$,

$$G(z) = \frac{\zeta}{1 - \zeta^2} \left[1 + 2 \sum_{n \geq 1} \zeta^n \mu_n \right].$$

Equating the two expressions gives

$$1 + 2 \sum_{n \geq 1} \zeta^n \mu_n = \frac{1 - \zeta^2}{1 + b\zeta^2}, \quad b := 1 - w.$$

Expanding the right-hand side as $(1 - \zeta^2) \sum_{k \geq 0} (-b)^k \zeta^{2k} = 1 + \sum_{k \geq 1} [(-b)^k - (-b)^{k-1}] \zeta^{2k}$ and using $(-b)^k - (-b)^{k-1} = -(1+b)(-b)^{k-1}$ gives $2\mu_{2k} = -(2-w)(-1)^{k-1}$ for $k \geq 1$ and $\mu_{\text{odd}} = 0$, which is (40). $\square$

Proposition A.6 (Closed-form return amplitude). [D] For $0 < w < 2$ the return amplitude $a_w(t) := P_{\text{pri}} e^{-iHt} P_{\text{pri}}$ of Theorem 6.2 is the real-valued function

$$a_w(t) = J_0(2t) + (2-w) \sum_{k \geq 1} (1-w)^{k-1} J_{2k}(2t), \quad (41)$$

the series converging absolutely and uniformly on $[0, \infty)$, with $a_w(0) = 1$. For $w = 1$ it reduces to

$$a_1(t) = J_0(2t) + J_2(2t) = \frac{J_1(2t)}{t}, \quad (42)$$

ρ_1 being the Wigner semicircle density on $[-2, 2]$. The impulse response of the form with attachment datum B is $g^{\text{ac}}(t) = -iB a_w(t)$, and $\|G\|_{\text{pk}} = |B| \int_0^\infty |a_w(t)| dt$.

Proof. By Proposition 3.4, $a_w(t) = \int e^{-i\lambda t} \rho_w(\lambda) d\lambda$. Substituting $\lambda = 2 \cos \theta$ and using the Jacobi–Anger expansion $e^{-2it \cos \theta} = \sum_{n \geq 0} (-i)^n \epsilon_n J_n(2t) \cos n\theta$ with $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n \geq 1$, and integrating term by term against ρ_w gives $a_w(t) = \sum_{n \geq 0} (-i)^n \epsilon_n J_n(2t) \mu_n$. Only even n survive by (40), and $(-i)^{2k} = (-1)^k$, so

$$a_w(t) = J_0(2t) + 2 \sum_{k \geq 1} (-1)^k J_{2k}(2t) \mu_{2k} = J_0(2t) + (2-w) \sum_{k \geq 1} (1-w)^{k-1} J_{2k}(2t)$$

which is (41). Since $|J_{2k}(2t)| \leq 1$ and $|1-w| < 1$ for $0 < w < 2$, the series is dominated by a convergent geometric series, giving absolute and uniform convergence; termwise integration is justified by the same domination. At $t = 0$ all J_{2k} with $k \geq 1$ vanish, so $a_w(0) = J_0(0) = 1$, consistent with $\mu_x(\mathbb{R}) = 1$. For $w = 1$ only $k = 1$ survives, and $J_0(x) + J_2(x) = 2J_1(x)/x$ gives (42). The last two assertions are Lemma 3.1 with $C = 1$, $J = 0$. $\square$

A.3 Verification of H_{edge} and the tail

Proposition A.7 (The witness satisfies H_{edge}). [D] For $0 < w < 2$, Hypothesis 3.6 holds with $V = \{-2, +2\}$, $r_{\pm 2} \equiv 0$, and

$$h_{+2}^-(\lambda) = \frac{w\sqrt{2+\lambda}}{2\pi(\lambda^2(1-w) + w^2)}, \quad (43)$$

real-analytic near $\lambda = 2$, with the mirror statement at $\lambda = -2$; in particular $h_{\pm 2}^\pm \in C^\infty \subset C^2$. Consequently Theorem 3.8(ii) applies, and

$$a_w(t) = A_w \frac{\cos(2t - \frac{3\pi}{4})}{t^{3/2}} + o(t^{-3/2}), \quad A_w = \frac{w}{\sqrt{\pi}(2-w)^2}, \quad (44)$$

so $\|G\|_{\text{pk}} < \infty$ for every $0 < w < 2$.

Proof. On $(-2, 2)$ the density (39) is real-analytic, its denominator being strictly positive by Proposition A.5; hence V contains no interior point and clause (i) of Hypothesis 3.6 holds. Near $\lambda = 2$ write $\sqrt{4-\lambda^2} = (2-\lambda)^{1/2}(2+\lambda)^{1/2}$, so that $\rho_w(\lambda) = (2-\lambda)_+^{1/2} h_{+2}^-(\lambda)$ with h_{+2}^- as in (43), real-analytic on a neighbourhood of $\lambda = 2$ with $h_{+2}^-(2) = w/(\pi(2-w)^2)$; the density vanishes identically for $\lambda > 2$, so $r_{+2} \equiv 0$ and $h_{+2}^+ \equiv 0$. The statement at $\lambda = -2$ follows since ρ_w is even. Equation (44) is the leading term of the contribution of the two edges, obtained from the Claim in the proof of Theorem 3.8(ii) retaining the principal term: each edge contributes $h(\pm 2) \Gamma(\frac{3}{2}) t^{-3/2}$ times a unimodular phase, and the two combine to $h_{+2}^-(2) \sqrt{\pi} t^{-3/2} \cos(2t - 3\pi/4)$, which is (44). $\square$

A.4 Peak gains and the two minimizers

Proposition A.8 (Computed peak gains). [D] With $\|a_w\|_1 := \int_0^\infty |a_w(t)| dt$, so that $\|G\|_{\text{pk}} = |B| \|a_w\|_1$:

$w = v^2$	$\ a_w\ _1$	$\ G\ _{\mathcal{H}^\infty}$ at $ B = 1$	ratio
0.25	4.01011	4	1.0025
0.50	2.05053	2	1.0253
0.80269	1.50674	1.24581	1.2095
1.00	1.60789	1	1.6079
1.21	1.91190	1.26582	1.5104
1.50	2.90611	2	1.4531

At $w = 1$, (42) gives the closed form

$$\|a_1\|_1 = \int_0^\infty \frac{|J_1(2t)|}{t} dt = \int_0^\infty \frac{|J_1(u)|}{u} du = 1.607889 \dots \quad (45)$$

The map $w \mapsto \|a_w\|_1$ attains its minimum on $(0, 2)$ at $w_* = 0.80269 \dots$ with value $1.50674 \dots$, and diverges as $w \downarrow 0$ and as $w \uparrow 2$.

Proof. The values are obtained from (41) by quadrature on the intervals between consecutive zeros of a_w , with the tail beyond a cutoff T supplied by (44); see §A.10 for the independent verification. Divergence as $w \downarrow 0$ is Proposition A.14 below; divergence as $w \uparrow 2$ follows from $A_w = w/(\sqrt{\pi}(2-w)^2) \rightarrow \infty$ in (44). Equation (45) is (42) with the substitution $u = 2t$. $\square$

This completes the proof of Proposition 5.6: by [Sch26d, Prop. A.8] the worst-case norm is minimized at $w = 1$ with value 1, while by Proposition A.8 the peak gain is minimized at $w_* = 0.80269 \dots$, where the worst-case norm equals $1.24581 \dots$

A.5 The L^1 transformation geometry on the witness

Proposition A.9 (Closed-form distances). [D] For the forms (38), with $\|a_1\|_1 = 1.60789$ and $\|a_{1/2}\|_1 = 2.05053$:

$$\begin{aligned} \delta_{\text{pk}}(G_1, G_3) &= 0, & \delta_{\text{pk}}(G_1, G_2) &= 0.80394, \\ \delta_{\text{pk}}(G_2, G_4) &= 1.56385, & \delta_{\text{pk}}(G_1, G_4) &= 1.20481, \end{aligned} \quad (46)$$

and consequently, on the three-edge graph of [Sch26d, Prop. A.2],

$$d_{\text{pk}}(x_1, x_3) = 0, \quad d_{\text{pk}}(x_1, x_2) = 0.80394, \quad d_{\text{pk}}(x_1, x_4) = 2.36779 \quad (47)$$

with $d_{\text{pk}}^\vee(x_1, x_4) = 1.56385$.

Proof. By Proposition A.6 the impulse responses are $g_j(t) = -iB_j a_{w_j}(t)$ with a_w real, so for scalar channels the unitary groups are the circle and

$$\delta_{\text{pk}}(G_i, G_j) = \min_{|c|=1} \int_0^\infty |c B_i a_{w_i}(t) - B_j a_{w_j}(t)| dt.$$

When $w_i = w_j$ the integrand factors and the minimum is $\|B_i\| - \|B_j\| \|a_w\|_1$, giving the first two values of (46) and, in particular, $\delta_{\text{pk}}(G_1, G_3) = 0$, consistent with Proposition 4.3 since $G_3 = e^{i\psi} G_1 \simeq G_1$. When $w_i \neq w_j$ the minimum is evaluated numerically over the phase; it is attained at $c = 1$ in both remaining cases. The path values (47) follow from Definitions 4.5 and 4.10 applied to the three-edge graph. $\square$

Remark A.10 (What the table witnesses). Comparison with [Sch26d, Prop. A.11], where $\delta_u(G_1, G_2) = \frac{1}{2}$, $\delta_u(G_2, G_4) = \frac{3}{2}$, $\delta_u(G_1, G_4) = 1$, $d_{\mathcal{H}^\infty}(x_1, x_4) = 2$ and $d_{\mathcal{H}^\infty}^\vee(x_1, x_4) = \frac{3}{2}$, establishes by exhibition:

- (i) the dominations of (23) are strict in general: $1 < 1.20481$ and $2 < 2.36779$;
- (ii) d_{pk} is a pseudometric and not a metric: $x_1 \neq x_3$ while $d_{\text{pk}}(x_1, x_3) = 0$, consistent with Proposition 4.3;
- (iii) inequality (23) is strict at the level of routes: $\delta_{\text{pk}}(G_1, G_4) = 1.20481 < 2.36779 = d_{\text{pk}}(x_1, x_4)$, the excess being the cost of routing through x_2 ;
- (iv) $d_{\text{pk}}^\vee < d_{\text{pk}}$ strictly, so the bottleneck and additive closures differ, as in [Sch26d, Rem. 5.16];
- (v) d_{pk} and d_C^* are incomparable: by [Sch26d, (106)], $d_C^*(x_1, x_3) = 1 > 0 = d_{\text{pk}}(x_1, x_3)$ while $d_C^*(x_1, x_2) = \frac{1}{10} < 0.80394 = d_{\text{pk}}(x_1, x_2)$.

The witness does *not* exhibit the strictness of Proposition 4.7: every form of (38) lies in $\mathcal{G}_1$, so d_{pk} is finite throughout and

the infinite locus is empty. That strictness is witnessed only by the separate construction in the proof of Proposition 4.7.

A.6 The joint survival radius on the witness

Example A.11 (A joint ball, and how tight it is). [D] Take $\delta = \frac{1}{4}$ and $\alpha = \frac{1}{10}$. By Proposition A.2, $\theta^* = \frac{2}{5}$ and $\kappa^* = 1$, so $\beta(\delta, \alpha) = \min\{4, 4\} = 4$, and the joint radius at x_1 of Corollary 5.3 is $4 - 1.60789 = 2.39211$. By (47) the ball contains all four forms, x_4 included, with margin $2.39211 - 2.36779 = 0.02432$. Raising the amplitude to $\alpha = 0.102$ gives $\beta = 3.92157$ and radius $2.31368 < 2.36779$, and x_4 leaves the ball. The certificate is therefore not merely nonvacuous on the witness but close to binding.

Remark A.12 (The conservatism of Theorem 5.2, quantified). At $\delta = \frac{1}{4}$, [Sch26d, Thm. 5.9] gives at x_1 a robust-stability radius $1/\delta - \|G_1\|_{\mathcal{H}^\infty} = 3$ in the metric $d_{\mathcal{H}^\infty}$, in which $d_{\mathcal{H}^\infty}(x_1, x_4) = 2$. Theorem 5.2 at the same δ and at $\alpha = \frac{1}{10}$ gives radius 2.39211 in the metric d_{pk} , in which $d_{\text{pk}}(x_1, x_4) = 2.36779$. The two radii are measured in different metrics and are not directly comparable; two dimensionless comparisons are. First, the fraction of the available radius consumed by the same pair of forms rises from $2/3 = 0.6667$ to $2.36779/2.39211 = 0.9898$. Second, the parameter at which x_4 leaves the ball changes character: under [Sch26d, Thm. 5.9] it is the environment level, x_4 leaving at $\delta = \frac{1}{3}$, whereas under Theorem 5.2 at $\delta = \frac{1}{4}$ the binding constraint is the disturbance amplitude, x_4 leaving at $\alpha = \theta^*/(\kappa^*(\gamma_{x_1}^{\text{pk}} + d_{\text{pk}}(x_1, x_4))) = 0.10061$. Both certificates admit all four forms at $(\delta, \alpha) = (\frac{1}{4}, \frac{1}{10})$, and the second certifies strictly more; what it costs is that almost none of the radius remains. This is the price recorded in Remark 5.5, and it is the reason [Sch26d, Thm. 5.9] is not superseded for the robust-stability half alone.

A.7 The bandwidth floor on the witness

Proposition A.13 (Floor and amplitude ceiling). [D] For every form of (38), $\sigma(H) = [-2, 2]$ and hence $L_x = 4$, so Theorem 6.2 reads $\|a_w\|_1 \geq \pi/4 = 0.78540$. The bound holds for every $w \in (0, 2)$, with least ratio $1.50674/0.78540 = 1.91845$ at $w = w_*$. With $\theta^* = \frac{2}{5}$ and $\kappa^* = 1$, the amplitude ceiling of Corollary 6.4 and the actual value $\alpha_{\max} = \theta^*/(\kappa^*\|G\|_{\text{pk}})$ are

form	$\ G\ _{\text{pk}}$	$\alpha_{\max}$	bound (34)
x_1	1.60789	0.24877	0.50930
x_2	0.80394	0.49755	1.01859
x_3	1.60789	0.24877	0.50930
x_4	2.05053	0.19507	0.50930

Proof. $\sigma(H) = [-2, 2]$ because the composite Hamiltonian is the half-line Jacobi matrix with off-diagonals $(v, 1, 1, \dots)$, whose essential spectrum is $[-2, 2]$ and which has no eigenvalue outside it for $0 < v^2 < 2$ by [Sch26d, Prop. A.9] with $\varepsilon = 0$. The floor and the tabulated values follow from Theorem 6.2, Corollary 6.3 with $\sigma_{\min}(C) = 1$ and $\sigma_{\min}(B) = |B|$, and Corollary 6.4. Note that the floor on $\int_0^\infty \|P_{\text{pri}} e^{-iHt} P_{\text{pri}}\| dt$ is independent of B , whereas the floor on the peak gain carries the factor $|B|$; this is why the bound for x_2 is twice that for the other forms. $\square$

A.8 Degenerate limits and the hypothesis register

Proposition A.14 (The two degenerate limits). [D] At $w = 0$ one has $W = 0$, $\Sigma \equiv 0$, $\Gamma \equiv 0$ and $G(z) = 1/z$, which is the transfer operator of the explicit model $\mathfrak{R}_{\text{ex}}$ of [Sch26c, App. B.7]; by [Sch26d, Prop. 4.22] and Theorem 3.8(i), both $\|G\|_{\mathcal{H}^\infty}$ and $\|G\|_{\text{pk}}$ are infinite. At $w = 2$ the denominator of (39) vanishes at $\lambda = \pm 2$ and $\rho_2(\lambda) = (\pi\sqrt{4 - \lambda^2})^{-1}$, the arcsine density, whose transform is $a_2(t) = J_0(2t) \notin L^1$; Hypothesis 3.6 fails there, the exponent at the edges being $-\frac{1}{2}$ rather than $+\frac{1}{2}$, and $\|G\|_{\mathcal{H}^\infty} = \infty$ as well.

Proof. At $w = 0$ the coupling $W = v|\cdot\rangle\langle\delta_0|$ vanishes, so $\Sigma \equiv 0$, $M(z) = z$ and $G(z) = 1/z$, which is the transfer operator of $\mathfrak{R}_{\text{ex}}$ computed in [Sch26c, App. B.7] and recalled in [Sch26d, Rem. 4.6]. By [Sch26d, Prop. 4.22] the gain is infinite, and by Theorem 3.8(i) so is the peak gain; directly, $\mu_x = \delta_0$ and $a_0(t) \equiv 1$, which is not integrable.

At $w = 2$ the denominator of (39) is $\lambda^2(1 - 2) + 4 = 4 - \lambda^2$, so $\rho_2(\lambda) = 2\sqrt{4 - \lambda^2}(2\pi(4 - \lambda^2))^{-1} = (\pi\sqrt{4 - \lambda^2})^{-1}$ on $(-2, 2)$, the arcsine density, whose Fourier transform is $a_2(t) = J_0(2t)$. Since $J_0(2t) \sim (\pi t)^{-1/2} \cos(2t - \pi/4)$, $a_2 \notin L^1(0, \infty)$ and $\|G\|_{\text{pk}} = \infty$. The exponent of ρ_2 at $\lambda = \pm 2$ is $-\frac{1}{2}$, so (18) fails and Hypothesis 3.6 does not hold. Finally $|M(\pm 2)|^2 = 4 - \lambda^2 = 0$ at $\lambda = \pm 2$, so M has a real zero and $\|G\|_{\mathcal{H}^\infty} = \infty$ by [Sch26d, Thm. 4.17]; equivalently, $w = 2$ is the endpoint excluded by the restriction $0 < v^2 < 2$ of [Sch26d, Prop. A.8]. $\square$

Remark A.15 (The witness does not settle Target 3.11). By Proposition A.14, the two criteria of §3 fail together at both ends of the family and hold together throughout its interior. The witness therefore exhibits no form with finite worst-case norm and infinite peak gain, and settles Target 3.11 neither way. This is consistent with Remark 3.10: the only separation available to this paper is the one constructed in Proposition 3.9(ii), which lies outside H_{pop} .

Hypothesis	Status on $\mathfrak{R}_{\text{wit}}$
$H_{\text{fib}}, H_{\text{lab}}, H_{\text{form}}$	verified, [Sch26d, Rem. A.7]
H_{ch}	verified, $k = q = 1$
$H_{\text{sec}}, H_{\text{pop}}, H_{\text{sp}}, H_{\text{cov}}$	verified, [Sch26d, Tab. 3]
H_{reg}	verified, [Sch26d, Prop. A.4]
H_{min}	verified, $n = 1$
H_{rec}	verified, scalar channels
H_{edge}	verified, Prop. A.7
H_{pk}	verified for $0 < w < 2$, Prop. A.7
$H_{\text{out}}, H'_{\text{out}}$	instantiated, Def. A.1, Prop. A.2
H_{um}	verified, $\theta^* = \frac{2}{5}$, $\kappa^* = 1$
H_{env}	instantiated, Rem. A.4
H_{ss}	not satisfied: Σ algebraic, not rational
$H_{\text{nuc}}, H_{\text{chem}}$	not instantiated, not used

A.9 What the witness does not establish

$\mathfrak{R}_{\text{wit}}$ as augmented here is a non-vacuity witness for §§3–6 and for nothing further. It does not instantiate H_{chem} : the forms (38) are not the rearranged frameworks of [Sch26d, Def. 2.28], no molecular datum enters them, and no statement of this appendix is of type [E]. It does not re-establish non-vacuity of the twelve axioms, which remains [Sch26c, Thm. 61]. It does not satisfy H_{ss} , the self-energy $\Sigma(z) = w g_s(z)$ being algebraic and not rational, so [Sch26d, Target 6.26] is inherited unchanged and §5 supplies no instance of switching between forms. It does not exhibit the strictness of Proposition 4.7, nor settle Target 3.11 (Remark A.15). And it exhibits no mixed population, so it says nothing about [Sch26d, Target 4.24].

A.10 Independent numerical verification

The closed form (41) was checked against direct adaptive quadrature of $\int e^{-i\lambda t} \rho_w(\lambda) d\lambda$ at twelve pairs (w, t) with $w \in \{0.25, 0.80, 1.21, 1.50\}$ and $t \in \{0.7, 3.3, 11.0, 47.0\}$; the two agreed to twelve significant figures. At $w = 1$ it was checked against (42) to machine precision for $t \leq 800$. The values of $\|a_w\|_1$ were computed by Gauss quadrature on the panels between consecutive zeros of a_w , with the tail beyond T supplied by (44); the results were stable to five significant figures across $T \in \{50, 100, 200, 400, 800\}$. The value (45) was independently obtained to ten significant figures by 25-digit quadrature over the first 1200 zeros of J_1 , giving 1.607888765. The envelope A_w of (44) was confirmed

against the numerical envelope of $a_w(t)t^{3/2}$ on $t \in [300, 320]$ to six significant figures at $w = 0.25, 0.80, 1.00$ and to five at $w = 1.50$. The distances (46) were obtained by minimizing over a uniform grid of 721 phases. Finally the peak gains $\|G_n\|_{\text{pk}} = 3.000, 3.943, 5.273, 7.148, 9.796, 13.538, 18.828$ of the all-pass family quoted in [Sch26d, Rem. 8.28], which Proposition 4.9 cites, were re-derived here from the closed form $g_n^{\text{ac}}(t) = (-1)^{n+1} 2e^{-t} L_{n-1}^{(1)}(2t)$ and confirmed to the precision quoted.

B Concordance, Chemistry Items, and Defect Ledger

B.1 Scope and method

This appendix records, for every prior statement of the Chrysene Formalism corpus that the present paper touches, whether the statement is *preserved* (used unchanged), *extended* (used together with a further hypothesis, without alteration of its content), *restated* (reformulated here in a manner the reviewer must check is equivalent), *corrected* (found to be false or type-incorrect as written, with the repair given), *not applicable*, or *superseded*. The classification is made against the criterion C of [Sch26b, Def. 2]: a statement of this paper is concordant when it neither contradicts a prior statement of the formalism nor silently weakens a structure a prior statement relies upon. Where a correction is recorded, the entry states what the prior statement asserts, why it fails, and what replaces it; the affected downstream results are named where known. The method is that of [Sch26d, App. B], and the locators below were verified against the sources at the concordance pass.

B.2 Concordance with the twelve-axiom paper

Remark B.1 (The one place where a convention is load-bearing). The causal convention e^{izt} , $\text{Im } z > 0$, of Axiom A_{11} is usually cosmetic. It is not here: it fixes the factor $-i$ rather than $+i$ in (8), and since the impulse response enters Definition 4.1 linearly — unlike (4), where only $\|g^{\text{ac}}\|$ appears — an error of sign would propagate into every distance of §4. The derivation is given in full as Step 2 of the proof of Lemma 3.1 for that reason.

B.3 Concordance with the transfer-geometry paper

Remark B.2 (The three corrections, in full). (a) [Sch26d, Def. 8.11]. The peak gain is there defined as $\|g\|_{L^1}$ and identified with the norm induced $\mathcal{W}_\infty \rightarrow L^\infty$. For non-scalar channels the induced gain is a row-sum functional of the matrix entries of g and is not unitarily invariant,

Prior statement	Disposition	Locus here
A_1 – A_{12}	preserved	throughout; no axiom added, Rem. 2.1
Defs. 1, 3, 5, 6	preserved	status labels, §1.5
A_{10} , Def. 56: J vanishes only under a specified model	preserved	Rem. 2.7
A_{10} , (377): Schur complement	preserved	Lem. 3.1, Step 1
A_{11} : causal convention e^{izt}	preserved	fixes the sign in (8)
Prop. 26: resolvent bound	preserved	Lem. 3.1
A_{12} , (419)–(420)	preserved	§2, Prop. 4.6
Def. 63: constraint boundary	preserved	Rem. 2.10
Prop. 30: d_C^* pseudometric	preserved; analogue proved	Prop. 4.6
Prop. 31: preservation criterion	extended	exact confinement also forces $\ G\ _{\text{pk}} = \infty$; [Sch26d, Prop. 4.22] with Prop. 2.4(iii)
Prop. 39: acyclicity	preserved	Rem. 2.1
Def. 80: prerequisite relation	preserved	Rem. 3.12
Thm. 61; Rem. 62	preserved, not re-established	App. A.1, §A.9
App. B.7: explicit model	superseded as a witness	Prop. A.14: the case $w = 0$
(444): four separations	preserved	Prop. 7.1(x)
Thm. 36	not applicable	no molecular datum enters
Thms. 40–49	not discharged	Tab. 1

Table 3: Concordance with [Sch26c].

so it cannot support an orbit construction. The repair is Definition 2.3, the operator-norm integral. No conclusion of [Sch26d, Thm. 8.13] is affected, because that theorem uses only the upper Young bound, which holds for both; this is verified in Step 3 of the proof of Theorem 5.2.

(b) [Sch26d, Hyp. 2.22]. As written, H_{out} requires $C_j(x) = c_j(y_x)$ for functions c_j of the output signal. With a single c_j independent of the form this is untenable whenever C_j is non-constant on Ω_{adm} and the undisturbed output vanishes, since then $c_j(0)$ would have to equal $C_j(x)$ for every x . The version used in [Sch26d, Thm. 8.13], and instantiated by Definition A.1, is the incremental H'_{out} of [Sch26d, Hyp. 8.12], in which the readout is per-form and only the increment factors through the output channel. Again no conclusion is affected; what is corrected is the statement

of the hypothesis, which any instance schema (Remark 8.3) must read incrementally.

(c) [Sch26d, Rem. 9.6]. That remark sketches the present construction and asserts that applying the orbit construction to the impulse response “yields a quantity dominating δ_u and hence a route-propagating version”. Both clauses are correct — Propositions 4.2(iv) and 4.6. The sketch is incomplete in two respects that this paper had to resolve. It does not record that the resulting distance is defined only on $\mathcal{G}_1 \subseteq \mathcal{G}$ (Proposition 3.9), so that a well-posedness theorem is required before the geometry exists at all; and it leaves the impression that the finer norm yields a finer separation, which Proposition 4.3 refutes. The disposition is *corrected* rather than *extended* because a reader following the sketch would construct an object on the wrong domain.

B.4 Chemistry and domain items

Remark B.3 (The chemistry surface of this paper is narrow). [E] Two items only are offered to the chemistry review of [Sch26b, §4.3], and the narrowness is structural rather than an omission: no molecular datum enters any proof, H_{chem} is not instantiated by the witness (§A.9), and the paper’s only statements of type [E] are Remarks 3.13 and 6.8. The two items are these.

(i) *Band-edge behaviour*. Hypothesis 3.6 asserts square-root vanishing of the local spectral density at the edges of the second fibration’s bands. For a stacked assembly of the kind H_{pop} abstracts, the chemistry review is asked whether a square-root edge is the expected generic behaviour, and whether any structural feature of the substituted chrysene stack would produce a flat band or a band-edge resonance instead — the two circumstances under which, by Proposition A.14 and [Sch26d, Hyp. 4.12], the hypothesis fails and the peak gain is infinite.

(ii) *The amplitude ceiling*. Remark 6.8 reads Corollary 6.4 as a ceiling set by the bandwidth of the second fibration. The chemistry review is asked to confirm only that the reading is not stronger than the mathematics: the ceiling is on what the certificate of Theorem 5.2(ii) can establish, not on what a physical assembly can withstand, and it is conditional in addition on the Identification Hypothesis Φ of [Sch26b]. Nothing further is claimed, and the review is asked not to supply molecular content that no statement here uses.

B.5 Self-identified defect ledger

Remark B.4 (The item the ledger does not close). Item 17. This paper makes the survival claim a single theorem for one form and along a transformation route, and leaves it a conjunction across switching. Items 15 and 16 are a

sharpening and a hypothesis to be weakened; item 17 is the one that limits what the discharge of [Sch26d, Target 8.26] means.

B.6 Bibliographic items

Remark B.5 (Status of the citation inconsistencies of [Sch26d, Rem. B.9]). Four items are recorded there. The first is now superseded: [Sch26c] self-cites as 1(2), 476–547, article 39, matching the listing in [Sch26b], not the 405–476 and article 38 that [Sch26d, Rem. B.9] reports. A different inconsistency remains in its place: [Sch26b] self-cites as 476–488, which overlaps 476–547. The remaining three items of [Sch26d, Rem. B.9] — the citation block of [Sch26b] describing the paper as a survey of Volume 1, Number 2 where Number 1 is meant; the closing paragraph of [Sch26b, §10] terminating mid-sentence; and the document metadata of that file — are outstanding at the time of writing. Separately, [Sch26d, Rem. B.6] locates the conjugated π -framework hypothesis $H_{\partial\pi}$ in “[Sch26c, App.]”; it is Definition 4 of [Sch26c, §2.1], with Remark 1 immediately following, which is explicit that $H_{\partial\pi}$ asserts no ring current, no equal bond order and no measured conductivity. None of these affects any result of this paper.

C Established Mathematical Literature

As in the literature appendices of [Sch26c, Sch26b, Sch26d], none of the subjects below is claimed to be novel. The purpose is to identify the established machinery each part of this paper draws upon, and to keep it distinct from the specific organization the Chrysene Formalism proposes.

C.1 Robust control and the two norms

The small-gain theorem used in Step 2 of the proof of Theorem 5.2, the bounded real lemma referred to in Target 7.5, and the exactness of the stability radius are in [ZDG96]. The peak-to-peak gain of Definition 2.3 and its optimal control theory are the ℓ^1 theory of [DP87], already cited in [Sch26d, App. C.1]; the obstruction noted in Target 7.5, that the peak-to-peak gain has no quadratic storage characterization, is standard there.

C.2 The Wiener algebra and absolutely convergent transforms

The proper inclusion $A(\mathbb{T}) \subseteq C(\mathbb{T})$ used in Proposition 3.9(ii), the inversion theorem for finite measures with integrable transform used in the proof of Theorem 6.2, and the sharpness of Bernstein’s theorem relevant to Target 3.11 are in [Kat04].

Wiener’s lemma and the inverse-closedness of the Wiener algebra, which is the reason (5) is an algebra norm, are there as well.

C.3 Local decay, limiting absorption, and oscillatory integrals

The $O(t^{-3/2})$ decay of Theorem 3.8(ii) is a local-decay estimate of the type developed for Schrödinger operators in [JK79] and, in the threshold-resonance cases, in [JMP84]; the exponent $\frac{3}{2}$ is the one associated with a square-root band edge. The van der Corput and stationary-phase estimates underlying the Claim in that proof are in [Ste93, Ch. VIII].

C.4 Floquet–Bloch theory and van Hove singularities

The direct-integral decomposition, the analyticity of band functions for a finite-range symbol, and the absolute continuity of the spectrum on non-flat bands are standard Floquet–Bloch theory; see [RS78] and [Kuc93]. The classification of the critical points of a band function and the resulting one-sided square-root corrections to the density of states, which is the form (18) records, is due to [vH53].

C.5 Operator-valued measures and Stone’s formula

The spectral theorem, Stone’s formula in the form used in the proof of Proposition 3.4, and the perturbation theory behind [Sch26d, Lem. 3.14] are in [RS80] and [Kat66].

C.6 Uncertainty principles for Fourier pairs

The argument of Theorem 6.2 — that a probability measure supported in an interval of length L cannot have a transform of small L^1 norm — is the elementary member of the family of uncertainty principles surveyed in [HJ94]. The extremal problem of Target 6.7 is of the type treated there, and is not solved by the elementary bound.

C.7 Bessel and Laguerre asymptotics

The Jacobi–Anger expansion, the identity $J_0(x) + J_2(x) = 2J_1(x)/x$, and the large-argument asymptotics used in Appendix A are in [Olv97]; the Laguerre form of the all-pass impulse response verified in §A.10 is standard there.

C.8 What is drawn from the Chrysene Formalism itself

Axioms A_1 – A_{12} , the status vocabulary, the explicit model and the prerequisite relation are [Sch26c]. The composite realization,

the embedded-mode criterion, the unitary-orbit geometry, the switching results, the two halves of the survival claim and the witness are [Sch26d]. The review architecture, the editorial classification and the non-implication conventions are [Sch26b], with the case study in [Sch26a]. None of these is external literature in the sense of this appendix; they are listed here so that the boundary between established machinery and the formalism’s own claims is visible in one place.

References

- [DP87] Munther A. Dahleh and J. Boyd Pearson. ℓ^1 -optimal feedback controllers for MIMO discrete-time systems. *IEEE Transactions on Automatic Control*, 32(4):314–322, 1987.
- [HJ94] Victor Havin and Burglind Jöricke. *The Uncertainty Principle in Harmonic Analysis*. Springer-Verlag, Berlin, Heidelberg, 1994.
- [JK79] Arne Jensen and Tosio Kato. Spectral properties of Schrödinger operators and time-decay of the wave functions. *Duke Mathematical Journal*, 46(3):583–611, 1979.
- [JMP84] Arne Jensen, Éric Mourre, and Peter Perry. Multiple commutator estimates and resolvent smoothness in quantum scattering theory. *Annales de l’Institut Henri Poincaré, Physique Théorique*, 41(2):207–225, 1984.
- [Kat66] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, Heidelberg, 1966.
- [Kat04] Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge University Press, Cambridge, third edition, 2004.
- [Kuc93] Peter Kuchment. *Floquet Theory for Partial Differential Equations*. Birkhäuser, Basel, 1993.
- [Olv97] Frank W. J. Olver. *Asymptotics and Special Functions*. A K Peters, Wellesley, MA, 1997.
- [RS78] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators*. Academic Press, New York, 1978.
- [RS80] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, New York, revised edition, 1980.
- [Sch26a] Charles D. Schaper. AI-augmented peer review by orthogonal constraint: A retrospective case study of mathematical, concordance, chemical, and meta-review. *Annals of the Chrysene Formalism*, 1(2):548–558, 2026. Schaper Systems Press.

- [Sch26b] Charles D. Schaper. The chrysene formalism as candidate mathematics and candidate biogenesis: A quantitative and categorical survey of volume 1, number 1. *Annals of the Chrysene Formalism*, 1(2):476–488, 2026. Schaper Systems Press.
- [Sch26c] Charles D. Schaper. The chrysene formalism revisited: A twelve-axiom foundation for molecular characterization, tensorial noncommutative geometry, and operator response. *Annals of the Chrysene Formalism*, 1(2):476–547, 2026. Schaper Systems Press.
- [Sch26d] Charles D. Schaper. Stable transfer geometry of the substituted chrysene network: h_∞ transformation metrics, dissipative well-posedness, and survival under persistent disturbance. *Annals of the Chrysene Formalism*, 1(2):559–600, 2026. Schaper Systems Press.
- [Ste93] Elias M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, NJ, 1993.
- [vH53] Léon van Hove. The occurrence of singularities in the elastic frequency distribution of a crystal. *Physical Review*, 89(6):1189–1193, 1953.
- [ZDG96] Kemin Zhou, John C. Doyle, and Keith Glover. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, NJ, 1996.

Prior statement	Disposition	Locus here
Defs. 2.3, 2.5, 2.7	preserved	§2
Defs. 3.2, 3.6, 4.1, 4.2	preserved	§2
Lem. 4.4; Prop. 4.5	preserved	Lem. 3.1
Prop. 4.13	extended	Prop. 3.4: the density is $\pi^{-1}M^{-1}\Gamma M^{-*}$
Rem. 4.14	preserved	not discharged, Prop. 7.1(xi)
Thm. 4.17	refined	Thm. 3.8
Props. 4.20, 4.22	preserved	Rem. 6.5
Def. 5.1; Props. 5.2, 5.6, 5.7, 5.14	preserved; analogues proved	§4
Rem. 5.23	extended	does not extend to d_{pk} , Prop. 4.7
Thm. 5.9; Cor. 5.10	preserved, not superseded	Rem. 5.5
Prop. 6.6; Thm. 6.13; Cor. 6.16; Thm. 6.19	preserved, untouched	Prop. 7.1(vii)
Prop. 7.2	preserved	Rem. 2.7
Prop. 7.10	preserved	Prop. 2.4(iv)
Prop. 8.15	preserved	Prop. 4.9
Hyp. 2.22 (H_{out})	corrected	Rem. A.3
Def. 8.11	corrected	Rem. 2.5
Thm. 8.13	preserved	proof of Thm. 5.2, Step 3
Prop. 8.18	extended	a fourth, strictly stronger, condition
Rem. 8.28	preserved	numerics re-derived, §A.10
Target 8.26	discharged	Thm. 5.2, Rem. 5.4
Rem. 9.6	corrected	see Rem. B.2
Tab. 3: register	extended	H_{out} , H_{env} now instantiated
Tab. 6, item 7	partially addressed	§5 now has an instance; [Sch26d, §6] still does not
App. A: witness	preserved and extended	Def. A.1, §A.2
Props. A.4, A.8, A.9, A.11	preserved	App. A

Table 4: Concordance with [Sch26d]. Three rows are disposed *corrected*.

	$\text{crit}(R)$	$\text{rev}(R)$	$\text{ver}(R)$
1	Def. 2.3, θ -dependent case, used $\ J\ $ rather than the Wiener norm of J ; the witness of Prop. 3.9 would then have finite peak gain	fully accepted; corrected	explicit: Prop. 3.9
2	Hyp. 3.6 stated with $h_* \in C^1$; the estimate requires C^2	fully accepted; corrected	explicit: second integration by parts, proof of Thm. 3.8
3	Hyp. 3.6 stated with a two-sided $ \lambda - \lambda_* ^{1/2} h_*$; false at a band edge, where ρ_x vanishes on one side	fully accepted; corrected to (18)	explicit: Prop. A.7
4	Prop. 3.9 asserted that its witness shows H_{edge} cannot be dropped; it satisfies neither H_{pop} nor the hypotheses of Prop. 3.4	fully accepted; claim withdrawn	explicit: Rem. 3.10, Target 3.11
5	provisional values for $\ a_w\ _1$ computed with a truncated cutoff and no $t^{-3/2}$ tail correction; three of five wrong in the third significant figure	fully accepted; closed form derived and values recomputed	explicit: Prop. A.6, §A.10
6	Rem. 6.6 asserted that the closest approach to the floor occurs at the minimizer of the peak gain, which is vacuous since the floor is constant along the family	fully accepted; rephrased	explicit: Rem. 6.6
7	§§5–6 initially had no instantiated example, H_{out} and H_{env} being recorded as not instantiated in [Sch26d, Tab. 3] — the defect reported as item 7 of [Sch26d, Tab. 6]	fully accepted; readout and environment class supplied	explicit: Prop. A.2, Rem. A.4
8	a cross-reference in the proof of Prop. 5.6 pointed at that proposition itself	fully accepted; repaired	explicit
9	(mathematical review) the proof of Prop. 2.4(iii) cited Lem. 3.1, creating a dependency of §2 on §3 and leaving the statement unproved for realizations that are not composite, such as the witness of Prop. 3.9(ii)	fully accepted; the proof now appeals to Def. 2.2 directly	explicit
10	(mathematical review) Def. 2.2 presupposed a feedthrough operator, which [Sch26d, Def. 4.1] supplies only for composite realizations	fully accepted; J defined by (2) and reconciled with [Sch26d, Def. 4.1]	explicit
11	(mathematical review) the second witness in the proof of Prop. 4.9 used a different network for each n , which does not refute a comparison asserted within one network	fully accepted; rebuilt as a single network with Ω_{adm} countably infinite	explicit
12	(mathematical review) Rem. A.12 compared a margin in $d_{\mathcal{H}^\infty}$ with a margin in d_{pk} and reported their ratio; the two are measured in different metrics and the ratio has no meaning	fully accepted; replaced by the fraction of radius consumed and by the critical amplitude	explicit: Rem. A.12, §9
13	(concordance review) the citation-keys remark announced a key shift for monograph volumes that this paper never cites and that do not appear in its bibliography	fully accepted; remark rewritten to record that no monograph is cited	explicit
14	(concordance review) the row of Tab. 3 for [Sch26c, Prop. 31] attributed the extension to Thm. 3.8(i), which assumes H_{pop} ; under exact confinement there is no band structure and the hypothesis is unavailable	fully accepted; attributed instead to [Sch26d, Prop. 4.22] with Prop. 2.4(iii)	explicit
15	the sharp constant of Thm. 6.2 is unknown; $c^* \in [\pi, 6.0270]$	accepted, not repaired	none: Target 6.7
16	Hyp. 2.9 is a genuine restriction on an infinite Ω_{adm} and no result here weakens it	accepted, not repaired	none: Target 7.6
17	Thm. 5.2 does not extend across switching	accepted, not repaired	none: Target 7.5

Table 5: Self-identified defect ledger, in the crit/rev/ver form of [Sch26b, Def. 3]. Items 1–8 were found in drafting, items 9–14 in the review passes of [Sch26b, §4.3]; items 15–17 are recorded as accepted and unrepaired.