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A Finite Axiomatic Framework for the Interior Chrysene Lattice and Protocell Organization

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Abstract

A finite interior chrysene lattice is defined by molecular cells, ordered attachment ports, compatible directional incidence, an explicit boundary, and specified algebraic and dynamical enrichments. We formulate finite counterparts F1–F12 of the twelve-axiom chrysene framework and prove a recognition theorem: a finite directional graph is an induced lattice region precisely when its edge increments integrate to distinct lattice coordinates and its nearest-neighbor incidence is complete. A $3 \times 3 \times 3$ construction gives a nonempty model with an interior cell and nontrivial plaquettes. Partial translations generate a finite matrix algebra; its tensor extension admits an explicit standard form and a separating spectral seminorm. A distinct edge representation identifies a geometric spectral distance with weighted graph distance. We then define functional realizations, typed partial operations and several inequivalent notions of equivalence. Nonisomorphic path and cycle components realize the same ordered-port control and terminal response, and this response equivalence is preserved under passive interconnection. A fixed AAA carrier supplies reference-invariant external addressing, while a fiber-product construction coordinates genetic, functional and membrane states. A concrete finite-stock model proves compatibility and viability on a specified interval. Theorems about these finite structures require no assertion that geometry alone determines chemistry, chirality, quantum coherence or biological function. Molecular implementations are characterized by an additional common-realization condition.

Keywords: finite chrysene lattice; axiomatic characterization; partial operators; functional realization; genetic addressing; membrane dynamics.

1 The objects and the problem

An axiomatic characterization specifies a class of structures before attaching an interpretation to its elements. For the interior chrysene lattice, the relevant structures include a labelled molecular graph, ordered attachment sites, finite directional incidence, tensor fibers, algebras, states and partially defined operations. Their types matter. A displacement is a vector, a covalent bond is a graph edge, a response is an operator depending on constitutive data, and a genetic address is an element of a finite quotient. These objects can be coordinated without being identified.

The present theory concerns finite regions with boundary. The term *interior chrysene lattice* refers to its localization within an enclosing compartment. Separately, a cell belongs to the *graph interior* when all six of its native directional neighbors occur in the region. The graph interior may be empty; our principal box example has a nonempty graph interior. Neither usage removes boundary cells or adds periodic identifications. The region is a finite discrete structure with a geometric realization in $\mathbb{R}^3$; it is not a three-dimensional topological manifold.

Article 40 supplies the native molecular signature and twelve-axiom framework [Sch26d]. Its local molecular assertions remain meaningful on a finite region. Its global translation, crossed-product and infinite spectral assertions require finite replacements. We therefore introduce F1–F12 with their own quantifiers and prove their consistency directly. The letters F distinguish these assertions from the source axioms A1–A12. An extension of a finite object to a source model is an additional mathematical statement; it is not part of the word "finite."

The theory has two principal tasks. First, it characterizes the finite ICL and proves an explicit existence theorem. Second, it describes functional components as realizations of selected structures and operations. A realization may preserve an ordered terminal interface while changing the internal graph. It may reproduce a response without preserving all internal states. These distinctions lead to an equivalence hierarchy and to precise composition theorems.

The genetic extension follows revised Article 75: the resident constitutional carrier is AAA, whereas the external, referenced address varies [Sch26b]. The membrane extension follows the separation of material constitution, atom ancestry, transport and

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finite operating state in revised Article 77 [Sch26a]. Both extensions share explicitly identified coordinates with the functional system. A fiber product expresses this agreement; a compatibility identity decides whether separately defined evolutions induce one evolution on it.

1.1 Order of construction

Section 2 first establishes the directional-graph recognition theorem. Section 3 states the finite axioms, Section 4 proves their consequences, and Section 5 gives a box realization, including its matrix and spectral data. Section 6 distinguishes geometric and algebraic metrics and studies restriction. Sections 7–9 develop functional realizations, transformations and explicit response-equivalent components. Sections 10 and 11 construct the genetic and membrane interfaces and a coordinated nonzero model. The appendices contain exact molecular data, source concordance, additional operator models, material transformations and a more detailed finite-count specialization.

The contribution is the finite typed ICL specification, its simultaneous native-data witness, and its explicit functional, genetic and compartment constructions. Several supporting arguments use standard finite mathematics: integration of graph increments, matrix units, standard-form left and right multiplication, Schur complements, elementary error correction, Lipschitz error propagation, finite-group orbit normalization and uniqueness of ordinary differential equations. They are proved here in the forms needed for the construction; no novelty claim is made for those general methods. The explicit spectral operators and comparison examples specify how those methods apply to this finite structure.

The axioms are a sufficient defining system; no minimality or logical independence is claimed. The finite existence construction proves simultaneous satisfiability. Counterexamples distinguish conclusions that require additional assumptions. Physical constants, energy functions and response laws enter as supplied data whenever used. This is necessary for a mathematical theory in which a theorem has the same meaning in every model of its stated hypotheses.

2 Recognition of a finite lattice region

Let $\mathcal{D} = \{\pm e_1, \pm e_2, \pm e_3\} \subset \mathbb{Z}^3$. A directed edge and its reverse will represent the two orientations of one undirected incidence; neither orientation is an extra bond.

Definition 2.1 (Directional cell graph)**.** A directional cell graph is a finite nonempty connected graph (C, E) without loops or multiple edges, together with a label $s(c, d) \in \mathcal{D}$ on each oriented edge, such that $s(d, c) = -s(c, d)$ and at most one edge leaving a cell has any specified label. The increment of a walk $\gamma = (c_0, \dots, c_m)$ is

$$I(\gamma) = \sum_{j=0}^{m-1} s(c_j, c_{j+1}).$$

A coordinate map is a map $a : C \to \mathbb{Z}^3$ satisfying $a(d) - a(c) = s(c, d)$ on every oriented edge.

Theorem 2.2 (Recognition and uniqueness)**.** *A directional cell graph has a coordinate map if and only if*

$$I(\gamma) = 0 \quad \text{for every closed walk } \gamma. \tag{1}$$

When it exists, the coordinate map is unique up to translation. It is injective if and only if no walk between distinct cells has increment zero. Its image is an induced nearest-neighbor lattice region if and only if, in addition,

$$a(d) - a(c) \in \mathcal{D} \implies \{c, d\} \in E. \tag{2}$$

Thus these three conditions characterize finite connected induced lattice regions up to a choice of origin.

Proof. If a coordinate map exists, each closed-walk increment telescopes to zero. Conversely, fix a root o and assign $a(o) = 0$. For any c, choose a walk from o to c and let $a(c)$ be its increment. Two choices differ by the increment of a closed walk, so (1) makes a well defined. Extending a root walk by one edge gives the coordinate identity. If a' is another coordinate map, $a' - a$ has equal values at adjacent cells, hence is constant by connectedness.
The increment of any walk from c to d is $a(d) - a(c)$. The stated separation condition is therefore equivalent to injectivity. Each existing edge is a nearest-neighbor pair by definition. Condition (2) supplies all such pairs in the image, giving exactly the induced lattice graph. The converse is immediate. □

It suffices to check zero increment on a fundamental cycle basis: choose a spanning tree, integrate labels on its paths, and check every remaining edge against the resulting coordinates. This is a finite recognition procedure. Checking only elementary plaquettes is insufficient on an arbitrary directional graph. An $n \times n$ periodic square graph with $n \geq 3$ has zero increment around each elementary square, but a winding cycle has increment ne_1. Such a graph fails (1) and cannot be treated as an open finite region.

Definition 2.3 (Boundary and interior)**.** For a recognized region $\Lambda = a(C)$, its boundary-incidence set is

$$\partial\Lambda = \{(k, \delta) : k \in \Lambda,\ \delta \in \mathcal{D},\ k + \delta \notin \Lambda\}.$$

Its interior is $\Lambda^\circ = \{k \in \Lambda : k + \mathcal{D} \subseteq \Lambda\}$. A region with nonempty graph interior has $\Lambda^\circ \neq \varnothing$. This is an additional property, not a requirement of every finite ICL. Every boundary incidence retains its axis, sign and missing-neighbor coordinate. Lateral incidences and stacking incidences have different molecular types.

Proposition 2.4 (Boundary count and partial translations)**.** *If $e(\Lambda)$ is the number of undirected cell incidences, then*

$$|\partial\Lambda| = 6|\Lambda| - 2e(\Lambda) \geq 6.$$

For $m \in \mathbb{Z}^3$, translation is the bijection

$$t_m : \Lambda \cap (\Lambda - m) \longrightarrow \Lambda \cap (\Lambda + m), \qquad k \longmapsto k + m.$$

On every domain where the intermediate points occur, $t_m t_n = t_{m+n}$. No nonzero translation acts as a total bijection of a nonempty finite region.

Proof. There are six possible outgoing incidences per cell, and every retained undirected edge accounts for two; this proves the count. For each axis a cell at its minimum has a missing negative incidence and a cell at its maximum has a missing positive incidence. These six typed incidences are distinct, proving the bound. The displayed translation has inverse t_{-m}. Addition gives the composition law on its actual domain. If a nonzero m preserved the finite region, iteration would produce the infinitely many distinct points $k + jm$, a contradiction. □

The equality of partial maps in the proposition does not identify their largest possible domains: the domain of $t_m t_n$ additionally requires the intermediate point $k + n$. This distinction persists for the corresponding partial shift operators.

3 A finite axiomatic structure

The following twelve clauses define a class of finite interior chrysene lattices. The numbering records the correspondence with the revised Article 40 axioms; it does not assert logical independence of the clauses. An object is finite in its cell set, graph data, tensor dimensions and structural configuration set. Its scalar fields remain $\mathbb{R}$ and $\mathbb{C}$. Consequently, its space of density matrices and its continuous-time trajectories need not be finite sets.

The molecular data, finite spatial geometry, tensor algebra, represented spectral geometry, directional Hamiltonians and structural transitions are separately typed parts of one structure. In particular, equality of underlying matrix-algebra dimensions is not an isomorphism of the whole structure.

3.1 Molecular and geometric clauses

Axiom 3.1 (F1: finite embedded molecular reduct)**.** There is a reference molecular reduct $\mathfrak{C} = (V_C, E_C, \iota, \lambda)$ with the following properties. The finite simple graph $G_C = (V_C, E_C)$ is the carbon-skeleton graph of an edge-connected union of exactly four cells of the regular hexagonal tiling. The embedding $\iota : V_C \to \mathbb{R}^3$ is injective. The label maps are

$$\lambda_{\mathrm{loc}} : V_C \to \mathcal{L}_{\mathrm{loc}} \sqcup \{\varnothing\},$$
$$\lambda_{\mathrm{chem}} : V_C \sqcup E_C \to \mathcal{L}_{\mathrm{chem}},$$
$$\lambda_{\mathrm{att}} : V_C \to \mathcal{L}_{\mathrm{att}} \sqcup \{\varnothing\}.$$

Each locant $j \in \{2, 3, 5, 6, 8, 9, 11, 12\}$ labels exactly one vertex v_j. The primary sites $P_C = \{v_3, v_6, v_9, v_{12}\}$ have non-null attachment labels; the distinguished dual sites are $P'_C = \{v_2, v_5, v_8, v_{11}\}$. The chemical carbon edges, directional connectors, and stacking relations are different relation types.

Axiom 3.2 (F2: reference and realized symmetry)**.** The canonical regular-polyhex embedding $\widehat{\iota}_{\mathrm{px}}$ of G_C in $\mathbb{R}^3$ has complete carbon-framework point group conjugate to C_{2h}. Let Γ_C^{core} be its Euclidean stabilizer. For the actual embedding ι, let Γ_C^{emb} preserve the embedded graph and chemical labels, and let Γ_C additionally preserve the primary site set and the attachment data, with their specified orthogonal covariance action. Thus $\Gamma_C \leq \Gamma_C^{\mathrm{emb}}$. Neither the actual embedding nor its attachment decoration is required to have the full canonical point group.

Axiom 3.3 (F3: ordered directional attachments)**.** Put

$$(v_x^-, v_x^+) = (v_3, v_9), \quad (v_y^-, v_y^+) = (v_{12}, v_6),$$
$$d_\alpha = \iota(v_\alpha^+) - \iota(v_\alpha^-).$$

The vectors d_x, d_y are linearly independent. Each primary site has a unit tangent $t_v \in S^2$ satisfying $t_{\varphi_g(v)} = R_g t_v$ for $g \in \Gamma_C$. No element of Γ_C takes the complete ordered x attachment datum $(v_3, v_9; d_x, t_3, t_9)$ to its ordered y counterpart $(v_{12}, v_6; d_y, t_{12}, t_6)$. The same independence and ordered directional inequivalence are required for the canonical reference displacement vectors under Γ_C^{core}. Orthogonality is not required at either level.

Axiom 3.4 (F4: finite directional bridge catalogues)**.** For $\alpha = x, y$ there is a finite nonempty set $\mathcal{N}_\alpha \subseteq \mathbb{Z}_{\geq 0}$ and a map $B_\alpha : \mathcal{N}_\alpha \to \mathbb{B}_\alpha(\mathfrak{C})$. An element of $\mathbb{B}_\alpha(\mathfrak{C})$ is a finite connected embedded graph with two distinct ordered terminal vertices, an injective vertex placement, terminal labels, and a declared compatibility relation with the ordered molecular interface $(\lambda_{\mathrm{att}}(v_\alpha^+), \lambda_{\mathrm{att}}(v_\alpha^-); t_{v_\alpha^+}, t_{v_\alpha^-})$. Write $b_\alpha(N) \in \mathbb{R}^3$ for its terminal displacement. A specified N-independent terminal displacement $c_\alpha \in \mathbb{R}^3$ gives total connector displacement $c_\alpha + b_\alpha(N)$. Compatibility is a structural relation. A chemical realization, if asserted, requires additional chemical data.

Axiom 3.5 (F5: exact directional closure)**.** There are vectors $a_x, a_y \in \mathbb{R}^3$ and distinguished catalogue entries N_x^*, N_y^* such that

$$a_\alpha = d_\alpha + c_\alpha + b_\alpha(N_\alpha^*), \qquad \alpha = x, y.$$

Equivalently, the closure residual $r_\alpha(N) = a_\alpha - d_\alpha - c_\alpha - b_\alpha(N)$ vanishes at N_α^*. This equality is an endpoint equation and asserts no minimal connector length or nearest-neighbor property of the molecular geometry.

Axiom 3.6 (F6: finite open lattice and tensor fibration)**.** There are a finite nonempty set $\Lambda \subset \mathbb{Z}^3$ and a third vector a_z such that $\det A \neq 0$ for $A = (a_x\ a_y\ a_z)$, and the nearest-neighbor graph induced by Λ is connected. The molecular placement

$$r : \Lambda \times V_C \longrightarrow \mathbb{R}^3, \qquad r(k, v) = \iota(v) + Ak$$

is injective. For every $k, k + e_\alpha \in \Lambda$ there is exactly the specified directional relation from (k, v_α^+) to $(k + e_\alpha, v_\alpha^-)$ when $\alpha = x, y$; for $\alpha = z$ there is instead a stacking relation with label $\eta_k \in \mathbb{Z}_2$. There are no wraparound identifications. The boundary-port set is

$$\partial_{\mathrm{p}}\Lambda = \{(k, \alpha, \epsilon) : k \in \Lambda,\ \alpha \in \{x, y, z\},\ \epsilon \in \{-1, 1\},\ k + \epsilon e_\alpha \notin \Lambda\}.$$

An omitted neighbor leaves this typed port; it does not erase the local molecular site or its attachment label. The four x/y ports correspond to v_3, v_9, v_{12}, v_6 in their stated order; the two z ports are stacking interfaces.
The six interface spaces $V_3, V_9, V_{12}, V_6, V_{z^-}, V_{z^+}$ and the internal space K_{int} are nonzero finite-dimensional complex Hilbert spaces. The local tensor Hilbert space is

$$\mathcal{K}_C = V_3 \otimes V_9 \otimes V_{12} \otimes V_6 \otimes V_{z^-} \otimes V_{z^+} \otimes K_{\mathrm{int}}.$$

There is a nonzero distinguished tensor $\Psi_C \in \mathcal{K}_C$ and a copy $\mathcal{K}_k$ at each $k \in \Lambda$. Specified bilinear contractions $V_9 \otimes V_3 \to \mathbb{C}$, $V_6 \otimes V_{12} \to \mathbb{C}$, and $V_{z^+} \otimes V_{z^-} \to \mathbb{C}$ for each registry label contract precisely the internal incidences. The x/y contractions are nonzero. Boundary factors remain uncontracted.
For $m \in \mathbb{Z}^3$, translation is the partial map

$$t_m : D_m \longrightarrow R_m,$$
$$k \longmapsto k + m,$$
$$D_m = \{k \in \Lambda : k + m \in \Lambda\},$$
$$R_m = D_m + m.$$

The registry is transported by $(m \cdot \eta)_{k+m} = \eta_k$ whenever both endpoints of the original and translated stacking edge are retained. Thus translation relates the original and transported decorated substructures; it is a symmetry of fixed decorations only when their labels agree. Composition agrees with addition whenever every required intermediate cell is present.

The adjective interior specifies location inside a chosen enclosure; it does not require a cell with all six neighbors. The graph-theoretic interior is $\Lambda^\circ = \{k \in \Lambda : k \pm e_\alpha \in \Lambda \text{ for all } \alpha\}$. The stronger three-axis nondegenerate subclass has $\mathrm{span}_{\mathbb{R}}(\Lambda - \Lambda) = \mathbb{R}^3$. The explicit box below has both properties, but neither is required of every finite ICL.

The intrinsic version of F6 uses a finite connected directed graph with oppositely labelled reverse edges, labels in $\{\pm e_x, \pm e_y, \pm e_z\}$, zero label sum around every closed walk, no zero-sum path between distinct vertices, and completeness of every available coordinate-neighbor incidence. Its coordinate map is unique up to translation. This is the coordinate-recognition theorem, not an assumption that a finite patch has a global translation action.

3.2 Algebraic and represented clauses

The finite cell-position factor below is a declared representation of the finite base. It is additional to the simultaneous tensor degrees of freedom. This choice makes partial cell shifts elements of a finite operator algebra while retaining the full regional tensor algebra.

Axiom 3.7 (F7: finite tensor and sequence algebra)**.** Fix an ordering of Λ, and put

$$\mathcal{K}_\Lambda = \bigotimes_{k \in \Lambda} \mathcal{K}_k, \qquad \mathfrak{Q}_\Lambda = \mathcal{B}(\mathcal{K}_\Lambda).$$

On $\ell^2(\Lambda)$ let $P_k = |k\rangle\langle k|$ and

$$S_m|k\rangle = \begin{cases} |k + m\rangle, & k + m \in \Lambda, \\ 0, & k + m \notin \Lambda. \end{cases}$$

The finite sequence algebra, with its named generators, is

$$\mathfrak{A}_\Lambda = C^*(P_k, S_{\pm e_x}, S_{\pm e_y}, S_{\pm e_z}) \otimes \mathfrak{Q}_\Lambda.$$

It has a specified positive normalized state ω. For $D \subseteq \Lambda$ the inclusion of a regional observable is $a \mapsto a \otimes I_{\Lambda\setminus D}$. A partial translation gives separately the canonical tensor-factor reindexing isomorphism

$$\begin{aligned} \beta_{m,D} : \mathfrak{Q}_D &\longrightarrow \mathfrak{Q}_{D+m}, \\ a \text{ at } k &\longmapsto a \text{ at } k+m, \\ D, D+m &\subseteq \Lambda. \end{aligned}$$

No global translation automorphism of $\mathfrak{Q}_\Lambda$ is assumed.

Axiom 3.8 (F8: perimeter cipher and represented commutant)**.** On a chosen oriented external perimeter cycle of G_C, the pairs

$$(3,2), \quad (6,5), \quad (9,8), \quad (12,11)$$

are all adjacent with the same orientation. Extending these pairings in both directions gives the cipher involution on the eight distinguished locants. It need not be a molecular graph automorphism. There are nonzero finite-dimensional dual interface Hilbert spaces V_2, V_8, V_{11}, V_5 and paired unitary maps $D_j : V_j \to V_{j-1}$ for $j = 3, 6, 9, 12$. In the ordered tensor factors their map is

$$\mathscr{D}_{\mathrm{dc}} = D_3 \otimes D_9 \otimes D_{12} \otimes D_6 \otimes I_{z^-} \otimes I_{z^+} \otimes I_{K_{\mathrm{int}}}, \qquad \Psi'_C = \mathscr{D}_{\mathrm{dc}} \Psi_C.$$

The first four target factors are ordered $(2, 8, 11, 5)$. The internal factor has a specified decomposition

$$K_{\mathrm{int}} = K_{\mathrm{aux}} \otimes \mathbb{C}^2_{\mathrm{dc}} \otimes \mathbb{C}^2_z, \qquad \Delta_0 = I_{\mathrm{aux}} \otimes X \otimes I_z, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The structural alphabet is $\Sigma_C = \mathbb{Z}_2^{\mathrm{dc}} \times \mathbb{Z}_2^z$. The local registry coordinate and edge registry labels are related only by a specified preparation rule, if one is imposed.
On the GNS Hilbert space $\mathcal{H}_\omega$ there is a conjugate-linear isometric involution J satisfying

$$\begin{aligned} J\mathfrak{M}_{\mathrm{pri}} J &= \mathfrak{M}'_{\mathrm{pri}}, \\ \mathfrak{M}_{\mathrm{pri}} &= \pi_\omega(\mathfrak{A}_\Lambda)'', \\ \mathfrak{M}_{\mathrm{dual}} &:= J\mathfrak{M}_{\mathrm{pri}} J. \end{aligned}$$

The antiunitary J, the linear cipher involution Δ_0, and the geometric interface map $\mathscr{D}_{\mathrm{dc}}$ have different types.

Axiom 3.9 (F9: finite separated spectral realization)**.** The state ω is faithful. There is a self-adjoint operator $D : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that

$$\{a \in \mathfrak{A}_\Lambda : [D, \pi_\omega(a)] = 0\} = \mathbb{C}I.$$

The represented spectral distance is

$$d_D(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a = a^*,\ \|[D, \pi_\omega(a)]\| \le 1\}.$$

Full domain, bounded commutators and compact resolvent hold automatically in finite dimension. The scalar-kernel condition is an explicit finite nondegeneracy requirement added here; it does not follow merely from the compact-resolvent clause of Article 40. No first-order real-spectral-triple condition or equality with Euclidean distance is imposed.

3.3 Hamiltonian, response and admissibility clauses

Axiom 3.10 (F10: finite directional Hamiltonian data)**.** For each $\alpha \in \{x, y\}$ and $N \in \mathcal{N}_\alpha$ there are finite-dimensional Hilbert spaces $\mathcal{H}_{L,\alpha}, \mathcal{H}^{(N)}_{B,\alpha}, \mathcal{H}_{R,\alpha}$ and the self-adjoint block Hamiltonian

$$H^{(N)}_\alpha = \begin{pmatrix} H_{L,\alpha} & (W^{(N)}_{L,\alpha})^* & \mathcal{J}^{(N)}_\alpha \\ W^{(N)}_{L,\alpha} & H^{(N)}_{B,\alpha} & W^{(N)}_{R,\alpha} \\ (\mathcal{J}^{(N)}_\alpha)^* & (W^{(N)}_{R,\alpha})^* & H_{R,\alpha} \end{pmatrix}.$$

The diagonal blocks are self-adjoint, and

$$\begin{aligned} W_L &: \mathcal{H}_L \to \mathcal{H}_B, \\ W_R &: \mathcal{H}_R \to \mathcal{H}_B, \\ \mathcal{J} &: \mathcal{H}_R \to \mathcal{H}_L. \end{aligned}$$

For $z \notin \operatorname{spec}(H_B)$ define

$$\begin{aligned} R_B(z) &= (zI - H_B)^{-1}, \\ \mathcal{T}(z) &= \mathcal{J} + W_L^* R_B(z) W_R : \mathcal{H}_R \to \mathcal{H}_L. \end{aligned}$$

A relation to the structural tensor representation is extra data, rather than an identification of these Hilbert spaces by notation.

Axiom 3.11 (F11: finite causal response)**.** There is a finite-dimensional response Hilbert space, a self-adjoint Hamiltonian H_{resp}, a specified normalized ground-state vector $|0\rangle$, and self-adjoint dipoles μ_x, μ_y, μ_z. With $\hbar > 0$ fixed, put

$$V_E(t) = -\sum_j E_j(t)\mu_j, \qquad \mu_i(t) = e^{iH_{\mathrm{resp}}t/\hbar} \mu_i e^{-iH_{\mathrm{resp}}t/\hbar},$$

$$\alpha_{ij}(z) = \frac{i}{\hbar}\int_0^\infty e^{izt}\langle 0|[\mu_i(t),\mu_j]|0\rangle\, dt, \qquad \operatorname{Im} z > 0.$$

Real-frequency values are asserted only where the corresponding limit exists. The zero-frequency value, a real-frequency boundary value, and an upper-half-plane value are distinct notions. Unequal directional components are not consequences of F3 alone.

Axiom 3.12 (F12: finite structural configurations and admissibility)**.** There is a nonempty finite set $\mathcal{X}$ and an injective map $j_{\mathrm{str}} : \mathcal{X} \to S(\mathfrak{A}_\Lambda)$. There is a symmetric loop-free relation $\mathscr{E}_0 \subseteq \mathcal{X}\times\mathcal{X}$ with symmetric weights $w : \mathscr{E}_0 \to (0,\infty)$. A finite family of functions $C_j : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ and thresholds $\tau_j \in \mathbb{R}$ defines the nonempty admissible domain

$$\Omega = \{x \in \mathcal{X} : C_j(x) < \tau_j \text{ for every } j\}.$$

Let d_w be the infimum of weighted lengths of paths in the induced admissible graph, with $d_w(x,y) = +\infty$ when no such path exists, and $d_w(x,x) = 0$. Independently put

$$d_D^{\mathrm{str}}(x,y) = d_D(j_{\mathrm{str}}(x), j_{\mathrm{str}}(y)).$$

Neither a kinetic law nor equality of these two distances is imposed.

The finite-catalogue, finite-configuration, nonempty-admissibility and spectral-separation requirements are declared choices of this finite theory. They should not be presented as automatic consequences of restricting an arbitrary infinite Article 40 model.

4 Consequences of the finite axioms

Proposition 4.1 (Finite boundary and partial shifts)**.** *Let E_Λ be the unoriented nearest-neighbor edge set of the finite cell graph. Then*

$$|\partial_{\mathrm{p}}\Lambda| = 6|\Lambda| - 2|E_\Lambda| \geq 6.$$

No nonzero $m \in \mathbb{Z}^3$ satisfies $\Lambda + m = \Lambda$. Moreover,

$$S_m^* = S_{-m}, \qquad S_m^* S_m = P_{D_m},$$
$$S_m S_m^* = P_{R_m}, \qquad P_D = \sum_{k\in D} P_k.$$

The product $S_m S_n$ agrees with S_{m+n} precisely on the basis vectors for which the endpoint and intermediate-domain conditions agree; in particular it agrees wherever $k, k+n, k+n+m \in \Lambda$.

Proof. Each cell has six possible oriented incidences; each internal edge uses two. For each signed coordinate direction, an extremal cell has a missing neighbor in that direction, proving the lower bound. If $\Lambda + m = \Lambda$ with $m \neq 0$, choose k maximizing $k\cdot m$; then $k+m$ would have a larger value. The operator identities follow by applying both sides to each basis vector. A missing intermediate cell can make $S_m S_n|k\rangle = 0$ even when $k+m+n \in \Lambda$. □

Proposition 4.2 (The finite generated algebra)**.** *For a connected finite cell graph,*

$$C^*(P_k, S_{\pm e_x}, S_{\pm e_y}, S_{\pm e_z}) = \mathcal{B}(\ell^2\Lambda).$$

Consequently, if $N = |\Lambda|$ and $d = \dim \mathcal{K}_\Lambda$, then $\mathfrak{A}_\Lambda \cong M_{Nd}(\mathbb{C})$, with $\mathfrak{Q}_\Lambda \cong M_d(\mathbb{C})$ as its named tensor subalgebra.

Proof. For neighboring cells $l = k + \epsilon e_\alpha$, $P_l S_{\epsilon e_\alpha} P_k = |l\rangle\langle k|$. Multiplying these matrix units along a connecting path gives $|l\rangle\langle k|$ for every pair of cells. The matrix units span the whole cell-position algebra. Tensoring with $\mathcal{B}(\mathcal{K}_\Lambda)$ gives the claim. □

For a position-dependent observable $F = \sum_k P_k \otimes A(k)$,

$$(S_m \otimes I)F(S_m^* \otimes I) = \sum_{k\in D_m} P_{k+m} \otimes A(k).$$

This is the finite position covariance formula. It is not the assertion that S_m translates every tensor factor of $\mathfrak{Q}_\Lambda$. In particular, abstract isomorphism of the full matrix algebras forgets the named molecular sites, ports, local subalgebras and partial shifts. Those data must remain in a structure-preserving isomorphism.

Proposition 4.3 (Finite standard-form duality)**.** *Let $\mathfrak{A} = M_n(\mathbb{C})$, $\tau(a) = \mathrm{Tr}(a)/n$, and $\mathcal{H} = L^2(\mathfrak{A},\tau)$. Set $L_a(x) = ax$, $R_b(x) = xb$, and $Jx = x^*$. Then J is an antiunitary involution and*

$$JL_aJ = R_{a^*}, \qquad L(\mathfrak{A})' = R(\mathfrak{A}), \qquad JL(\mathfrak{A})J = L(\mathfrak{A})'.$$

Proof. Traciality gives $\|x^*\|_2 = \|x\|_2$, and direct multiplication gives the first identity. Every right multiplication commutes with every left multiplication. Conversely, if T commutes with all L_a, then $T(a) = T(L_a I) = L_a T(I) = aT(I)$, so $T = R_{T(I)}$. □

Proposition 4.4 (A nondegenerate finite spectral operator)**.** *In Proposition 4.3, let $P_I x = \tau(x)I$ and $D = \lambda P_I$ for $\lambda > 0$. Then $D = D^*$, $JD = DJ$, and the F9 scalar-kernel condition holds. For $a = a^*$,*

$$\|[D, L_a]\| = \lambda\sqrt{\tau(a^2) - \tau(a)^2} = \lambda\|a - \tau(a)I\|_{2,\tau}.$$

For ordinary trace-one density matrices ρ, σ,

$$d_D(\varphi_\rho, \varphi_\sigma) = \frac{\sqrt{n}}{\lambda}\|\rho - \sigma\|_{\mathrm{HS}}, \qquad \varphi_\rho(a) = \mathrm{Tr}(\rho a).$$

Equivalently, for τ-normalized densities $r = n\rho$ and $s = n\sigma$, $d_D(\varphi_r, \varphi_s) = \|r - s\|_{2,\tau}/\lambda$.

Proof. Since $\|I\|_{2,\tau} = 1$, P_I is an orthogonal projection. It commutes with J. If $[P_I, L_a] = 0$, evaluation on I gives $\tau(a)I - a = 0$; the reverse implication is immediate. For self-adjoint a, set $a_0 = a - \tau(a)I$. Relative to $\mathcal{H} = \mathbb{C}I \oplus (\mathbb{C}I)^\perp$, the only nonzero blocks of $[P_I, L_a]$ are the rank-one row $x \mapsto \langle a_0, x\rangle I$ and column $I \mapsto -a_0$. Hence its norm is $\|a_0\|_{2,\tau}$. Let $\delta = \rho - \sigma$; it is self-adjoint and traceless. Scalars in a make no contribution. Cauchy–Schwarz gives

$$|\mathrm{Tr}(\delta a_0)| \le \|\delta\|_{\mathrm{HS}}\|a_0\|_{\mathrm{HS}} \le \sqrt{n}\|\delta\|_{\mathrm{HS}}/\lambda.$$

For $\delta \neq 0$, equality is attained at $a_0 = \sqrt{n}\,\delta/(\lambda\|\delta\|_{\mathrm{HS}})$. The case $\delta = 0$ is immediate. □

Remark 4.5 (Meaning and limits of the spectral witness)**.** The matrix D acts on the standard-form Hilbert space of dimension n^2, not on the defining n-dimensional matrix space. The multiplicity is essential: a single self-adjoint D on the defining representation of M_n, $n > 1$, cannot have scalar commutant, since it commutes with itself (and a scalar D commutes with every matrix). The displayed D is a valid finite spectral operator and commutes with J. It is not claimed to obey the real-spectral-triple first-order condition $[[D, L_a], R_b] = 0$. Indeed, with $a = b = Z$ a traceless involution, $[[P_I, L_Z], R_Z](I) = 2I$. The operator establishes finite metric nondegeneracy; it is a chosen spectral decoration rather than a reconstruction of molecular bond lengths.

4.1 A spectral operator constructed from native finite generators

The trace-projection operator gives a closed spectral-distance formula. A second construction attaches the spectral operator directly to the named cell and cipher generators, without asserting that its distance equals the cell path distance.

Let Λ be a connected finite patch with N cells, one internal factor $\mathbb{C}^2_{\mathrm{dc}} \otimes \mathbb{C}^2_z$ per cell, and one-dimensional interface factors. Let $\mathfrak{N}_\Lambda = M_N \otimes M_{4^N} = M_n$. The following matrices are Hermitian:

$$P_k \otimes I \quad (k \in \Lambda), \qquad A_e = (|k\rangle\langle l| + |l\rangle\langle k|) \otimes I \quad (e = \{k, l\} \in E_\Lambda),$$

$$I_N \otimes X_{k,b}, \quad I_N \otimes Z_{k,b} \quad (k \in \Lambda,\ b \in \{\mathrm{dc}, z\}).$$

Here $X_{k,b}, Z_{k,b}$ are the Pauli operators on the indicated local bit, with identity on every other tensor factor. Enumerate this finite family as $G_1, \ldots, G_r$, and choose strictly positive real coefficients $c_1, \ldots, c_r$. They can distinguish axis-labelled edges or be assigned as explicit functions of the connector lengths. On $L^2(M_n, \tau_n)$ define

$$D_{\mathrm{loc}} = \sum_{j=1}^{r} c_j L_{G_j} R_{G_j}.$$

Proposition 4.6 (Native-generator spectral structure)**.** *The operator D_{loc} is self-adjoint, satisfies $JD_{\mathrm{loc}} = D_{\mathrm{loc}}J$ for $Jx = x^*$, and has*

$$[D_{\mathrm{loc}}, L_a] = 0 \quad \Longleftrightarrow \quad a \in \mathbb{C}I.$$

It therefore supplies another F9 realization on the same finite standard-form representation, with a finite separating state metric.

Proof. Every G_j is Hermitian, and its left and right multiplications commute; thus each $L_{G_j}R_{G_j}$ is self-adjoint. Conjugation by J exchanges these two commuting factors, proving the stated reality relation.

The listed matrices are linearly independent. The cell diagonal and cell edge matrices occupy distinct matrix entries. The local Pauli matrices have zero internal trace and are pairwise orthogonal for the Hilbert–Schmidt product, so they are independent of each other and of the cell matrices tensored with identity. This argument does not add a separate identity matrix to the list of all P_k, which would introduce a linear dependence.

The map from $M_n \otimes M_n^{\mathrm{op}}$ to $\mathrm{End}(L^2(M_n, \tau_n))$ given by $a \otimes b \mapsto L_aR_b$ is a linear isomorphism: left and right matrix units send one matrix-unit basis vector to another and span all endomorphisms. Hence

$$0 = [D_{\mathrm{loc}}, L_a] = \sum_j c_j L_{[G_j, a]} R_{G_j}$$

and linear independence of the right factors force $[G_j, a] = 0$ for every j.

The cell projections and edge matrices generate M_N: multiplying $P_kA_eP_l$ produces directed edge matrix units, and connected paths produce all cell matrix units. The local Pauli pairs generate every local M_2 and consequently the full tensor factor. Together the listed G_j generate M_n. Their common commutant within M_n is therefore $\mathbb{C}I$. Conversely, scalar a commutes with D_{loc}. The finite separating-metric conclusion follows from the scalar-kernel criterion. □

The numerical metric now depends on the chosen positive coefficients. This construction does not turn unequal geometric port lengths into a unique spectrum or into a uniquely prescribed optical response. It does show that finite spectral nondegeneracy

can be realized using exactly the named finite cell incidences and local cipher factors, rather than an unrelated matrix list.

Proposition 4.7 (Separation and finite admissible paths)**.** *F9 makes d_D a finite metric on $S(\mathfrak{A}_\Lambda)$. For F12, d_w is an extended metric on Ω; its finite distances are attained by simple paths. On each connected component of the admissible graph it is an ordinary finite metric. The injective map j_{str} makes d_D^{str} a finite metric on all of $\mathcal{X}$. These assertions do not identify d_D^{str} with d_w.*

Proof. On the finite-dimensional quotient of self-adjoint matrices by scalars, $a \mapsto \|[D, \pi(a)]\|$ is a norm by F9. Its unit ball is compact; a difference of states vanishes on scalars, so its supremum on that ball is finite and attained. Distinct states differ on a self-adjoint matrix, which can be rescaled to prove positive distance. Symmetry and the triangle inequality follow from the supremum formula. For d_w, deleting a cycle strictly decreases the weight of a path. Only finitely many simple paths join a fixed pair of vertices, so a reachable minimum is attained. The positive minimum edge weight separates distinct connected vertices. The remaining metric properties follow from path reversal and concatenation. Distinct components have infinite distance by definition. □

Proposition 4.8 (Finite directional resolvent identities)**.** *F10 gives a rational matrix-valued function on $\mathbb{C} \setminus \mathrm{spec}(H_B)$, with*

$$\|\mathcal{T}(z) - \mathcal{J}\| \le \frac{\|W_L\|\,\|W_R\|}{\mathrm{dist}(z, \mathrm{spec}(H_B))}.$$

After ordering the terminal spaces together and the bridge space last, write $H = \left(\begin{smallmatrix} H_T & V^* \\ V & H_B \end{smallmatrix}\right)$. Then the terminal Schur complement is $zI - H_T - V^*(zI - H_B)^{-1}V$; wherever both sides are defined,

$$P_T(zI - H)^{-1}P_T^* = [zI - H_T - V^*(zI - H_B)^{-1}V]^{-1}.$$

Proof. The finite spectral theorem gives the resolvent bound and rationality. Eliminating the bridge variable in the two block equations for $(zI - H)u = f$ gives the displayed terminal equation. Its off-diagonal effective-Hamiltonian block is $\mathcal{J} + W_L^* R_B(z) W_R$. □

Proposition 4.9 (Finite causal response)**.** *The F11 integral converges absolutely and defines a holomorphic function on $\mathrm{Im}\, z > 0$, with*

$$|\alpha_{ij}(z)| \le \frac{2\|\mu_i\|\,\|\mu_j\|}{\hbar\, \mathrm{Im}\, z}.$$

If $H_{\mathrm{resp}}|n\rangle = E_n|n\rangle$ and $\Delta_n = E_n - E_0$, its finite spectral expansion is

$$\alpha_{ij}(z) = \sum_n \left[\frac{\langle 0|\mu_i|n\rangle\langle n|\mu_j|0\rangle}{\Delta_n - \hbar z} + \frac{\langle 0|\mu_j|n\rangle\langle n|\mu_i|0\rangle}{\Delta_n + \hbar z} \right].$$

The expression is evaluated on the upper half-plane; cancellations should be retained if $\Delta_n = 0$. No finite real value at a resonant frequency follows.

Proof. Unitary evolution preserves $\|\mu_i\|$, and the commutator norm is at most $2\|\mu_i\|\|\mu_j\|$. The exponential has absolute value $e^{-t\,\mathrm{Im}\, z}$, proving convergence and the bound. Differentiation on compact subsets is dominated by an integrable polynomial times a decaying exponential, proving holomorphy. Insertion of the finite eigenbasis gives exponentials $e^{\mp i\Delta_n t/\hbar}$; integrating them with the specified Fourier–Laplace convention gives the stated denominators and signs. □

5 An explicit finite realization

5.1 The molecular node and the open box

Use the native eighteen-vertex, twenty-one-edge graph and canonical coordinates displayed in Appendix A. The exact calculation there gives molecular diameter $\sqrt{43}$, complete canonical point group C_{2h}, centroidal covariant tangents, and uniformly oriented primary-to-dual perimeter pairings. Its ordered displacements satisfy

$$d_x = (1/2, -7\sqrt{3}/2, 0), \quad d_y = (7/2, -\sqrt{3}/2, 0),$$
$$\|d_x\|^2 = 37, \quad \|d_y\|^2 = 13, \quad d_x \cdot d_y = 7.$$

Their cross product is $12\sqrt{3}e_z$, proving independence, and their unequal lengths prove directional inequivalence.

Set

$$\Lambda = \{-1, 0, 1\}^3, \qquad a_\alpha = 8e_\alpha \quad (\alpha = x, y, z).$$

For $\alpha = x, y$ choose $\mathcal{N}_\alpha = \{1\}$, $c_\alpha = 0$, and an abstract two-terminal one-edge bridge with displacement $b_\alpha(1) = 8e_\alpha - d_\alpha$. Assign its endpoint labels and a singleton compatibility relation matching its ordered molecular interface. Set every retained stacking registry to zero. The placement is injective because the difference between distinct cell origins has norm at least eight, exceeding the molecular diameter. Exact closure follows directly from the displacement equation.

Proposition 5.1 (Finite box counts)**.** *This finite geometry has* 27 *molecular cells,* 486 *carbon vertices,* 567 *intramolecular carbon edges,* 36 *directional x/y connectors,* 18 *stacking relations and* 54 *external ports. Its cell graph has one*

vertex with all six neighbors, 36 *elementary square plaquettes, and* 8 *elementary cubes.*

Proof. There are $3^3 = 27$ cells and 18 vertices and 21 carbon edges per cell. In each positive coordinate direction there are $(3-1)3^2 = 18$ internal cell edges. Thus the total cell-edge count is 54, of which 36 are directional and 18 are stacking relations. The boundary count is $6 \cdot 27 - 2 \cdot 54 = 54$. Only $(0,0,0)$ has both neighbors on every axis. Each of the three coordinate-plane orientations has $2^2 \cdot 3 = 12$ elementary squares; there are 2^3 elementary cubes. □

5.2 Tensor, algebra and spectral realization

Take every interface factor equal to $\mathbb{C}$, take $K_{\mathrm{aux}} = \mathbb{C}$, and put $K_{\mathrm{int}} = \mathbb{C}^2_{\mathrm{dc}} \otimes \mathbb{C}^2_z$. Let every contraction and every paired interface map be ordinary scalar multiplication or the identity, as appropriate. Let $\Psi_C = |0\rangle_{\mathrm{dc}} \otimes |0\rangle_z$, with scalar interface factors suppressed. The contracted regional tensor is nonzero. Then

$$q = 4^{27}, \qquad \mathfrak{Q}_\Lambda = M_q(\mathbb{C}), \qquad n = 27q, \qquad \mathfrak{A}_\Lambda = M_n(\mathbb{C}).$$

The named local cipher matrices X and Z satisfy $[X, Z] \neq 0$. Choose normalized trace τ_n, its standard-form Hilbert space, $Jx = x^*$ and $D = \lambda P_I$. Propositions 4.3 and 4.4 prove every F8–F9 represented requirement, including nondegenerate finite distance. These identities can be checked symbolically on factors; constructing an n^2 by n^2 dense array is neither necessary nor asserted.

5.3 Nonzero directional response data

Use one-dimensional left, bridge and right spaces for each directional motif. Set $H_L = H_R = 0$, $W_L = W_R = 1$, $\mathcal{J} = 0$, and $H_{B,x} = 1$, $H_{B,y} = 2$. Then

$$H_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\mathcal{T}_x(z) = \frac{1}{z-1}, \quad \mathcal{T}_y(z) = \frac{1}{z-2}.$$

These are chosen finite Hamiltonian data for the respective native ordered interfaces.

For F11 identify the response space explicitly with one local internal factor, with ordered basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle$, and declare units with $\hbar = 1$. Put

$$H_{\mathrm{resp}} = \mathrm{diag}(0,1,2,3), \qquad |0\rangle = |00\rangle,$$
$$\mu_x = |00\rangle\langle 10| + |10\rangle\langle 00|,$$
$$\mu_y = |00\rangle\langle 01| + |01\rangle\langle 00|,$$
$$\mu_z = 0.$$

The finite response formula gives

$$\alpha_{xx}(z) = \frac{2}{1-z^2},$$
$$\alpha_{yy}(z) = \frac{4}{4-z^2},$$
$$\alpha_{zz}(z) = 0, \quad \alpha_{ij}(z) = 0 \quad (i \neq j).$$

In particular $\alpha_{xx}(0) = 2$ and $\alpha_{yy}(0) = 1$ exist and differ. The example proves that the finite axioms admit nonzero anisotropic response. The chosen gaps and dipoles are not derived from the unequal molecular displacement lengths and are not an experimental calibration.

5.4 A nontrivial structural domain

Let $\mathcal{X} = \{0,1\}^\Lambda$ vary the cipher bits and keep all local registry bits and geometric stacking labels zero. For $s \in \mathcal{X}$ let $|s, 0_z\rangle \in \mathcal{K}_\Lambda$ be its computational basis vector, and define the ordinary density matrix

$$\rho_s = \frac{I_{27}}{27} \otimes |s, 0_z\rangle\langle s, 0_z|, \qquad j_{\mathrm{str}}(s)(a) = \mathrm{Tr}(\rho_s a).$$

These states are distinct. Let two configurations be adjacent exactly when one cipher bit differs, assign every edge weight one, and impose

$$C(s) = \sum_{k \in \Lambda} s_k < 2.$$

The admissible graph is a star on 28 vertices: the all-zero configuration and the 27 single-excitation configurations. Its center-to-leaf distance is one and its leaf-to-leaf distance is two. For any distinct configurations,

$$\|\rho_s - \rho_t\|_{\mathrm{HS}} = \sqrt{2/27}, \qquad d_D^{\mathrm{str}}(s,t) = \frac{\sqrt{2q}}{\lambda}.$$

Thus choosing $\lambda = \sqrt{2q}$ gives $d_D^{\mathrm{str}}(s,t) = 1$ for every distinct pair, while the admissible path metric still has leaf-to-leaf distance two. This is an explicit same-algebra failure of metric identification; neither distance is degenerate.

Theorem 5.2 (Existence of a nontrivial finite ICL)**.** *The preceding data satisfy F1–F12. The model has genuine three-axis finite geometry, internal plaquettes, noncommuting local observables, exact commutant duality, finite separating spectral distance, nonzero unequal directional response, and a nonsingleton admissible transition graph.*

Proof. The displayed molecular graph, complete reference stabilizer and ordered port computations give F1–F3 and the geometric part of F8. The declared bridge catalogue, exact endpoint equations, injective placement and finite box incidences give F4–F6. The scalar interface factors and explicit

two-bit internal factor supply the tensor and cipher data. The finite generator theorem and trace state give F7. Standard-form duality gives the remaining F8 requirement, and the trace-projection Dirac calculation gives F9. The displayed self-adjoint motif matrices and their resolvents give F10. The finite response Hamiltonian, dipoles, and convergent causal integral give F11. Finally, the distinct density matrices, one-bit graph, unit edge weights and strict count constraint give F12. Each asserted nontriviality has been established by a displayed calculation above. □

5.5 A finite temporal realization

Proposition 5.3 (Deterministic finite Hamiltonian evolution)**.** *If $H(t)$ is a continuous self-adjoint matrix on a finite-dimensional Hilbert space, the initial-value problem*

$$i\hbar\dot{U}(t) = H(t)U(t), \qquad U(0) = I,$$

has a unique unitary solution on every compact interval on which H is defined. Thus $\rho(t) = U(t)\rho(0)U(t)^*$ remains positive with trace one. If a fixed orthogonal projection P commutes with $H(t)$ for every t, then an initial state supported in P remains supported in P.

Proof. The matrix equation is a finite linear system with continuous coefficients, so existence and uniqueness hold. Differentiating U^*U gives zero by self-adjointness. Positivity and trace are preserved by unitary conjugation. The commutator $UP - PU$ solves a homogeneous linear initial-value problem with zero initial condition, hence is zero. □

For the preceding structural model let P_Ω project onto the span of the all-zero and single-excitation basis vectors in $\mathcal{K}_\Lambda$, and let X_c flip the central cipher factor. The explicitly chosen Hamiltonian

$$H_{\mathrm{adm}} = P_\Omega X_c P_\Omega$$

commutes with P_Ω. With $\hbar = 1$, it evolves the all-zero vector as

$$e^{-itH_{\mathrm{adm}}}|0\rangle = \cos t\,|0\rangle - i\sin t\,|e_c\rangle.$$

At $t = \pi/2$ its density matrix is the central single-excitation state. On the full matrix state space use $U_{\mathrm{full}}(t) = I_{27} \otimes e^{-itH_{\mathrm{adm}}}$ and evolve ρ_0 by its unitary conjugation. Define the enlarged *support-admissible* state set by $(I_{27} \otimes P_\Omega)\rho(I_{27} \otimes P_\Omega) = \rho$. The evolution is nonconstant and preserves this support-admissible set; its sampled endpoints belong to $j_{\mathrm{str}}(\Omega)$ and follow an F12 edge. Intermediate superpositions are not being identified with elements of the finite configuration set $\mathcal{X}$. A transition relation alone would not have determined this Hamiltonian or the duration of the pulse.

6 Geometry restriction and finite metrics

The finite algebraic metric of F9 is a metric on algebraic states. A geometric distance between cells is a different construction. It becomes spectral only after a representation and operator tied to the cell edges have been specified.

6.1 An exact geometric spectral construction

Let (Λ, E) be a finite connected cell graph, and give each undirected edge e a length $w_e > 0$. Let $C_\Lambda = \mathbb{C}^\Lambda$, with pointwise multiplication and complex conjugation. Choose one orientation $e = (u_e, v_e)$ for each edge and put

$$\mathcal{H}_E = \bigoplus_{e\in E} \mathbb{C}^2,$$

$$\pi_E(f) = \bigoplus_{e\in E} \begin{pmatrix} f(u_e) & 0 \\ 0 & f(v_e) \end{pmatrix},$$

$$D_E = \bigoplus_{e\in E} \frac{1}{w_e} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The maximum over an empty edge set is defined to be zero. When the graph has a single vertex, add one scalar summand representing that vertex and put $D_E = 0$. More generally, scalar zero-operator summands represent any isolated vertices if disconnected graphs are allowed. This keeps π_E faithful.

Theorem 6.1 (Geometric spectral distance)**.** *For real-valued f,*

$$L_E(f) = \|[D_E, \pi_E(f)]\| = \max_{\{u,v\}\in E} \frac{|f(u) - f(v)|}{w_{uv}}.$$

On a connected graph its kernel consists exactly of constant functions. The spectral distance between evaluation states is the weighted shortest-path distance:

$$\sup_{L_E(f)\le 1} |f(u) - f(v)| = d_E(u, v).$$

For distinct components the distance is infinite.

Proof. Each commutator block has norm $|f(u_e) - f(v_e)|/w_e$, and a finite direct sum has the largest block norm. The kernel assertion follows by connectedness. Along any path, an admissible f changes by at most the sum of its edge lengths; minimizing over paths gives the upper bound. For a fixed u, the function $f(x) = d_E(u, x)$ changes by at most w_{xy} across every edge, by the triangle inequality. It is admissible and attains the upper bound at v. On distinct components, a function constant separately on each component has seminorm zero and arbitrary difference between the components. □

For a rectangular lattice box with constant edge weight 8, this distance is $8\|u - v\|_1$. It differs from the Euclidean distance $8\|u - v\|_2$ except in special directions. Neither is generally the state distance of the trace-projection operator on $\mathfrak{A}_\Lambda$. The theorem proves one precise metric identification; it does not impose that identification on F9 or F12.

6.2 Restriction and two different algebra maps

For $\Omega \subseteq \Lambda$, retain all cells of Ω and precisely those directional incidences with both ends in Ω. Boundary incidences must be recomputed. If Ω is nonempty and connected, the structural F1–F6 data restrict to another finite ICL skeleton. A cut may destroy connectedness, so arbitrary subsets belong to a larger class of finite molecular regions.

Fix tensor-factor orders and write

$$\mathfrak{A}_\Lambda = \mathfrak{A}_\Omega \otimes \mathfrak{A}_{\Lambda\setminus\Omega}.$$

The molecular-algebra map $q \mapsto q \otimes I$ is an injective unital $*$-homomorphism and preserves normalized traces. Its left inverse

$$E_\Omega = \mathrm{id} \otimes \tau_{\Lambda\setminus\Omega}$$

is a trace-preserving conditional expectation; in general it is not multiplicative. For example, a nonzero traceless self-adjoint matrix b in the removed factor has $E_\Omega(I\otimes b) = 0$ but $E_\Omega(I\otimes b^2) = \tau(b^2)I \neq 0$.

On the cell-position factor let $V : \ell^2(\Omega) \to \ell^2(\Lambda)$ be the coordinate inclusion. Then

$$j_\Omega(A \otimes q) = VAV^* \otimes (q \otimes I) \tag{3}$$

is an injective $*$-homomorphism of the combined algebras, with $j_\Omega(I) = P_\Omega \otimes I$. It is an embedding into the cell-position corner, tensored with the molecular unital inclusion, and is not unital when $\Omega \neq \Lambda$. Compression in the reverse direction is not generally multiplicative: with $u \in \Omega$ and $v \notin \Omega$, both compressed matrix units E_{uv} and E_{vu} vanish, while the compression of $E_{uv}E_{vu} = E_{uu}$ does not.

These maps prevent two common ambiguities. The finite tensor inclusion and the cell-position inclusion have different units, and forgetting a region is not an isomorphism. A chosen F9 operator, F10 Hamiltonian or F11 response law need not restrict compatibly; any such compatibility must be supplied as an intertwining condition. The existence of each finite enrichment can instead be established independently by the finite construction.

6.3 Frame changes and symmetry

A common Euclidean isometry transports all cell placements, terminal vectors, attachment tangents and boundary coordinates. It therefore gives an isomorphic geometric presentation when the transformed data are included. A rigid motion of the whole object is distinct from an internal symmetry fixing the object. In particular, the two native port directions have different lengths and nonzero dot product. A prescribed quarter-turn is a map to a rotated presentation, not an automorphism exchanging those native ordered channels.

The molecular carbon embedding has spatial stabilizer C_{2h}, proved in Appendix A. The stabilizer of an enriched structure is the subgroup preserving its additional decorations. A state, port order, field or asymmetric attachment may reduce it. Conversely, an arbitrary symmetry of an undecorated graph need not preserve its geometric or dynamical enrichment.

7 Functional realizations and equivalence

We now define the structures to be realized. The word "functional" means that both states and actions are specified, not that a physical activity is inferred from a shape.

Definition 7.1 (Finite-interface dynamical system)**.** A finite-interface dynamical system is

$$\mathcal{S} = (X, U, \{f_u\}_{u\in U}, O, o),$$

where X is a state set, U is a finite input alphabet, $f_u : \mathrm{Dom}(f_u) \subseteq X \to X$ are partial maps, O is an observation space and $o : X \to O$. A legal word $u_1 \cdots u_n$ at x is one for which every successive partial map is defined. The empty word acts as the identity. A structural enrichment consists of specified sorts, relations and maps on these states, such as an ordered port set and a map to its geometric configurations.

The state set may be finite, continuous or hybrid. "Finite" in a finite ICL refers to cell and molecular sets and finite-dimensional fibers and algebras, not to the cardinality of $\mathbb{R}$, $\mathbb{C}$ or continuous time. Every operator carries its own domain even when the underlying graph is finite.

Definition 7.2 (Realization and simulation)**.** Let $\mathcal{S}$ and $\mathcal{T} = (Y, U, \{T_u\}, O, o_T)$ have the same input and observation types. An *encoded realization* is an injection $E : X \to Y$ such that

$$E(x) \in \mathrm{Dom}(T_u) \iff x \in \mathrm{Dom}(f_u),$$
$$T_u E(x) = E f_u(x),$$
$$o_T E = o.$$

A decoder D on a neighborhood of $E(X)$ may additionally satisfy $DE = \mathrm{id}_X$. Error correction is an extra condition on D, not part of injectivity.

An *exact quotient simulation* is a surjection $\pi : Y \to X$ such that

$$\mathrm{Dom}(T_u) = \pi^{-1}(\mathrm{Dom}(f_u)), \qquad \pi T_u = f_u \pi, \qquad o_T = o\pi.$$

It may identify many internal target states. A one-sided inclusion of domains with the same intertwining identity gives only a sound simulation of the transitions that exist; it need not lift every source input word.

For an encoded realization the image is invariant under all legal operations. The weaker one-step equation $DT_u E = f_u$ alone would not establish repeated execution: $T_u E(x)$ might leave the encoding image, and the next operator could act differently there.

Theorem 7.3 (Composition of realizations)**.** *Encoded realizations and exact quotient simulations each compose. Their identity maps are identities. An encoded realization preserves legality, every finite action word and its observations. An exact quotient simulation preserves and lifts legality and all finite observed words from any state above the given source state.*

Proof. For encoded maps $E : X \to Y$ and $F : Y \to Z$, domain equivalence applied twice gives the domain equivalence for FE, and $R_u FE = FT_u E = FEf_u$. Observation compatibility also composes. For quotient maps, preimages of domains compose and $\pi\sigma R_u = \pi T_u \sigma = f_u \pi\sigma$. Induction on word length proves the assertions for finite words. At each step the domain equivalence supplies precisely the next legal operation. Identity maps satisfy all conditions. □

7.1 The equivalence hierarchy

The following notions refer to specified data.

1. A *structural isomorphism* is a family of bijections on all chosen sorts preserving and reflecting every specified relation and commuting with every specified map. An isomorphism of finite ICLs must therefore specify whether the algebraic, spectral and response enrichments are included.
2. An *isomorphism of a reduct* applies after designated sorts or relations have been forgotten. It preserves only the remaining signature.
3. A *dynamical conjugacy* is a bijection satisfying the encoded-realization identities in both directions. It compares specified dynamics, not merely state cardinalities.
4. An *input–output equivalence* of two initialized systems means equality of their legal input words and observed output traces. Hidden internal states need not correspond bijectively.

A full structural isomorphism including the dynamics induces a dynamical conjugacy and hence input–output equivalence. An encoded realization gives input–output equivalence to its invariant image. Quotient simulation may discard information and is generally directed rather than symmetric.

Proposition 7.4 (Strictness of the distinctions)**.** *Input–output equivalence does not imply a graph isomorphism or dynamical conjugacy. Equality of a coarse genetic output does not imply equality of addresses. Equality of carbon counts does not imply molecular graph isomorphism.*

Proof. A three-vertex path and a four-cycle have different vertex counts and cycle ranks, yet Section 9 assigns them the same terminal response. More abstractly, replace any state set X by $X \times \{0, 1\}$, let every operation fix the second coordinate, and let observations forget it. Projection is an exact quotient simulation; if X is nonempty and finite, no bijective conjugacy exists. Section 10 gives distinct GCG and GCU addresses with the same Ala output. Finally, a path on four vertices and a cycle on four vertices have equal vertex counts and different edge counts. Labelling every vertex carbon does not alter that obstruction. □

7.2 Approximate realization

Let O be a normed observation space. If F_i are ideal observed maps and $\widetilde{F}_i$ are implemented maps on a common invariant observation domain, assume

$$\|F_i(x) - F_i(y)\| \le L_i \|x - y\|, \qquad \|\widetilde{F}_i(x) - F_i(x)\| \le \varepsilon_i.$$

Then an initial discrepancy e_0 yields

$$e_n \le \Big(\prod_{i=1}^{n} L_i\Big) e_0 + \sum_{i=1}^{n} \Big(\prod_{j=i+1}^{n} L_j\Big) \varepsilon_i. \tag{4}$$

Indeed $e_i \le L_i e_{i-1} + \varepsilon_i$, and induction gives the formula. The common invariant domain is indispensable: an error can otherwise change a guard or make a subsequent map undefined. Exact predicates such as atom identity and ordered-port incidence require exact compatibility, or a separately defined approximate notion; a numerical norm does not silently weaken them.

8 Typed transformations and their invariants

An operator is defined by a domain, codomain and rule. Calling a map an operator does not make it linear, invertible or everywhere defined. The following types permit finite ICL structures and functional components to be transformed without confusing their mathematical objects.

8.1 Structural transformations

Let Reg be the class of finite molecular regions with typed ports and boundary records, and let $\mathsf{ICL}_0 \subset \mathsf{Reg}$ be its finite ICL skeletons satisfying F1–F6.

1. *Restriction* takes $(\mathcal{L}, \Omega)$ to its induced region $\mathcal{L}|_\Omega$. Its codomain is Reg; it lies in ICL_0 exactly when the retained native data and finite-region conditions hold.
2. *Compatible gluing* identifies selected boundary ports of two regions. The domain consists of pairs whose labels, order, placement and connection data agree, whose identified atom labels agree, and whose other atom sets are disjoint. The codomain is initially Reg. It belongs to ICL_0 only if the recognition, incidence-completeness, molecular conditions hold after gluing.
3. *Bond deletion* acts on a specified molecular edge set. It generally leaves ICL_0 if it deletes a required native carbon edge. A relation-edge deletion and a covalent-bond deletion are distinct maps.
4. *Assembly* maps a finite collection of material graphs and an explicit identification or bond-addition prescription to a new graph. The map includes every affected element, bond-order and attachment label. A graph-theoretic assembly is a chemical preparation only when an admissible molecular path is also supplied.
5. *Release* forgets designated noncovalent constraints or deletes specified attachment edges. Its output retains all remaining atoms and bonds. A release map alone specifies no folding trajectory.

Proposition 8.1 (Gluing criterion)**.** *Suppose two recognized finite regions use one common molecular and tensor reference and translation matrix, with coordinate maps agreeing on common cells. Assume the union is nonempty and connected, has distinct coordinates for distinct unshared cells, has all coordinate-neighbor incidences, and has an injective combined molecular placement. If the native decorations agree on overlaps and compatible bridge, stacking and interface-contraction data are supplied for every incidence, including every new cross-piece incidence, their glued object is a finite ICL skeleton.*

Proof. The coordinate maps define one injective map on the identified union. The edge labels are its increments, so all periods vanish; the hypotheses supply exactly the neighboring edges. The recognition theorem applies. Molecular and bridge conditions hold on either piece and agree on overlaps. The stated connectedness completes the structural requirements. □

Gluing is not automatically closed in ICL_0. An inconsistent coordinate identification can create a nonzero-period cycle; overlapping placements can identify different atoms; and two boundary cells can become coordinate neighbors without the required connector. Each is an explicit obstruction.

8.2 Operations on a common state

A finite material state can be represented by a fixed atom universe V, immutable element and origin maps, a current bond-order map, allocation or compartment labels, and additional control and continuous coordinates. A transformation may change current bonds without changing atom identity. This representation distinguishes survival of a molecule from ancestry of its atoms.

Definition 8.2 (Admissible partial operation)**.** An admissible operation $T : \mathrm{Dom}(T) \subseteq X \to Y$ has a specified guard defining its domain, an explicit update rule, declared coordinates that it may change, and stated invariants. When composition is considered on a common state space, its complete read coordinates include every coordinate used by its guard and by its update rule. Conserved quantities are functions c for which $c(Tx) = c(x)$ on the domain.

Theorem 8.3 (Composition and noninterference)**.** *Let T_i be a finite sequence of partial operations on one common state description. Suppose each operation preserves a predicate P and conserved quantity c, and each intermediate output belongs to the domain of the next operation. Their composite preserves P and c. If a finite nonnegative inventory r changes by $r \mapsto r - \nu_i(x)$, with $\nu_i(x) \geq 0$ and guard $r \geq \nu_i(x)$ componentwise, then all intermediate inventories remain nonnegative.*
Two operations commute on a shared domain if their changed coordinates are disjoint and each preserves every coordinate and predicate read by the other's guard and update. Under these conditions the two orderings have the same domain there and the same result.

Proof. The first assertions follow by induction through the stated domains. For the last assertion, either operation leaves the other guard true and leaves its update values unchanged. Disjoint changed coordinates therefore receive the same final values in either ordering, and all other coordinates are fixed. □

Disjoint changed coordinates alone do not imply commutation: changing an enabling flag can make another operation undefined even if that operation changes only a different coordinate. Likewise, two operations drawing from a shared finite substrate require the shared amount to be included in their read and changed coordinates.

8.3 Conservative extension and finite construction

Definition 8.4 (Conservative extension)**.** A system $\mathcal{S}'$ extends $\mathcal{S}$ conservatively on a domain Y_0 when Y_0 is invariant under the lifted old operations and there is a surjective projection $p : Y_0 \to X$ such that old operations remain defined exactly on the preimages of their old domains and satisfy

$$pT'_u = f_u p, \qquad o' = op.$$

Any new operation intended to leave the old system unchanged must satisfy $pT'_v = p$ on its domain. Shared concentrations, boundary conditions or material amounts are part of p whenever old operations read them.

Proposition 8.5 (Finite extension)**.** *Suppose each new component has disjoint atom support outside a fixed preserved region, every old operation is extended by the displayed projection identity, and all shared-variable domain conditions hold. Then adding finitely many components successively gives a conservative extension for every old legal word. If each addition uses at least one fresh atom from a fixed finite eligible set, only finitely many additions are possible.*

Proof. Composition of the projection identities proves preservation for each old word. Induction over the number of additions gives the finite assertion. Freshness makes the used atom sets pairwise disjoint; a finite set has only finitely many nonempty disjoint subsets. □

The definition admits nontrivial new behavior without changing the old interpretation. It does not assert that every extension is conservative: altered membrane permeability or shared substrate depletion may change the domain of an old operation. Such a change must be represented explicitly.

8.4 Mechanical constraints and prescribed geometry

Let q belong to a finite-dimensional displacement space with rigid-body motions removed. A mechanical enrichment supplies a symmetric stiffness matrix $K(q_0)$, loads and boundary constraints. If a continuous replacement parameter $u \in [0, 1]$ satisfies

$$v^T K(u)v \geq k_*\|v\|^2 \quad (k_* > 0)$$

on that constrained space, then every equilibrium $K(u)q = F$ obeys $\|q\| \leq \|F\|/k_*$. Indeed Cauchy–Schwarz gives $k_*\|q\|^2 \leq q^T F \leq \|q\|\|F\|$. This is a constitutive matrix condition. Graph connectivity alone does not prove it.

Pinning restricts configurations to a specified nonempty constraint set, and a relaxation operator additionally requires a chosen energy and evolution law. A rotation transports positions and attached vectors together; an imposed rotation need not be a symmetry of the unrotated object. These operator types, together with electronic field response and material conversion, are developed in Appendix C.

8.5 Finite loom and geometric templates

A finite loom structure consists of an ordered template-site set, a specified attachment map from selected ICL ports to those sites, a current product graph, and partial incorporation, pinning and release maps. Its state includes the occupied sites and current reading position. A finite assembly is a legal composition of these maps. The attachment map must satisfy the declared connector and enclosure constraints; the product graph must satisfy its material invariants. The composition theorem preserves those conditions when each map preserves them and each output belongs to the next domain. The following helical and sheet placements provide explicit choices for the template geometry, while the addressed sequence construction in Section 10 supplies a possible control order.

Definition 8.6 (Finite helical template)**.** Choose an integer $N \geq 2$, radius $r > 0$, axial increment $h \neq 0$, and angle $0 < |\theta| < 2\pi$. A finite helical template is the ordered placement

$$q_k = (r\cos(k\theta), r\sin(k\theta), kh), \qquad 0 \leq k < N,$$

together with declared attachment labels at these sites. A realization in an ICL consists of admissible attachment maps to these positions, in one chosen common frame, with any required connector lengths and enclosure conditions.

Proposition 8.7 (Exact helical geometry)**.** *The sites are distinct and are successive iterates of the screw isometry $S(q) = R_z(\theta)q + he_z$. Their pair distances satisfy*

$$\|q_k - q_l\|^2 = 4r^2 \sin^2\left(\frac{(k-l)\theta}{2}\right) + (k-l)^2h^2.$$

The associated continuous helical interpolation has signed pitch $2\pi h/\theta$. After centering the axial coordinate, the finite template lies in a ball of radius $\sqrt{r^2 + ((N-1)h/2)^2}$.

Proof. The nonzero axial increment separates distinct indices. Rotation by θ and translation by h give the successor identity. The planar chord has squared length $4r^2\sin^2((k-l)\theta/2)$, perpendicular to the axial difference $(k-l)h$. A full turn of the continuous interpolation changes the axial coordinate by $2\pi h/\theta$. Centering its endpoint heights bounds the axial coordinate by $|(N-1)h|/2$, proving the enclosure bound. □

These are geometric equalities for supplied parameters. They do not select an amino-acid sequence, hydrogen-bond pattern, handedness-selection process, or stable α-helical fold. In particular a discrete sample need not contain two sites separated by exactly one full turn.

Definition 8.8 (Finite sheet template)**.** Let $a, b \in \mathbb{R}^3$ be linearly independent and let m, n be positive integers. A finite sheet template has sites

$$\begin{aligned} q_{ij} &= q_0 + ia + jb, \\ 0 &\le i < m, \\ 0 &\le j < n, \\ \eta_{ij} &= (-1)^{i+j}. \end{aligned}$$

It records the two ordered nearest-neighbor directions and the alternating labels η_{ij}. The vectors need not be orthogonal.

Linear independence makes the site map injective. Adjacent distances in the two directions are $\|a\|$ and $\|b\|$, and a diagonal displacement has squared length $\|a\|^2 + \|b\|^2 + 2a \cdot b$. Thus the construction can retain the actual nonorthogonal ICL directions. An alternating label is structural data, not a proof of a peptide β-sheet. A peptide interpretation must add residue identities, backbone orientation, admissible torsions, interactions, and post-release stability.

8.6 Hinge and rotor realizations

A hinge realization consists of two specified rigid components, an attachment axis and pivot, and a relative angular coordinate ϑ in a declared interval. A rotor uses $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$, together with any additional occupancy or drive-state coordinates. Rotations about the specified axis define their configuration maps. Allowed angular motion does not by itself provide a torque, direction of motion, reset, or useful work.

For example, a separately supplied mechanical realization can have

$$I\ddot{\vartheta} + c\dot{\vartheta} + V'(\vartheta) = \tau_{\text{drive}} - \tau_{\text{load}}, \qquad I > 0, \quad c \ge 0.$$

Here a periodic potential is used for a rotor. Along every differentiable solution,

$$\frac{d}{dt}\left(\frac{I}{2}\dot{\vartheta}^2 + V(\vartheta)\right) = \tau_{\text{drive}}\dot{\vartheta} - \tau_{\text{load}}\dot{\vartheta} - c\dot{\vartheta}^2.$$

The identity follows by multiplying the equation of motion by $\dot{\vartheta}$. It gives the exact energy balance for $W_{\text{out}} = \int \tau_{\text{load}}\, d\vartheta$ and separates geometric rotation from work production. A chemical drive further needs its own reaction states, source of free energy, coupling law, and recovery mechanism.

8.7 Catalytic pockets and reaction-network realizations

A pocket can be specified by a finite set of labelled boundary sites, a configuration-space region, and an accessibility/binding predicate for a named substrate. Its geometric description is independent of a turnover law. For a positive formal catalytic realization, specify a finite species set, a stoichiometric matrix N, nonnegative reaction rates $v(x)$, and

$$\dot{x} = Nv(x)$$

on a declared nonnegative invariant domain. Atom- or element-composition rows ℓ with $\ell N = 0$ are conserved, since $d(\ell x)/dt = \ell N v(x) = 0$. Current atom-resolved realization additionally requires the origin maps of the individual reactions.

For example, the explicitly supplied cycle

$$C + S \;\rightleftharpoons\; CS \;\longrightarrow\; C + P$$

has recovered free catalyst in its productive step and conserved total catalyst $x_C + x_{CS}$. With rates $v_1 = k_1 x_C x_S$, $v_{-1} = k_{-1} x_{CS}$ and $v_2 = k_2 x_{CS}$, where all rate constants are positive, the nonnegative orthant is invariant and $\dot{x}_P = k_2 x_{CS}$. At every state with $x_{CS} > 0$ the product flux is strictly positive. These conclusions follow from the specified reaction network and rate law. Identifying the symbol C with an ICL pocket requires an additional common molecular realization; pocket shape alone proves none of the rate constants, selectivity, or catalytic recovery conditions.

9 Explicit functional component realizations

The following construction proves that selected native structures and nontrivial actions can have distinct internal realizations. It does not identify a path or cycle with a complete chrysene molecule.

9.1 Ordered ports and site graphs

At one interior cell select the ordered ports

$$(p_x^-, p_x^+, p_y^-, p_y^+) = (r_3, r_9, r_{12}, r_6).$$

The structural reduct records these four positions, their order, the displacement vectors d_x, d_y, the reference frame and a channel connecting each terminal pair. It forgets the other molecular vertices.

For each channel, representation P is a path $0 - 1 - 2$ with terminals $0, 2$; representation L is a cycle $0 - 1 - 2 - 3 - 0$ with opposite terminals $0, 2$. Place the path midpoint halfway between the corresponding ports. Place the two cycle interior vertices at that midpoint plus and minus a sufficiently small

nonzero in-plane perpendicular vector. Different blocks may occupy different parallel planes. An observation subtracting each declared plane offset recovers the original ordered ports and displacement. Because only finitely many sites are involved, offsets can be selected to avoid all coincidences. This construction is covariant under a common frame change.

For two channels the material graph may join their roots by an additional edge. That edge is not included in the response network specified below. The material graph and the constitutive response network are separate pieces of data.

9.2 Encoding an actual operation

Let $X = \{0,1\}^2$, with state (m, r), and define

$$u(m, r) = (m, r \oplus m).$$

For one bit put

$$E_P(b) = (b, b, b), \qquad E_L(b) = (b, b, 1-b, 1-b).$$

Apply these encodings blockwise. The decoder for P is majority. For L, first complement the last two sites and then take strict majority, rejecting ties. Let Y_j be the union of Hamming balls of radius one around the two codewords in each block, with the two-block product convention. Define

$$T_j = E_j u D_j : Y_j \longrightarrow E_j(X) \subseteq Y_j.$$

For the dynamical comparison, take $O = X$, $o = \mathrm{id}_X$ and $o_j = D_j$. The fixed ordered-port geometry is a common structural enrichment of every state. The terminal-response observations below are functions of this decoded observation.

Theorem 9.1 (Selected structural and dynamical realization)**.** *For $j = P, L$, $D_j E_j = \mathrm{id}_X$, and decoding corrects one site error in each block. The maps E_j are encoded realizations of u, while $D_j : Y_j \to X$ are exact quotient simulations of T_j onto u. Both preserve the specified ordered-port reduct. The action is nonidentity.*

Proof. The two three-bit codewords differ in all three positions, so their radius-one balls are disjoint and majority is correct. After the specified complementation, the four-bit words are constant words, at distance four; a single error leaves at least three correct sites. Thus $D_j E_j = \mathrm{id}$, and both decoders are defined on all of Y_j. Consequently $T_j E_j = E_j u$ and $D_j T_j = u D_j$. All domains are the full declared state sets. The geometric observation is independent of the register update. Finally $u(1,0) = (1,1)$. □

9.3 Terminal response and contextual equivalence

Let the decoded bit of a channel be b and write $g = 1 + b > 0$. Give each edge of P conductance $2g$ and each edge of L conductance g. Kirchhoff's law places every interior vertex at the mean terminal potential. The path has one series branch with conductance g; the cycle has two branches each with conductance $g/2$. Thus both give

$$I = g(V_+ - V_-), \qquad K_B = g\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5}$$

For the two channels the response is the direct sum with $g_x = 1 + m$ and $g_y = 1 + r$. These values distinguish the control states. They are supplied conductance laws, rather than consequences of carbon adjacency.

More generally, let a passive finite network have boundary vertices B, interior vertices I and weighted Laplacian

$$K = \begin{pmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{pmatrix}.$$

Assume every connected component meets B. Then K_{II} is positive definite: its quadratic form is the sum of edge energies after boundary values are set to zero, and a zero value would force a constant zero value on each component. Eliminating the interior potentials gives the Dirichlet-to-Neumann map

$$\Lambda_B = K_{BB} - K_{BI} K_{II}^{-1} K_{IB}. \tag{6}$$

Theorem 9.2 (Contextual terminal equivalence)**.** *Suppose two passive networks have the same ordered boundary vertices and the same map Λ_B. Attach either network to any one common passive context along those vertices, without changing its interior vertices. Whenever the resulting Dirichlet problem has a unique interior solution, the two assemblies have the same external terminal response.*

Proof. Eliminate the module's interior vertices first. Equation (6) replaces either module by the same boundary current law. The context contributes identical edge energies and currents in either case. The reduced equations on all remaining vertices are therefore identical. Uniqueness gives identical solutions and external currents. Equivalently, eliminating the remaining variables is the same Schur-complement operation applied to equal reduced matrices. □

This theorem is stronger than equality for one chosen voltage. It permits replacement in every context of the stated class. It remains weaker than graph isomorphism, equality of internal dissipation at every individual edge, or equality of arbitrary dynamical responses. Time-dependent capacitive or quantum behavior would require its own boundary operator and comparison domain.

9.4 Structural information that does not survive

The path and cycle have different cycle ranks; the internal state encodings have different lengths. Their equivalence retains only

the ordered structural reduct, decoded action and terminal law. A realization of a peptide or lipid component would likewise need a specific covalent graph, conformation domain, attachment map, observation and response. The existence of these finite site-graph realizations alone does not supply those additional structures.

10 A genetic interface on the fixed AAA carrier

The genetic structure is an additional typed structure over the ICL. Its reference action is not the local dual-cipher operator of F8 or the standard-form involution J. Revised Article 75 fixes the resident carrier type to AAA while permitting a variable external address [Sch26b, fixed-carrier architecture and addressed registry].

Let $M = \mathbb{F}_2^6$. Define $F : M \to M$ by complementing all bits and $R : M \to M$ by reversing their order. They are commuting involutions. With complement-reference tag $a \in \mathbb{F}_2$ and direction-reference tag $d \in \mathbb{F}_2$, put

$$\Omega = M \times \mathbb{F}_2^2 \times \{AAA\}.$$

The group $G = \langle F, R \rangle$ acts by

$$F(u, a, d, t) = (Fu, a+1, d, t), \qquad R(u, a, d, t) = (Ru, a, d+1, t).$$

The resident type t is fixed by both actions. Pair consecutive bits using

$$E(00) = A, \quad E(01) = C, \quad E(10) = G, \quad E(11) = U.$$

Theorem 10.1 (Normal form and genetic action)**.** *Every addressed orbit has one representative* $(n, 0, 0, AAA)$ *with* $n = F^a R^d u$*. Consequently*

$$\Omega/G \simeq \mathbb{F}_2^6 \simeq \{A, C, G, U\}^3.$$

Any map $h : \mathbb{F}_2^6 \to U$ *into the input alphabet of a functional system defines a reference-invariant dispatch* $[u, a, d, AAA] \mapsto h(F^a R^d u)$*. Its associated partial action is well defined precisely on the states in the domain of the dispatched operation.*

Proof. Applying $F^a R^d$ makes both tags zero and gives the stated word. An element of G preserving both zero tags is the identity, proving uniqueness. The formula is therefore constant on each orbit. Applying a well-defined dispatch to a specified family of partial maps gives an action only on the indicated operation domains. □

There are 64 fully referenced addresses. If only the complement tag is retained, normalization leaves the residual reversal action on six bits and gives $(64 + 8)/2 = 36$ classes. If both tags are omitted, the four group elements fix respectively $64, 0, 8, 8$ words, giving 20 classes. These are quotients of different retained data. Their cardinalities do not identify the unreferenced classes with the 20 biological amino acids.

The transported reversal of a three-letter word is not ordinary word reversal. Reversing six bits reverses pair order and interchanges the two bits of each pair; hence

$$ER = \tau \text{ rev } E,$$

where τ exchanges C and G and fixes A, U. This identity is a direct test of any claimed implementation of the reference convention.

10.1 An explicit dispatch and its retained information

On the selected address domain, put

$$M_0 = \{100110, 100111\},$$
$$h_{\text{init}} : M_0 \longrightarrow X = \{0, 1\}^2,$$
$$h_{\text{init}}(n) = (1 - n_6, 1).$$

The normalized words encode GCG and GCU and initialize $(1, 1)$ and $(0, 1)$ respectively. This state-valued initialization map has a different type from the operation selector h in Theorem 10.1. To express initialization as an action, adjoin input symbols prep_x for $x \in h_{\text{init}}(M_0)$, with total maps

$$f_{\text{prep}_x}(y) = x \quad (y \in X),$$
$$T_{j,\text{prep}_x}(z) = E_j(x) \quad (z \in Y_j), \quad j = P, L.$$

Then $T_{j,\text{prep}_x} E_j = E_j f_{\text{prep}_x}$ and $D_j T_{j,\text{prep}_x} = f_{\text{prep}_x} D_j$, so the same component encodings and decoders realize these preparation actions. The operation selector is now $h_{\text{op}}(n) = \text{prep}_{h_{\text{init}}(n)}$ on M_0. The normal-form argument makes this a reference-invariant partial dispatch, with address domain exactly the orbits whose normalized word lies in M_0. No preparation is assigned to other addresses. The subsequent operation $u(m, r) = (m, r \oplus m)$ acts on the initialized register; it is not the initialization map. All of these maps retain the fixed resident AAA tag.

The two external codons have the same standard Ala output. Therefore their distinction cannot be reconstructed from that output alone. More generally, if $q : M_0 \to Q$ forgets information, the initialization h_{init} factors as $\bar{h}q$ if and only if it is constant on each fiber of q. Necessity follows from substitution; sufficiency follows by defining $\bar{h}(q(n)) = h_{\text{init}}(n)$. This elementary factorization criterion identifies exactly when a proposed coarse genetic state retains enough information for an intended action.

Revised Article 75 gives a stronger lower bound for retaining both its complete registry and its designated standard output map: the required common refinement has 61 states [Sch26b, minimal registry-preserving refinement]. That bound concerns the joint retained maps. It is neither a lower bound on the number of amino-acid species nor a claim that the two-bit component implements complete translation.

10.2 Sequence operations as a further extension

A sequence-controlled assembly requires a state $(c_1 \cdots c_n, j, \Gamma)$, a reading position j, current product graph Γ, and a partial map for each legally selected incorporation. A joining map has domain consisting of a current terminal group, an eligible incoming residue, an atom map and the required noncarbon constituents; its codomain includes the joined graph and every coproduct. Incrementing j is a separate control update. Start, STOP, release and invalid-address behavior are additional partial maps.

If Q_c is a family of supplied residue-selection maps and J is a supplied joining map, the repeated construction is

$$(c_1 \cdots c_n, j, \Gamma) \longmapsto (c_1 \cdots c_n, j+1, J(\Gamma, Q_{c_j}(x))),$$

only where both component domains and the atom supports are compatible. The composition theorem then gives a finite sequence realization. The existence of the address quotient does not itself supply Q_c, J, a chemical rate, or an origin of the biological code.

11 Coordinated membrane and genetic structure

A compartment introduces localization, transport and a common physical state. These cannot be inferred from the finite cell graph. We first define the coupling and then give a nonempty example in which its conditions hold simultaneously.

11.1 The shared-state construction

Let X_L, X_G, X_F, X_M be lattice, genetic, functional and membrane state sets, with maps $q_i : X_i \to Z$ to a common description of shared variables. The shared variables include every coordinate read by more than one subsystem: named ports, control outputs, material amounts, placement, accessible volume or boundary conditions as appropriate. Define

$$X = X_L \times_Z X_G \times_Z X_F \times_Z X_M = \{(x_i) : q_i(x_i) = q_j(x_j) \text{ for all } i, j\}. \tag{7}$$

This set may be empty unless the data are compatible.

Proposition 11.1 (Discrete compatibility)**.** *Partial subsystem updates T_i define an update on X if and only if, for every point in their proposed common domain,*

$$q_i(T_i x_i) = q_j(T_j x_j) \quad \text{for all } i, j.$$

On that domain the product update is unique. If additive subsystem invariants include each shared material contribution exactly once and their interface fluxes cancel, the corresponding total is conserved.

Proof. The displayed condition is exactly membership of the product output in (7). The component rules determine that output uniquely. Adding their invariant identities cancels the stated internal fluxes; no shared term is duplicated. □

For continuous coupling, a sufficient condition avoids any unproved regularity of a fiber product. Suppose X_i and Z are open subsets of finite-dimensional Euclidean spaces, q_i are C^1, and locally Lipschitz vector fields v_i and w satisfy

$$Dq_i(x_i)v_i(x_i) = w(q_i(x_i)). \tag{8}$$

If subsystem solutions start with equal q_i values, their projections solve the same initial-value problem $\dot{z} = w(z)$. Uniqueness makes these projections equal for as long as all solutions remain defined. Thus the product trajectory stays in X. This proves compatibility on the common existence interval; it asserts neither unlimited continuation nor smoothness of an arbitrary fiber product. A vector field depending on several subsystem states instead needs one explicitly defined common state and its own existence argument, as in the example below.

11.2 Amounts transport and enclosure

A membrane state includes total enclosure volume V, occupied volume U, accessible water volume $v = V - U > 0$, membrane area A, internal amounts n, external amounts b and a common parameter set. For a stoichiometric matrix S and boundary fluxes, use

$$\dot{n} = S\, r(n, b, v) + j_{\text{in}} - j_{\text{out}}. \tag{9}$$

If $cS = 0$ for a row c of constituent counts, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(cn) = c\, j_{\text{in}} - c\, j_{\text{out}}.$$

Matching external withdrawal and cumulative export restore a closed total. An atom-mapped refinement is a separate requirement: a real-valued amount is not a fractional labelled atom. Concentration coordinates $x_i = n_i/v$ satisfy $\dot{x}_i = \dot{n}_i/v - x_i\dot{v}/v$.

One optional salt-equilibrium and water enrichment, used in revised Article 77, is

$$n_+ - n_- = Q,$$
$$n_+ n_- = c_s^2 v^2,$$
$$\dot{v} = \kappa A \left[\frac{N}{v} + \sqrt{(Q/v)^2 + 4c_s^2} - 2c_s - \omega \right].$$

Here N counts the declared independently translating nonpermeant particles, Q the fixed negative charge magnitude and $c_s > 0$ the bath salt concentration. The ionic term is nonnegative and is at most $Q^2/(4c_s v^2)$ by rationalization. Immobilized constituent atoms are not counted as independent ideal solute particles. These are supplied transport and constitutive laws, not finite-lattice axioms.

11.3 A jointly realized nonzero model

Fix the 27-cell ICL of Theorem 5.2, its protected native current structure, one AAA carrier and either the P or L component. Use the preparation actions of Section 10: GCG initializes $(m, r) = (1, 1)$ and GCU initializes $(m, r) = (0, 1)$. Put $k_m = 1+m$. Keep this decoded value fixed during the following evolution. The membrane subsystem reads the same m and the component's same observed constitutive coefficient $g_x = 1 + m$.

Let s_P, s_L, p, ℓ denote normalized amounts of two substrate classes, retained functional product and membrane material. The two conversions $S_P \to P$ and $S_L \to L$ occur together with extent ξ. Supply the constitutive rate

$$\begin{aligned} \dot{\xi} &= k_m(1 + p)(1 - \xi), \\ p &= \ell = \xi, \\ s_P &= s_L = 1 - \xi, \\ \xi(0) &= 0. \end{aligned} \tag{10}$$

The two material identities $s_P + p = 1$ and $s_L + \ell = 1$ are exact. The parameter $1 + p$ makes current retained product affect the rate; the address affects the distinct coefficient k_m. All four amounts are immobilized in this model. Their declared total occupied volume is $U = 1$, independent of conversion; the additional membrane area is supplied by L through

$$\begin{aligned} A &= 500 + \ell, \\ N &= 20, \\ Q &= 0, \\ \dot{v} &= \frac{A}{100}\left(\frac{20}{v} - \frac{1}{10}\right), \\ v(0) &= 200, \\ V &= v + 1. \end{aligned}$$

These choices describe a finite-amount, continuous kinetic enrichment, with a constant-volume material constitutive law. They do not identify P with a demonstrated catalyst or L with a particular synthesized lipid.

Here is an explicit choice of the shared-state maps in this example. Let $Z_0 = \{0, 1\} \times [0, 1) \times (0, \infty)$ with coordinates $z = (m, \xi, v)$. Let $a(m)$ be the fully normalized GCG address for $m = 1$ and the fully normalized GCU address for $m = 0$. Define four state lifts

$$\begin{aligned} \sigma_L(z) &= (z, \mathcal{L}_0), \\ \sigma_G(z) &= (z, a(m), AAA), \\ \sigma_F(z) &= (z, E_j(m, 1)), \\ \sigma_M(z) &= (z, 1 - \xi, 1 - \xi, \xi, \xi, \\ &\qquad 500 + \xi, 1, 20, 0, v + 1), \end{aligned}$$

where $\mathcal{L}_0$ is the fixed native lattice and the membrane entries following z are ordered as $(s_P, s_L, p, \ell, A, U, N, Q, V)$. For each subsystem take $X_i = \sigma_i(Z_0)$ and $q_i(\sigma_i(z)) = z$. The first coordinate makes every lift injective, so these read maps are well defined and the fiber product is canonically identified with Z_0. The common evolution on each fixed-m component is

$$\dot{m} = 0, \qquad \dot{\xi} = (1 + m)(1 - \xi^2), \qquad \dot{v} = \frac{500 + \xi}{100}\left(\frac{20}{v} - \frac{1}{10}\right). \tag{11}$$

Lifting this trajectory by the four σ_i gives exactly the subsystem evolutions and equal shared readings. Thus compatibility is explicit for this chosen enrichment; it is not asserted for arbitrary separately specified subsystem laws.

Theorem 11.2 (Common realization and viability)**.** *The declared lattice, genetic, functional and membrane data admit one joint realization of (7). Their evolution is uniquely defined for every finite $t \geq 0$, with*

$$\begin{aligned} \xi(t) &= \tanh(k_m t), \\ 0 \leq \xi(t) &< 1, \\ v(t) &= 200, \\ V(t) &= 201, \\ 500 \leq A(t) &< 501. \end{aligned}$$

Both substrate amounts stay positive at every finite time, the native current lattice and AAA type remain unchanged, and a common capsule enclosure contains the scaled lattice with positive clearance. At time one, the GCG model satisfies $p > 9/10$, while the GCU model has $p < 4/5$. Removing only the factor $1 + p$ from the GCG rate gives $p(1) < 9/10$.

Proof. Substitution of $p = \xi$ reduces the extent equation to $\dot{\xi} = k_m(1-\xi^2)$. The displayed hyperbolic tangent is its solution, and local uniqueness follows from the smooth vector field. It remains in $[0, 1)$, so it continues for every finite time and gives nonnegative amounts and the two conservation identities. At $v = 200$ the water right-hand side is zero for every A, so the constant solution is unique on $v > 0$. The native and carrier coordinates have zero updates by construction. All subsystem readings of m, p, ℓ, A, v are the same coordinates, establishing the compatibility identities on this nonempty common state. For an enclosure with area A and total volume V, a capsule with radius $\rho > 0$ and cylindrical length $L \geq 0$ obeys

$$A = 2\pi\rho L + 4\pi\rho^2, \qquad V = \pi\rho^2 L + \frac{4}{3}\pi\rho^3.$$

Eliminating L gives $V = A\rho/2 - 2\pi\rho^3/3$. This is strictly increasing for $0 < \rho < \sqrt{A/(4\pi)}$ and ranges up to $A^{3/2}/(6\sqrt{\pi})$. The bound $36\pi\, 201^2 < 500^3$ follows already from $\pi < 22/7$, so a unique capsule with positive L exists throughout. It has $\rho \geq 2V/A > 402/501 > 4/5$.
In native coordinates the box cells are $k \in \{-1, 0, 1\}^3$. The molecular centroid is $c = (3/4, -\sqrt{3}/4, 0)$ and every native

vertex is at distance at most $\sqrt{43}/2$ from c. Thus all core vertices lie within $8\sqrt{3} + \sqrt{43}/2 < 18$ of c. Expressing these lengths in compartment units by a declared factor $1/100$ puts them inside radius $18/100$. An additional clearance of $2/100$ gives radius $1/5 < 4/5$. The remaining occupied material can be placed in a measurable region of volume one outside this small clearance ball, since the enclosure volume is 201. This proves localization and a positive formal clearance, not atomic hard-sphere packing or bilayer stability.
Finally $\tanh x = (e^{2x}-1)/(e^{2x}+1)$. Since $e < 3$, $e^2 < 9$, giving $\tanh 1 < 4/5$. Since $e > 8/3$, $e^4 > 19$, giving $\tanh 2 > 9/10$. Without product feedback, the same GCG coefficient gives $\dot{p} = 2(1-p)$ and $p(1) = 1 - e^{-2} < 9/10$ because $e^2 < 10$. □

The comparison changes one named information or feedback connection at a time. It proves a role for those connections in this constitutive model. It is not a molecular mechanism inferred from addressing, a proof of catalytic turnover, or a complete genetic translation system. The finite-count construction of Appendix E retains an analogous control structure while replacing continuous amounts by threshold-driven integer events.

12 Mathematical conclusions and realization conditions

The finite ICL is now a defined class with a nonempty, explicitly nontrivial model. Its structural content is not exhausted by a matrix dimension or a carbon count. Named molecular sites, ordered ports, exact connector equations, finite incidence and boundary, local tensor factors and partial shifts remain part of the object. The recognition theorem derives lattice coordinates from directional graph data, and the box construction supplies closed cycles, a genuine graph-interior cell and nontrivial algebraic and response enrichments.

Functional components are obtained by specified realizations of selected structures. Their interpretation maps determine what is preserved. The path and cycle examples show how an ordered interface, a nonidentity logical action and an entire passive terminal-response law can survive a change of internal graph. The genetic quotient determines a reference-invariant external address on a fixed AAA type. Shared-state compatibility then coordinates that address with a component and a membrane model. None of these theorems requires a false implication from graph equivalence to chemical or biological equivalence.

12.1 A common molecular realization

A molecular realization supplies one environment, one collection of actual constituents, and one state interpretation on a domain containing the complete proposed operation. It must implement current molecular graphs and attachments, native geometry, admissible transformations, constitutive response, membrane transport and all shared-variable relations simultaneously. Consecutive paths must share their complete endpoint configuration, including conformation, charge, solvent and remaining constituents. Endpoint formula agreement alone is insufficient.

Proposition 12.1 (Quantitative realization condition)**.** *Suppose a formal finite execution has a common molecular realization whose interpreted trajectory remains within ε of its formal trajectory in one normed observation domain. Suppose all required discrete identities and equality constraints are respected exactly. For each scalar inequality $g_j > 0$, assume g_j is K_j-Lipschitz on that domain, has formal margin $m_j > 0$, and has an additional constitutive discrepancy at most δ_j. If*

$$K_j\varepsilon + \delta_j < m_j$$

for every j, the interpreted execution satisfies all those inequalities.

Proof. At each time, the discrepancy in g_j is at most $K_j\varepsilon + \delta_j$ by the Lipschitz estimate and triangle inequality. Subtracting it from the formal lower bound m_j leaves a positive value. Uniform hypotheses give the conclusion throughout the stated execution. Exact predicates transfer by their separate hypothesis. □

An exact closure equation, atom identity or ordered-port incidence cannot generally be preserved by an arbitrary small perturbation. It must be enforced by the realization or replaced by a separately defined approximate class. Likewise, the formal template, cavity, rotor or Hamiltonian families require their stated constitutive data to become physical mechanisms.

Proposition 12.2 (Symmetry does not force handedness)**.** *Let an involution R represent reflection on a state space, and suppose a deterministic evolution has unique solutions and is equivariant under R. If the initial state is fixed by R, its entire trajectory is fixed by R.*

Proof. Reflecting a solution gives another solution with the same initial condition. Uniqueness makes them identical. □

A handedness-selection mechanism therefore requires asymmetric data, a dynamical instability together with a selection mechanism, or another explicitly supplied symmetry-breaking ingredient. The finite native symmetry alone supplies none of these. Similar distinctions apply to energy minima, electronic decoupling, catalysis and folding: the appropriate energy, Hamiltonian, rate or evolution law must first be part of the structure.

The resulting programme is constructive. Additional functional components can be formulated as finite typed objects,

compared at the desired level of equivalence, composed under explicit compatibility conditions and tested against quantitative realization margins. This gives the finite lattice a precise role in a theory of coordinated protocell organization while leaving each physical realization as a determinate mathematical and experimental problem.

A Native molecular identities and boundary conventions

The following coordinate and adjacency lists are the native Article 40 explicit-model data. They fix the finite molecular object used in the existence construction and the ordered interface used by its functional realizations.

A.1 Coordinates adjacency and exact symmetry

Use carbon vertices $v_1, \dots, v_{12}, w_1, \dots, w_6$. Their integer coordinate pairs are

j	1	2	3	4	5	6	7	8	9	10	11	12
X_j	−2	−1	1	2	4	5	5	4	2	1	−1	−2
Y_j	2	3	3	2	0	−1	−3	−4	−4	−3	−1	0

$$(w_1, w_2, w_3, w_4, w_5, w_6)$$
$$= ((-1,1),(1,-1),(1,1),(2,-2),(2,0),(4,-2)),$$
$$\iota(X,Y) = (X/2,\ \sqrt{3}Y/2, 0).$$

The oriented external perimeter is

$$(v_1, v_2, v_3, v_4, w_3, w_5, v_5, v_6, w_6, v_7, v_8, v_9,$$
$$v_{10}, w_4, w_2, v_{11}, v_{12}, w_1),$$

closed back to v_1, with fusion chords $\{w_1, w_3\}, \{w_2, w_5\}, \{w_4, w_6\}$. All carbon vertices and carbon edges have their common chemical type; all primary sites have the same non-null attachment type. Locant names are as displayed. The centroid is $c = (3/4, -\sqrt{3}/4, 0)$; take $t_j = (\iota(v_j) - c)/\|\iota(v_j) - c\|$. The canonical carbon point group is C_{2h} and the molecular diameter is $\sqrt{43}$. Indeed, the unique smallest-radius pair about c is $\{w_2, w_5\}$ with squared radius $1/4$, and the unique largest-radius pair is $\{v_2, v_8\}$ with squared radius $43/4$. Choose their radius vectors $u = (-1/4, -\sqrt{3}/4, 0)$ and $v = (-5/4, 7\sqrt{3}/4, 0)$. They are linearly independent and $u \cdot v = -1$. A planar orthogonal stabilizer must send u to ϵu and v to δv, with $\epsilon, \delta \in \{-1, 1\}$. Preservation of their nonzero inner product forces $\epsilon = \delta$, hence the planar stabilizer is contained in $\{I, -I\}$. The half-turn preserves the displayed graph, so equality holds. The normal sign can independently be either sign, giving exactly the spatial group C_{2h}. Twice the largest radius is $\sqrt{43}$, attained by the opposite pair v_2, v_8, which also proves the diameter. Finally,

$$d_x = (1/2, -7\sqrt{3}/2, 0), \qquad d_y = (7/2, -\sqrt{3}/2, 0),$$
$$\|d_x\|^2 = 37, \quad \|d_y\|^2 = 13, \quad d_x \cdot d_y = 7, \quad d_x \times d_y = 12\sqrt{3}e_z.$$

Each primary-to-dual pairing is exactly one backward step on the displayed oriented perimeter. These finite identities certify the ordered molecular geometry and F8 perimeter incidence.

A.2 The half-turn and locant types

In the integer coordinates used there, the centroid is $(3/2, -1/2)$ and the half-turn is

$$(X, Y) \longmapsto (3 - X, -1 - Y).$$

It exchanges $v_1 \leftrightarrow v_7$, $v_2 \leftrightarrow v_8$, $v_3 \leftrightarrow v_9$, $v_4 \leftrightarrow v_{10}$, $v_5 \leftrightarrow v_{11}$ and $v_6 \leftrightarrow v_{12}$. On fusion vertices it exchanges $w_1 \leftrightarrow w_6$, $w_2 \leftrightarrow w_5$ and $w_3 \leftrightarrow w_4$. It preserves carbon adjacency and the primary and dual site sets separately. It reverses each ordered primary pair; it does not identify the x and y pair. In particular, their unequal squared lengths 37 and 13 obstruct any isometry exchanging them.

The fusion-locant correspondence used by revised Article 77 is

$$w_3 = 4a, \quad w_5 = 4b, \quad w_6 = 6a,$$
$$w_4 = 10a, \quad w_2 = 10b, \quad w_1 = 12a.$$

A carbon locant is a structural identifier. A chemical element label, attachment type, tensor factor and register bit are distinct labels even when attached to the same cell.

A.3 Exact bridges and typed missing neighbors

For the explicit spacing $a_x = 8e_x, a_y = 8e_y$, the single-entry abstract bridge displacements are

$$b_x = (15/2, 7\sqrt{3}/2, 0), \qquad b_y = (-7/2, 8 + \sqrt{3}/2, 0).$$

The molecular endpoint displacement is d_α, the terminal correction is zero, and $d_\alpha + b_\alpha = 8e_\alpha$ exactly. These are ordered endpoint relations. A one-edge abstract bridge does not claim an actual covalent linker of that length.

For a missing negative x neighbor the retained molecular attachment is v_3; for positive x it is v_9. The negative and positive y attachments are v_{12} and v_6. Missing z neighbors leave stacking interfaces, not extra carbon vertices. Missing neighbors do not turn non-null primary attachment labels into null ones.

For the seven-cell star $\{0, \pm e_x, \pm e_y, \pm e_z\}$, the corresponding counts are 126 core carbons, 147 core edges, four directional connectors, two stacking relations and 30 boundary incidences. It has no elementary internal plaquette. It is a valid smaller finite region, while the 27-cell box displays the cycle conditions nonvacuously. Neither example establishes a universal molecular minimality theorem.

A.4 Decorated stock and current structure

The source catalogue's T_2 constituent has a hydroxyethoxy arm at each primary site, with formula $C_{26}H_{28}O_8$. Thus 27 distinct T_2 constituents contain 702 carbons, of which 486 are native core carbons and 216 are arm carbons. This is a summed constituent inventory; connecting them by the abstract relation structure does not by itself assert a synthesized single molecule.

A molecular reduct forgets hydrogens, arm atoms and a particular Kekulé bond-order decoration while retaining core adjacency and native attachment descriptors. An operation preserving that reduct must preserve the current required core edges. Merely retaining the origins of carbon atoms after the core has been severed does not preserve the native molecular reduct.

B Correspondence with the twelve-axiom source

The correspondence records scope, rather than an implication that every finite restriction satisfies every source axiom. Article 40 is the controlling native source; revised Article 75 controls genetic addressing; revised Article 77 controls the distinction between molecular material, ancestry and compartment operation.

A1 to F1
The finite labelled molecular reduct and the four-hexagon carbon framework are retained. Actual embedding, chemical labels and attachment descriptors remain distinct data.

A2 to F2
The canonical C_{2h} framework symmetry is retained. Actual embedded and decorated stabilizers may be smaller. The superseded D_{2h} claim is not used.

A3 to F3
Ordered port pairs, covariant attachment tangents, linear independence and inequivalence are retained. No orthogonality is introduced; the exact native dot product is seven.

A4 to F4
Bridges retain their ordered terminals, embedding and compatibility data. This finite theory chooses finite nonempty catalogues explicitly.

A5 to F5
Exact directional closure is retained as an equality. Neither arbitrary approximate closure nor a claim of energetic optimality follows from it.

A6 to F6
The full $\mathbb{Z}^3$ lattice is replaced by a finite induced connected region with all missing-neighbor ports recorded. Translation has a domain and range; no total translation action is imposed. The tensor and registry data have finite regional versions.

A7 to F7
The infinite inductive-limit/crossed-product construction is replaced by a finite cell-shift algebra tensored with the regional molecular algebra. Its full-matrix form is proved. The finite and infinite algebras are not identified.

A8 to F8
The oriented perimeter pairings, paired interface maps and internal two-bit alphabet are retained at finite scope. Standard-form commutant duality is proved on the chosen finite representation. The linear cipher and antiunitary commutant map remain distinct.

A9 to F9
Self-adjointness, bounded commutators and compact resolvent are finite-dimensional conditions here. A scalar-kernel condition is added to guarantee finite state separation. The trace-projection example proves this new requirement; it does not prove an infinite spectral theorem or a first-order real-spectral-triple identity.

A10 to F10
Finite typed Hamiltonian blocks and terminal resolvents are supplied on their own Hilbert spaces. They are not inferred from an unweighted carbon graph or identified with F9's operator.

A11 to F11
A causal upper-half-plane response is defined from supplied Hamiltonian and dipole data. The explicit finite example is nonzero; it is not inherited from the zero-response choice in the source's particular full existence model.

A12 to F12
Finite structural states inject into algebraic states; a positive weighted symmetric graph and strict constraints define admissibility. The finite theory requires a nonempty admissible set. Its path metric and spectral metric are distinct, as the explicit example proves.

The superseded *Axioms, Operators, Equivalences* text poses additional physical questions but does not establish independence of its postulates or the claimed automatic consequences. In the present theory, chirality selection requires a symmetry-breaking mechanism; coherent transport requires a channel or Hamiltonian model; thermodynamic selection requires an energy and equilibrium problem; catalytic turnover requires a reaction law and active-state recovery; and peptide conformation requires molecular and solvent dynamics. A derivative of energy has units of power and is not itself a velocity. Equal constituent counts are conservation statements, not structural isomorphisms.

These corrections do not exclude deterministic models. They place each deterministic rule at its correct mathematical type and separate a chosen constitutive assumption from a consequence proved from that assumption.

A separate companion source concordance gives the item-by-item disposition of the superseded axioms, dependent claims, operators and equivalences. The definitions and proofs required by the present article are contained in this manuscript.

C Typed operational families and physical representations

An operation on a finite ICL need not be an endomorphism of its molecular structure. Restriction changes its carrier set; scission changes its current incidence; a register action changes information; and a physical evolution acts on a separately specified configuration or Hilbert space. Each operation therefore has a declared domain, codomain, and meaning. The families below retain useful questions from the earlier operator programme while replacing its unsupported geometric-to-physical implications.

C.1 Scission, release, pinning, and geometric action

Let $\Gamma = (V, E, \lambda)$ be a finite current atom-and-bond graph with immutable atom labels λ. Let $P \subseteq V$ be a protected atom set, and let E_P be its designated protected bonds. The graph also records bond orders, formal charge, molecular ownership, and boundary-valence context. These current data are distinct from a historical record of its preparation.

Definition C.1 (Admissible scission map)**.** For a specified set K of editable bonds, Sciss_K is the partial map that removes those bonds, retains the same labelled atom set, and updates ownership and boundary contexts. Its domain consists of states in which the bonds are present, $K \cap E_P = \varnothing$, and the prescribed resulting boundary contexts are permitted. A bond-order reduction is a separate version with the same declared data.

Proposition C.2 (Separation and protected structure)**.** *The map* Sciss_K *preserves every atom origin and every protected bond. Two surviving vertices* u, v *become disconnected if and only if every original* u–v *path uses an edge of* K*.*

Proof. The atom set and its labels are unchanged, and no protected bond is removed. A path survives precisely when all its edges lie outside K. Thus a surviving u–v path exists exactly when some original such path avoids K. □

This is a graph statement. It supplies neither a chemical reaction path nor a dynamical removal mechanism. Article 40 explicitly distinguishes constraint violation, metric disconnection, Hilbert-space projection, and physical removal [Sch26d, labels `eq:four-boundary-levels` and `rem:projection-not-scission-new`].

Let $Q \subseteq (\mathbb{R}^3)^V$ be an admissible configuration space. For an anchor set $A \subseteq V$ and positions $a = (a_i)_{i \in A}$, put

$$Q(A, a) = \{q \in Q : q_i = a_i \text{ for all } i \in A\}.$$

Definition C.3 (Pinning and constraint release)**.** A pinning operation replaces Q by the nonempty constrained set $Q(A, a)$. Release of a subset of anchors replaces $Q(A, a)$ by $Q(A', a|_{A'})$, where $A' \subseteq A$. An actual attachment or detachment of a molecular anchor additionally requires a current-material transition.

Proposition C.4 (Release inclusion and frame covariance)**.** *One has* $Q(A, a) \subseteq Q(A', a|_{A'})$ *whenever* $A' \subseteq A$*. For a common Euclidean isometry* g*, the simultaneous transformation of configurations and anchor positions satisfies*

$$g(Q(A, a)) = (gQ)(A, ga).$$

Proof. Every configuration satisfying all the anchor equations satisfies the selected subset. Since g is invertible, the equation $q_i = a_i$ is equivalent to $gq_i = ga_i$; applying this at every anchor proves the second equality. □

The inclusion does not select a post-release fold or prove relaxation to one. Such a conclusion requires a potential, dynamics, initial state, and a proved stability or convergence property. A rigid rotation is likewise distinct from an internal conformational change. In particular, a 90-degree frame rotation does not interchange the unequal, nonorthogonal native primary vectors, and does not by itself convert a peptide helix into a sheet.

C.2 Stoichiometric and catalytic transformations

A condensation operation has specified substrate and product graphs, a map of every atom origin, proton/electron and other reagent sources, charge balance, permitted boundary contexts, and a guarded current-state update. A catalytic version also specifies the catalyst state before and after the transformation and a condition enabling another cycle. Equality of net charge is not a proof of proton-coupled electron transfer, a lowered activation barrier, or a selected covalent product.

The same distinction applies to a cavity: a geometric binding region is structural data; catalytic turnover additionally needs a reaction pathway, rate/selectivity data, and a recovered active state. Revised Article 77 separates assembly, preparation, production, and operation for this reason [Sch26a, labels `arc:def-interface` and `arc:composition`].

C.3 Field-dependent physical Hamiltonians

A physical realization introduces its own Hilbert space. For example, with real field coordinates F_i, bounded self-adjoint operators H_0, μ_i and A_{ij} define the self-adjoint family

$$H(F) = H_0 - \sum_i F_i \mu_i - \frac{1}{2} \sum_{i,j} F_i F_j A_{ij}.$$

The units, parameter domain, and interpretation of these coefficients are part of the realization. A scalar quadratic energy approximation is another possible model, but is not an automatic consequence of a port embedding. Moreover, dE/dt has dimensions of power; a velocity requires its own kinematic law, such as $\dot{q} = v$ together with a specified force/mobility relation.

For terminal spaces $\mathcal{H}_L, \mathcal{H}_R$ and bridge space $\mathcal{H}_B$, let

$$\begin{aligned} H_B &= H_B^*, \\ W_L &: \mathcal{H}_L \to \mathcal{H}_B, \\ W_R &: \mathcal{H}_R \to \mathcal{H}_B, \\ J &: \mathcal{H}_R \to \mathcal{H}_L. \end{aligned}$$

Assume these operators are bounded. The terminal operator is

$$T(z) = J + W_L^*(zI - H_B)^{-1} W_R, \qquad z \in \rho(H_B).$$

This is the off-diagonal Schur-complement construction of Article 40, not a geometric attachment vector [Sch26d, labels `ax:physical-H-transfer` and `rem:transfer-schur-block`].

Proposition C.5 (Typed unitary covariance)**.** *Let U_B, U_L, U_R be unitaries between the corresponding unprimed and primed spaces. If*

$$\begin{aligned} H_B' &= U_B H_B U_B^*, \\ W_L' &= U_B W_L U_L^*, \\ W_R' &= U_B W_R U_R^*, \\ J' &= U_L J U_R^*, \end{aligned}$$

then $T'(z) = U_L T(z) U_R^*$ on their common resolvent domain.

Proof. The identity $zI - H_B' = U_B(zI - H_B)U_B^*$ gives $(zI - H_B')^{-1} = U_B(zI - H_B)^{-1}U_B^*$. Substitution in T' proves the result by cancellation of adjacent unitary factors. □

Proposition C.6 (Response stability away from the bridge spectrum)**.** *Suppose two bridge Hamiltonians act on the same space, both resolvents at z have norm at most $1/\delta$ with $\delta > 0$, and the contacts and direct coupling are unchanged. Then*

$$\|T'(z) - T(z)\| \le \frac{\|W_L\| \, \|W_R\|}{\delta^2} \, \|H_B' - H_B\|.$$

Proof. The resolvent identity is

$$(zI - H_B')^{-1} - (zI - H_B)^{-1} = (zI - H_B')^{-1}(H_B' - H_B)(zI - H_B)^{-1}.$$

Multiply by the two contacts and use submultiplicativity. □

The distance from the spectrum is essential. An exact or approximate graph correspondence provides none of the Hamiltonian/contact hypotheses by itself.

C.4 Causal response and a nonzero finite example

In units $\hbar = 1$, choose a finite-dimensional physical Hilbert space, a self-adjoint H, a density matrix ρ, and bounded observables A, B. For $\operatorname{Im} z > 0$, define

$$\chi_{AB}(z) = i \int_0^\infty e^{izt} \operatorname{Tr}(\rho[A(t), B]) \, dt, \qquad A(t) = e^{itH} A e^{-itH}.$$

The integral converges absolutely because its integrand is bounded in absolute value by $2\|A\|\|B\|e^{-t \operatorname{Im} z}$.

Proposition C.7 (Covariance and nontriviality of the finite response)**.** *A simultaneous unitary conjugation of H, ρ, A, B preserves χ_{AB}. With the Pauli matrices X, Y, Z, $\omega > 0$,*

$$H = \omega Z/2, \quad \rho = |1\rangle\langle 1|, \quad A = B = X,$$

one has

$$\chi_{XX}(z) = \frac{2\omega}{\omega^2 - z^2}, \qquad \chi_{XX}(i\eta) = \frac{2\omega}{\omega^2 + \eta^2} > 0 \quad (\eta > 0).$$

Proof. Unitary conjugation commutes with exponentiation and commutators, and the trace is invariant. In the example, $X(t) = \cos(\omega t)X - \sin(\omega t)Y$, so $[X(t), X] = 2i \sin(\omega t)Z$. Since $\operatorname{Tr}(\rho Z) = -1$, the integrand after multiplication by i is $2e^{izt}\sin(\omega t)$. Integrating the two exponentials yields the formula. □

This is a supplied finite response model. It is independent of the zero response in Article 40's particular explicit full-model witness, and it is not a measured chrysene response or a bulk dielectric tensor. Article 40 requires a separate limit and boundary-independence argument for a bulk susceptibility [Sch26d, labels `def:bulk-susceptibility-new` and `rem:no-fragment-bulk`].

C.5 Projection and dynamical decoupling

Proposition C.8 (Exact and approximate invariant sectors)**.** *Let H be bounded self-adjoint and let $P = P^* = P^2$. Put $U(t) = e^{-itH/\hbar}$. If $[H, P] = 0$, the two sectors $P\mathcal{H}$ and $(I - P)\mathcal{H}$ are invariant under $U(t)$. More generally,*

$$\|(I - P)U(t)P\| \le \frac{|t|}{\hbar}\|[H, P]\|.$$

Proof. If the commutator vanishes, P commutes with every term of the norm-convergent exponential series. In general, differentiating $U(-s)PU(s)$ and integrating gives $\|U(-t)PU(t) - P\| \le |t|\|[H, P]\|/\hbar$. Multiplication by unitaries and by the two orthogonal projections gives the displayed leakage bound. □

This theorem describes amplitude transfer between declared Hilbert sectors. It does not delete a molecular bond, remove matter, or force a physical projection to occur. For unbounded Hamiltonians the corresponding domain and reducing-subspace conditions must be retained; see Article 40 `prop:dynamical-preservation` [Sch26d].

C.6 Finite quantum channels and encoded actions

For a finite Hilbert space H, write $\mathcal{D}(H)$ for its positive trace-one operators. A quantum channel from H to K is a linear map

$$\Phi(\rho) = \sum_j A_j \rho A_j^*, \qquad A_j : H \to K, \qquad \sum_j A_j^* A_j = I_H.$$

The Kraus expression makes complete positivity explicit, and the final identity gives trace preservation. Two such channels compose with Kraus operators $B_i A_j$. This is a different operator type from a register map or a terminal resolvent.

Proposition C.9 (Exact finite encoded channel)**.** *Let $V : H \to K$ be an isometry, $P = VV^*$, and $\sigma \in \mathcal{D}(H)$. Define*

$$\mathcal{E}(\rho) = V\rho V^*, \qquad \mathcal{D}(\eta) = V^*\eta V + \mathrm{Tr}((I - P)\eta)\sigma.$$

These are channels and $\mathcal{D}\mathcal{E} = \mathrm{id}$. If unitaries U on H and W on K satisfy $WV = VU$, then for $\mathcal{T}(\eta) = W\eta W^*$,

$$\mathcal{T}\mathcal{E}(\rho) = \mathcal{E}(U\rho U^*), \qquad \mathcal{D}\mathcal{T}\mathcal{E}(\rho) = U\rho U^*.$$

Proof. The encoder has the single Kraus operator V. For the decoder, write $\sigma = \sum_a \lambda_a |s_a\rangle\langle s_a|$ and choose an orthonormal basis w_j of $(I - P)K$. Kraus operators are V^* and $\sqrt{\lambda_a}|s_a\rangle\langle w_j|$; their adjoint products sum to $P + (I - P) = I_K$. The formula follows by summation. Substitution of $V^*V = I$ proves $\mathcal{D}\mathcal{E} = \mathrm{id}$, and substitution of $WV = VU$ proves both dynamical identities. □

A concrete native-factor example is $H = \mathbb{C}^2$, $K = \mathbb{C}^2_{\mathrm{dc}} \otimes \mathbb{C}^2_z$, $V\psi = \psi \otimes |0\rangle$, $U = X$ and $W = X \otimes I$. It gives a nonidentity coherent action as a supplied finite mathematical channel. Its physical implementation still requires actual coherent preparation, evolution and decoding.

A classical commutative observation algebra cannot carry the full matrix algebra by an injective multiplicative map: such a map would send every commutator to zero, contradicting the nonzero commutator of X and Z. The classical address realizations and the channel realization above therefore preserve different structures. Trace preservation alone does not establish coherent transfer; complete dephasing is trace preserving but removes off-diagonal density-matrix entries.

D Finite material transformations and ancestry

A useful structural transformation can be defined without asserting that it is a chemically accessible reaction. Let V be a fixed finite atom set with immutable element map $e : V \to \mathcal{E}$ and origin map $o : V \to \mathcal{O}$. A material graph has a symmetric bond-order function

$$b : \binom{V}{2} \to \{0, 1, 2, 3\}$$

and prescribed valence bounds $\nu(v)$. Its allowed graph domain satisfies $\sum_{w \neq v} b(\{v, w\}) \le \nu(v)$, with explicit charge and boundary contexts when required. This finite graph domain permits unsaturated intermediate vertices; it is therefore broader than a chemical reaction mechanism.

Proposition D.1 (Finite split and join construction)**.** *Suppose two bond-order functions b_0, b_1 on the same labelled atom set satisfy the same valence bounds. Deleting the positive bond-order excess of b_0 over b_1 one unit at a time, then adding the positive excess of b_1 over b_0 one unit at a time, gives a finite path within that graph domain. It preserves all atom, element and origin identities. Bonds required to remain unchanged throughout can be protected if their orders are equal in b_0, b_1 and they are excluded from the edits.*

Proof. During deletion every vertex degree sum decreases. At the start of joining the bond function is the pointwise minimum of b_0, b_1. Every intermediate joining function is bounded above by b_1, so its degree sum satisfies the target valence bound. The number of unit edits is finite. None changes the vertex set or its immutable labels; protected equal bonds are untouched. □

This proposition proves a finite atom-preserving graph path, with a stated intermediate class. It does not provide reaction barriers, electronic states, stereochemistry, solvent compatibility or a time-dependent mechanism. If the permitted domain instead requires every intermediate vertex to have an exact saturated valence and a restricted reaction grammar, the argument need not apply.

D.1 An explicit productive stoichiometry

The earlier Article 85 construction used a donor $D_2 = C_{22}H_{20}O_4$, with two arm carbons allocated to a glycine product and twenty carbons retained in a remnant $R = C_{20}H_{16}O_2$. A finite block is

$$12D_2{+}10NH_3 \longrightarrow 2(\mathrm{Gly})_4{+}2\mathrm{Gly}{+}10R{+}2D_2^{\mathrm{spent}}{+}10H_2{+}6H_2O.$$

Here Gly $= C_2H_5NO_2$ and $(\mathrm{Gly})_4 = C_8H_{14}N_4O_5$. Both sides contain $(C, H, N, O) = (264, 270, 10, 48)$. Donors 1–4 supply the retained tetramer, donors 5–8 the released tetramer, donors 9–10 the free glycine graphs, and donors 11–12 remain spent carriers. Carbon keys include the donor identity and arm or core locant, so the ten remnants and all products have distinct, persistent ancestry.

This supplies a concrete element-count specialization of a conversion network. Applying Proposition D.1 requires complete atom maps and admissible endpoint bond graphs, not merely these formulas. A chemically restricted preparation requires still more. The genetic address can select a component's rate law without changing this block's sequence; sequence-selective covalent synthesis is the separate extension described in Section 10.

D.2 Material and functional equivalence

A material map preserves atom identities and its declared conserved quantities. A functional observation may forget almost all of those identities. Conversely, a conserved carbon total can remain fixed while every native core edge changes. Material conservation and functional realization are therefore separate conditions that a common molecular realization must satisfy together.

E A finite-count closed-loop specialization

This appendix retains a separate seven-cell realization with explicitly chosen footprint and kinetic laws, not the atom inventory of the 27-cell existence model. Its purpose is to show how the operator and coordination definitions accommodate integer-count events and a replacement. The material preparation is represented here at the element-balanced, count level by the block of Appendix D; no chemical rate is inferred from that block. An atom-resolved realization additionally requires complete compatible atom maps, admissible labelled endpoint bond graphs, a declared editing domain, and the complete reagent and resource data. The count identities below do not supply those data.

E.1 State and threshold clocks

Set $M = 100$ and $0 \le t \le 4$. Let B, D be the numbers of processing blocks and single-chain deactivations, initially zero. Define normalized amounts

$$b = 130 - 10B/M, \quad c = 30 - 2B/M, \quad p = (B - D)/M,$$
$$w = D/M, \quad l = B/M, \quad h = z = 2B/M,$$
$$r = 10B/M, \quad S = 10B/M.$$

A block consumes ten donor and two carrier packets and forms one retained tetramer, one lumen tetramer, two free residues, ten remnants and two spent carriers. Deactivation moves one retained tetramer from p to w without changing its atoms.

Let $I(t) = 1$ on each first half of a unit interval and zero on its second half. Put $\dot{s} = 2I(1-s) - s$, $s(0) = 0$. The decoded address gives $m = 1$ for GCG and $m = 0$ for GCU. A feedback indicator $f \in \{0, 1\}$ and an attenuation coefficient $99/100 \le \gamma(t) \le 1$ determine

$$J = \left(10 + 20f\gamma m \frac{p}{1+p}\right) \frac{b}{1+b} \frac{c}{1/10 + c} \left(\frac{19}{20} + \frac{s}{20}\right). \tag{12}$$

The clocks are closed-loop thresholds:

$$B(t) = \left\lfloor \frac{M}{10} \int_0^t J(u)\,\mathrm{d}u \right\rfloor, \qquad D(t) = \left\lfloor M \int_0^t p(u)\,\mathrm{d}u \right\rfloor. \tag{13}$$

Equivalently integrate two phase coordinates and reset each at one. At a simultaneous event process a block before deactivation. The latter removes the least-labelled active chain token. Token labels fix a deterministic event order; they do not specify a covalent atom map. Remaining-stock and fresh-identity conditions define the event domains. An initial component and a second disjoint component at $t = 2$ use the same decoded register. Their interpolation on $[2, 21/10]$ gives the stated attenuation bound; retired material remains present.

E.2 Constituent and compartment data

For this specialization only, the original fixed material contains 81,314 carbons: 81,000 in a baseline reserve, 182 in seven T_2 constituents and 132 in the carrier reserve. Two replacement packets contain 44 carbons each. There are 13,000 donor and 3,000 carrier packets with 22 carbons each. With $K = 1$ after initial assembly and $K = 2$ after replacement,

$$C_{\mathrm{in}} = 81314 + 44K + 264B,$$
$$C_{\mathrm{out}} = 22(13000 - 10B) + 22(3000 - 2B) + 44(2 - K).$$

Thus $C_{\mathrm{in}} + C_{\mathrm{out}} = 433402$. These constants belong to the seven-cell example; replacing it by 27 T_2 cells would add 520 carbons and require new geometric and preparation data. The native core graph is retained as a protected coordinate with zero update, so its current protected bonds are unchanged by all count events. This is distinct from an atom-resolved realization of the newly produced material.

The declared footprint and osmotic laws are

$$A = 518+2K+\frac{8}{5}S, \quad U = 8+K, \quad N = 24+\frac{3}{10}S, \quad v = V-U,$$

$$\dot{V} = \frac{A}{100}\left[\frac{N}{v} + \sqrt{4+16/v^2} - 2 - \frac{1}{10}\right], \qquad V(0) = 200. \tag{14}$$

Total volume V is continuous at replacement; accessible volume v decreases by one when U increases. All retained molecular groups are included in U and A. The freely translating solute and fixed charge values are supplied constitutive data. Salt and water are declared boundary services.

A finite normalized usable-energy reserve is

$$E(t) = 160-S(t)-\int_0^t [2I(u)(1-s(u))+p(u)]\,\mathrm{d}u-8-10\mathbf{1}_{[2,4]}(t).$$

The initial assembly requires eight energy units and replacement ten. Processing and continuous maintenance use the displayed terms. These are specified dimensionless service requirements, not measured free energies; event guards require the remaining reserve to cover the complete next event.

Theorem E.1 (Finite-count bounds)**.** *For the stated finite realization, the threshold system has a unique execution through time four, with finitely many events. For every allowed attenuation function that is piecewise continuous, with finitely many breakpoints, one has*

$$0 \le D \le B \le 1000, \quad 0 \le p \le 151/50,$$
$$b \ge 30, \quad c \ge 10, \quad S \le 100,$$
$$520 \le A \le 682, \quad 24 \le N \le 54,$$
$$100 \le V \le 600, \quad 90 \le v \le 591, \quad E \ge 548/25.$$

For GCG with product feedback, $p(t) \ge 8/5$ throughout

$$\frac{66035921}{16597218} \le t \le 4.$$

With either neutral decoding $m = 0$ or removed feedback $f = 0$, the same initial state and replacement schedule give $p(t) < 51/50$ throughout the interval.

Proof. The interval $0 \le s \le 1$ is invariant. A deactivation clock cannot advance when $p = 0$; at a clock hit an active chain exists, including at a simultaneous production event. Hence $0 \le D \le B$ until a material guard could fail. Define the absolutely continuous variable

$$Y(t) = \int_0^t (J/10 - p)\,\mathrm{d}u.$$

The two floor remainders in (13) give $p = Y + e$ with $|e| < 1/100$. From $J \le 30$ follows $Y' \le 3 - Y + 1/100$, whence $Y \le (301/100)(1 - e^{-t})$ and $p \le 151/50$. Substituting this into the gain term improves the rate bound to $J \le 5030/201$. Therefore

$$B \le \left\lfloor \frac{100}{10}\, 4\, \frac{5030}{201} \right\rfloor = 1000.$$

This proves all stock lower bounds and precludes any stock guard failure or infinite sequence of processing and deactivation events.

The lower availability factor is

$$\mu = \frac{30}{31}\frac{100}{101}\frac{19}{20} = \frac{2850}{3131}.$$

For the full run, when $Y \le 161/100$ one has $0 \le p < 81/50$, and

$$Y' \ge F(p) = \mu\left(1 + \frac{99}{50}\frac{p}{1+p}\right) - p.$$

The concave function F has its minimum on $[0, 81/50]$ at an endpoint; the smaller endpoint value is

$$\delta = F(81/50) = \frac{8298609}{20508050} > 0.$$

Thus Y reaches $161/100$ by time $(161/100)/\delta = 66035921/16597218 < 4$ and cannot cross downward through that level. The floor-error bound yields $p \ge 8/5$ thereafter. If either $m = 0$ or $f = 0$, then $J \le 10$, so $Y \le (101/100)(1 - e^{-t})$ and $p < 51/50$.

The amount identities give the area, occupied-volume and particle bounds. At $V = 100$ the water bracket is at least $24/100 - 1/10 > 0$. At $V = 600$, since $v \ge 590$ and $\sqrt{4 + 16/v^2} - 2 \le 4/v^2$, the bracket is at most

$$\frac{54}{590} + \frac{4}{590^2} - \frac{1}{10} = -\frac{1473}{174050} < 0.$$

The volume interval is invariant across flows and the continuous-volume replacement. In particular $v \ge 90$, so the intervening differential equations are locally Lipschitz in their state on a regular compact region. The prescribed reset order and transverse clock hits give uniqueness and continuation through time four. Finally,

$$E \ge 160 - 100 - 4(2 + 151/50) - 18 = 548/25 > 0,$$

so the energy guards cannot fail. The constituent identity is a direct sum of the displayed terms and is unchanged by deactivation. □

This specialization illustrates discrete material operations, continuous transport, exact finite conservation and a selected information-to-response coupling. Its numerical assumptions are not part of F1–F12. The general framework permits other kinetic, membrane and preparation choices without changing the meaning of the native lattice axioms.

F Finite verification and reproducibility

The accompanying source includes a standard-library Python checker, `certificates/verify_all.py`. It checks the finite native coordinates and adjacency, the box and boundary counts, the directional identities, representative partial-shift and spectral calculations, the complete finite register-error domains, the addressed quotient counts, and the rational inequalities used in the operating specializations. It produces a machine-readable result file alongside a short terminal summary. The original checker is retained unchanged and passes 2,593 exact checks. A separate journal-revision checker, `certificates/verify_revision.py`, passes 4,416 exact regression checks. It tests the typed preparation maps on their full finite component domains, reference normalization, the count-level conservation convention, and exact samples of the explicit shared-state construction. These repeated finite identities are not independent theorem proofs or an atom-resolved chemical realization. Its output is `revision_verification_results.json`.

The graph-recognition, full-matrix generation, standard-form commutant, spectral-separation, realization-composition and compatibility theorems are proved in the text. Finite calculations check their explicit instances; they do not replace the general proofs. The enormous tensor dimension in the box example is represented symbolically by its factors. No dense matrix of dimension $27 \cdot 4^{27}$ is constructed.

The LaTeX project is self-contained. Compile `main.tex` with pdfLaTeX, or run `build.sh`; repeated passes resolve the internal references. No shell escape, external bibliography service or network access is required. The document uses ordinary theorem environments, relative file paths and an internal bibliography.

Source comparisons use the supplied current manuscripts of Articles 40, 75 and 77. The earlier Article 85 count realization is retained only at its explicitly declared scope in Appendix E. The present finite F1–F12 and the common model in Theorem 11.2 are new constructions.

G Corpus context and joint realization certificates

This appendix relates the finite construction to the source corpus and specifies what additional evidence is needed when its formal objects are assigned molecular interpretations. The relationships below use the controlling sources already identified in the text and the supplied Article 83 manuscript [Sch26h].

G.1 The source statements retained

The role of Article 40 is the molecular signature and the twelve-axiom framework [Sch26d]. The finite object in this paper retains explicitly named local data but replaces global translation assertions by partial maps and finite-dimensional representations. Appendix B gives the clause-by-clause comparison. In particular, the existence of a finite object is not an extension theorem into an infinite source model. The finite F1–F12 and their model must be judged by their own quantified statements and proofs.

The genetic and material interfaces use different retained information. Revised Article 75 supplies the fixed AAA carrier and the distinction between that carrier and its externally referenced address [Sch26b]. Revised Article 77 supplies the separation between material constitution, carbon ancestry, transport and operating state [Sch26a]. In the present construction, equality of a decoded output does not identify two full addresses, and equality of a material amount does not identify labelled atoms. Consequently neither an address quotient nor a conservation row can substitute for all the data needed by a shared-state model. The maps to the common state in Section 11 must retain every quantity that another subsystem actually reads.

The supplied bibliography identifies Articles 40, 75 and 77 by their published journal metadata. References [Sch26d, Sch26b, Sch26a] also preserve the controlling revision descriptions used in the scientific body. The designation "Article 85 predecessor" refers only to the earlier construction supplied with this manuscript. It must not be identified with the differently titled published Article 85 in the supplied corpus bibliography. No result from that published work is imported under the predecessor label.

G.2 An exact finite recognition certificate

Theorem 2.2 gives a direct certificate that can be checked before any molecular interpretation is attempted. Given a finite directional cell graph, choose a root o and a spanning tree T. Integrate the signed unit labels along the unique tree paths and write these integer vectors as $a(c)$, with $a(o) = 0$. Check the following three finite conditions:

1. Every oriented non-tree edge obeys $a(d) - a(c) = s(c, d)$.
2. Distinct vertices have distinct coordinate triples.
3. Whenever two recorded triples differ by a signed coordinate unit vector, the corresponding cell incidence is present.

The tree edges satisfy the first identity by construction. If the remaining edges satisfy it, the coordinate increments telescope

along every closed walk, not only along visible plaquettes. The second and third checks give precisely injectivity and induced incidence. These checks therefore certify the conclusion of the recognition theorem; conversely every recognized region supplies such a certificate after choosing an origin and a spanning tree.

A failed certificate also has a finite witness. A failed edge identity gives a nonzero-period closed walk by adjoining that edge to the two tree paths. A repeated coordinate gives two distinct cells joined by a zero-increment walk. A missing incidence gives a specific pair of cells at unit coordinate separation. These failures concern different requirements, so a successful cycle check alone does not certify an induced lattice region. Once all three tests pass, boundary incidences are obtained by testing the six neighbors of each recorded triple; the identity $|\partial\Lambda| = 6|\Lambda| - 2e(\Lambda)$ is an additional count check, not a replacement for these incidence tests.

The resulting certificate contains graph labels, integer coordinates and finite comparisons. It does not certify that a proposed connector is chemically accessible, that the cell geometry is an energy minimum, or that a membrane encloses a stable molecular assembly. Those are separate claims with different evidence requirements.

G.3 Material and evidential interfaces

Article 83's source map links its vessel graph, cut-set enumeration, carbon budget and closure identity to Article 62 [Sch26f, Sch26h]. It links the per-period material ledger, membrane contribution, closure radius, osmolyte count and forcing period to Article 72 [Sch26c, Sch26h]. These are distinct inputs: a graph transformation fixes an incidence change, a material ledger tracks constituent demand, and an operating law specifies how a compartment changes in time. The amount equations and protected-atom maps of the present paper preserve that separation. A valid count identity cannot by itself certify a rate law or transport coefficient.

The same source map identifies Article 80 as the source of its status labels and Article 82 as the source of its measured-status and concordance discipline [Sch26e, Sch26g, Sch26h]. Here these works provide corpus context as described in Article 83; no theorem or numerical result from them is assumed in F1–F12. The relevant methodological requirement is to identify a formal quantity with a specified physical quantity before a measurement can test the model. In particular, a fitted constitutive parameter, an exact finite identity and a measured response have different roles even when they appear in one calculation.

G.4 Membrane parameters are additional data

Article 83 compares a proposed chrysene-derived membrane product with structural, permeability and electrical data and keeps the transfers between different compound classes explicit [Sch26h]. Its introduction distinguishes a named product from a demonstrated transformation, identifies transport as a separate requirement, and reports a chemiosmotic limitation under its stated comparison model. Its source discussion also distinguishes measured values from estimated or calculated rows and does not treat fatty-acid vesicle permeation as a direct measurement of the proposed phospholipid membrane. These qualifications are essential when using that source.

In this paper the constitutive choices $A = 500+\xi$, $U = 1$, $N = 20$ and the stated water law produce an internally compatible formal model. Article 83 does not turn these normalized choices into measured parameters for a particular material. To give them such an interpretation one must specify units, molecular composition, the meaning of occupied and accessible volume, and a transfer rule for each empirical parameter. The physical preparation must also account for reducing-equivalent inputs and other constituents; carbon ancestry alone does not close an electron or energy ledger. This is why the atom-preserving graph paths of Appendix D remain broader than chemically accessible pathways.

A useful parameter record therefore identifies the quantity, its unit, the material and environmental conditions for which it is supplied, and whether it is a definition, an exact identity, a measurement, a constitutive assumption or a transferred estimate. If a transfer is used, its discrepancy belongs in the realization error budget. This record is an obligation for a physical implementation, not a new hypothesis required for the abstract finite existence theorem.

G.5 One certificate for a complete operation

A proposed molecular execution needs one joint witness rather than separate witnesses for its parts. Such a witness specifies the initial constituents and environment; all intermediate states; the legal domain of every operation; the ordered ports and protected molecular identities; the shared-state readings; and the laws and parameters used for dynamics, transport and enclosure. Interface fluxes must cancel with each shared material contribution counted once. Consecutive steps must agree on their complete common endpoint, including the constituents that remain available for the next step.

For a finite observed execution, the error bound (4) makes part of this obligation quantitative. Write

$$E_k = \left(\prod_{i=1}^{k} L_i\right) e_0 + \sum_{i=1}^{k} \left(\prod_{j=i+1}^{k} L_j\right) \varepsilon_i, \qquad E_0 = e_0.$$

Under the common-domain hypotheses of Section 7, the implemented and formal observations differ by at most E_k at step k. Suppose a required scalar inequality at that step has formal margin $m_{j,k} > 0$, Lipschitz constant $K_{j,k}$ and additional constitutive discrepancy at most $\delta_{j,k}$. Then the finite family of conditions

$$K_{j,k} E_k + \delta_{j,k} < m_{j,k}$$

is sufficient to preserve those inequalities at the recorded steps, by Proposition 12.1. A claim over a continuous operating interval additionally needs uniform bounds throughout that interval; sampling endpoints alone does not supply such a bound. Exact atom identities, discrete incidence relations and shared-state equalities still require their separate exact compatibility conditions.

This certificate distinguishes three achievements: recognizing a finite lattice, realizing specified formal actions on a common state, and implementing those actions in one molecular environment. The first two have explicit constructions in the present paper. The third requires the additional joint evidence just described. The distinction allows the finite theorems, source comparisons and empirical tests to be used together without assigning any of them a stronger conclusion than their hypotheses support.

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