ANNALS OF THE CHRYSENE FORMALISM
RESEARCH ARTICLE | AI-AUGMENTED PEER-REVIEWED | VOL. 1, NO. 4 | NOVEMBER 2026

The Dual Cipher, Structurally Induced GF(4) Multiplication, and Real Spectral Structure: Companion-Matrix Realization of the Register Automorphism, Frobenius Reversal, and the Order-One Commutant Condition

Charles D. Schaper, Ph.D.
Chief Editor, Annals of the Chrysene Formalism
San Francisco, California, USA
editor@chrysene.com

Abstract

The two-bit register map $\chi(a, b) = (b, a + b)$ determines an explicit companion-matrix construction of GF(4). Its matrix satisfies $M_\chi^2 + M_\chi + I = 0$, and evaluation at a selected cyclic unit transports operator composition to a multiplication on the structural alphabet. Register reversal becomes Frobenius, the simultaneous flip becomes translation by one, and alignment is the field trace. The resulting action on three registers has fixed-point counts $(64, 0, 8, 8)$ and 20 orbits. The RNA dictionary is evaluated consistently through the same map. A perimeter locant correspondence commutes with inversion, while the dimer-to-node identification is formulated as a unitary coordinate isomorphism with a separate molecular-realization hypothesis.

At the spectral level, a symmetrized support filtration yields a self-adjoint compact-resolvent operator with bounded algebraic commutators and a compatible real involution. An explicit local-unitary counterexample shows that both the original and symmetrized filtrations fail the global order-one condition. Separately, the finite matrix bimodule operator $L(H) + \epsilon' R(H)$ satisfies the real sign and order-one relations and retains them under self-adjoint inner fluctuations. Even doubling realizes the KO-sign patterns for dimensions 0 and 6. These results establish the finite field and internal real spectral constructions while identifying the global order-one extension and molecular implementation as distinct targets. Exact verification and finite Lean certificates accompany the proofs.

Keywords: Chrysene Formalism; dual cipher; GF(4) finite field; companion matrix; Frobenius automorphism; Tomita–Takesaki commutant; real spectral triple; order-one condition; KO-dimension.

1 Introduction

1.1 From four states to a field and a real spectral triple

Axiom A_8 of Article 40 [Sch26a] equips each cell of the Chrysene Tensor Space with two binary structural coordinates: the primary-to-dual cipher coordinate $\epsilon \in \mathbb{Z}_2^{\mathrm{dc}}$, which records whether the local tensor section sits on the primary locants $S_{\mathrm{pri}} = \{3, 6, 9, 12\}$ ($\Psi_{\mathbb{C}}, \epsilon = 0$) or the one-step perimeter-displaced dual locants $S_{\mathrm{dual}} = \{2, 5, 8, 11\}$ ($\Psi'_{\mathbb{C}} = \mathcal{D}_{\mathrm{dc}}\Psi_{\mathbb{C}}, \epsilon = 1$), and the stacking registry coordinate $\eta \in \mathbb{Z}_2^z$ [Sch26a, Defs. 30–35]. Together they form the native structural alphabet

$$\Sigma_{\mathbb{C}} = \mathbb{Z}_2^{\mathrm{dc}} \times \mathbb{Z}_2^z = \{s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}\}, \qquad |\Sigma_{\mathbb{C}}| = 4,$$

with componentwise modulo-2 addition $\boxplus$ making $(\Sigma_{\mathbb{C}}, \boxplus) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ [Sch26a, Def. 36, Prop. 12]. At the operator-algebraic level, Axiom A_8 introduces the conjugate-linear isometric involution $J_{\mathrm{dc}} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ ($J_{\mathrm{dc}}^2 = I$) mapping the primary von Neumann sequence algebra $\mathfrak{M}_{\mathrm{pri}} = \pi_\omega(\mathfrak{A}_{\mathbb{C}})''$ onto its commutant $\mathfrak{M}_{\mathrm{dual}} = J_{\mathrm{dc}}\mathfrak{M}_{\mathrm{pri}}J_{\mathrm{dc}} = \mathfrak{M}'_{\mathrm{pri}}$ [Sch26a, eqs. (259)–(262)].

Article 40 is scrupulous about what Axiom A_8 and Axiom A_9 (the spectral triple $(\mathfrak{A}_{\mathbb{C}}^\circ, \mathcal{H}_\omega, D_{\mathbb{C}})$) do not establish by themselves:

(i) *The Dual-Cipher* GF(4) *Problem ([Sch26a, Rem. 20, Def. 82, Thm. 41, App. C.10]).* Although $(\Sigma_{\mathbb{C}}, \boxplus) \cong (\mathrm{GF}(4), +)$ as an additive group, cardinality four does not supply a field multiplication. Article 40 defines a binary operation $\boxtimes_{\mathfrak{R}} : \Sigma_{\mathbb{C}} \times \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ to be *structurally induced* when it is generated by an explicit composition law of cipher and registry operations rather than transported

Citation: Schaper, C. D. (2026). The Dual Cipher, Structurally Induced GF(4) Multiplication, and Real Spectral Structure: Companion-Matrix Realization of the Register Automorphism, Frobenius Reversal, and the Order-One Commutant Condition. *Annals of the Chrysene Formalism*, 1(4):1847–1855.
https://chrysene.com/index.php/acf/article/view/92
© 2026 Charles D. Schaper. Distributed under the CC Attribution-NonCommercial-NoDerivatives 4.0. First published by Schaper Systems Press in the *Annals of the Chrysene Formalism*, Vol. 1, No. 4 (2026). ISSN 3144-2021.

through an arbitrary bijection, and poses as Theorem 41 the derivation of a structurally induced multiplication satisfying $\mathbb{F}_2[\omega]/(\omega^2+\omega+1)$.

(ii) *The Physical Referent of the Cipher Coordinate ([Sch26c, Sch. 6.10, Target 14.5]).* Article 84 proves that on a catalyst–substrate register pair $(\eta_{\mathrm{cat}}, \eta_{\mathrm{sub}}) = (a, b) \in \mathbb{Z}_2^2$, the filing's purine and hydrogen-bonding bits are the components of the additive automorphism $\chi : (a, b) \mapsto (b, a + b)$ [Sch26c, Prop. 6.8], while noting in Scholium 6.10 and Target 14.5 that reconciling this two-site register datum with the single-site alphabet $\Sigma_{\mathfrak{C}}$ remains open.

(iii) *The Real Spectral-Triple and KO-Dimension Problem ([Sch26a, Rem. 26, Thm. 45, Rems. 51–52]).* Article 40 asks whether the commutant involution J_{dc} preserves $\mathrm{Dom}(D_{\mathfrak{C}})$, satisfies the sign relation $J_{\mathrm{dc}} D_{\mathfrak{C}} = \epsilon' D_{\mathfrak{C}} J_{\mathrm{dc}}$, and obeys Connes' order-one condition $[[D_{\mathfrak{C}}, \pi_\omega(a)], J_{\mathrm{dc}} \pi_\omega(b^*) J_{\mathrm{dc}}^{-1}] = 0$, and leaves open the even grading Γ and KO-dimension of the resulting geometry.

1.2 What is established here, by kind

The field construction and frame-orbit counts are exact finite algebra. The chosen register map χ and unit $s_{1,0}$ fix the multiplication uniquely; molecular realizability of the elementary gates is a separate question. The corrected RNA dictionary is derived explicitly rather than inferred from names of field elements.

At the operator level we prove two distinct results. First, a symmetrized filtration gives a self-adjoint, compact-resolvent operator with bounded algebraic commutators and a compatible real involution. It still fails the order-one condition, as an explicit local-unitary witness shows. Second, a finite matrix bimodule admits an exact real spectral triple, inner fluctuations and even KO-sign realizations. The finite theorem does not complete the infinite crossed-product construction.

The register-to-node map is a unitary coordinate isomorphism. Relating its first bit to a physical change of attachment locants requires the independently stated realization hypothesis H_{dimer}. This separates algebraic construction, imported nomenclature and experimental interpretation.

2 Conventions, nomenclature, and non-implications

Convention 2.1 (Status labels and documentary marks)**.** Every numbered statement carries one of four labels: **[A]** finite, algebraic, or axiom-free; **[D]** derived under hypotheses named and registered in Appendix A; **[C]** imported from a cited source; **[E]** a molecular or patent reading, entering no proof. Scholia are unlabelled and enter no proof. The filing [Sch25] is cited via Article 65's bundle H_{mat} [Sch26b] as amended by clause (i) of [Sch26d] and clause (j) of [Sch26c]. All numerical macros are generated and explicitly checked by `verify_a92.py` (blocks U1–U7, Appendix B). Numbered references to earlier articles retain the source manuscript's attribution; the present audit verifies the displayed statements here, not every theorem number in the earlier documents.

Notation 2.2 (Cipher, field, and operator data)**.** $S_{\mathrm{pri}} = \{3, 6, 9, 12\}$, $S_{\mathrm{dual}} = \{2, 5, 8, 11\}$, $\delta_{\mathrm{dc}} : S_{\mathrm{pri}} \to S_{\mathrm{dual}}$ ($3 \mapsto 2, 6 \mapsto 5, 9 \mapsto 8, 12 \mapsto 11$), and σ_{dc} the cipher involution of [Sch26a, Defs. 30–31]; $\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\mathrm{dc}} \times \mathbb{Z}_2^{z}$ with elements $s_{\epsilon,\eta} = (\epsilon, \eta)$ and addition $\boxplus$; $\chi : (a, b) \mapsto (b, a + b)$ the register automorphism of [Sch26c, Prop. 6.8]; $F_{\mathrm{dc}}|\epsilon\rangle_{\mathrm{dc}} = |\epsilon + 1\rangle_{\mathrm{dc}}$ and $F_z|\eta\rangle_z = |\eta + 1\rangle_z$; $\mathfrak{A}_{\mathfrak{C}} = \mathfrak{Q}_{\mathfrak{C}} \rtimes_\alpha \mathbb{Z}^3$ the sequence algebra, $\mathcal{H}_\omega = L^2(\mathfrak{A}_{\mathfrak{C}}, \omega)$ the GNS space of the faithful trace $\omega = \tau \circ E$, $J_{\mathrm{dc}}[A] = [A^*]$ the commutant involution, and $\pi_{\mathrm{dual}}(b^{\mathrm{op}}) = J_{\mathrm{dc}} \pi_\omega(b^*) J_{\mathrm{dc}}^{-1}$ [Sch26a, Def. 39, App. B.5].

Proposition 2.3 (Non-implications)**. [A]** *None of the following is asserted:*

(i) No Euclidean symmetry for δ_{dc}. *In accordance with [Sch26a, Rem. 17, eq. (282)], $\sigma_{\mathrm{dc}} \notin \Gamma_{\mathfrak{C}}$; the one-step shift δ_{dc} is a graph-perimeter bijection on $\partial G_{\mathrm{chr}}$, not a rigid motion in $E(3)$.*

(ii) No steric disjointness from GF(4). *Endowing $\Sigma_{\mathfrak{C}}$ with $\boxtimes_\chi$ is an algebraic operation on $K_{\mathrm{dc}} \otimes K_z$ and does not imply $V_{\mathrm{sub}}(\Psi_{\mathfrak{C}}) \cap V_{\mathrm{sub}}(\Psi'_{\mathfrak{C}}) = \emptyset$ without a 3D exclusion-volume model ([Sch26a, Rem. 18, eq. (279)]; Target 6.3).*

(iii) No order-one satisfaction by the uncalibrated filtration projection Q_n. *Proposition 5.3 proves that the witness filtration operator $D_{\mathfrak{C}} = \sum n Q_n$ of [Sch26a, Def. 92] violates Connes' order-one condition, as does its symmetric counterpart. The finite split theorem is a separate result (Theorem 5.4).*

(iv) No collapse of lattice, spectral, and KO dimensions. *As recorded in [Sch26a, Rem. 52], the base rank of $\mathbb{Z}^3$ (3), the analytic spectral dimension d_{sp}, and the algebraic KO-dimension in $\mathbb{Z}_8$ are distinct invariants (Theorem 5.6).*

3 The Dual-Cipher GF(4) Field Theorem (Theorem 41 of Article 40)

Article 40's Definition 82 and Theorem 41 ask whether the composition of cipher and registry operations in a Chrysene realization determines a structurally induced multiplication $\boxtimes_{\mathfrak{R}} : \Sigma_{\mathfrak{C}} \times \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$ satisfying $\mathbb{F}_2[\omega]/(\omega^2+\omega+1)$ [Sch26a]. We show that Article 84's register coordinate automorphism χ [Sch26c, Prop. 6.8] generates that exact field law after identifying the

source and target coordinate spaces and choosing the cyclic unit $s_{1,0}$. The canonicity is relative to these declared choices.

Lemma 3.1 (Companion-matrix realization and minimal polynomial of χ)**. [A]** *Let $\chi : \mathbb{F}_2^2 \to \mathbb{F}_2^2$ be the register automorphism $\chi(a,b) = (b, a+b) \pmod 2$ of [Sch26c, Prop. 6.8].*

(a) In the basis $e_1 = s_{1,0} = (1,0)^T$, $e_2 = s_{0,1} = (0,1)^T$ of $\Sigma_{\mathfrak{C}} = \mathbb{F}_2^2$, the matrix of χ is the companion matrix of the irreducible polynomial $P(\lambda) = \lambda^2 + \lambda + 1 \in \mathbb{F}_2[\lambda]$:

$$M_\chi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_2), \qquad M_\chi^2 + M_\chi + I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod 2, \tag{1}$$

with cyclic order $\mathrm{ord}(M_\chi) = 3$ *($M_\chi^2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $M_\chi^3 = I_2$).*

(b) In terms of the elementary factor operations on $\Sigma_{\mathfrak{C}} = \mathbb{Z}_2^{\mathrm{dc}} \times \mathbb{Z}_2^{z}$—the controlled-addition gate $C_{\mathrm{dc}\to z}(\epsilon,\eta) = (\epsilon, \epsilon+\eta)$ with matrix $C_{\mathrm{dc}\to z} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and the factor swap $S_{\mathrm{dc}\leftrightarrow z}(\epsilon,\eta) = (\eta,\epsilon)$ with matrix $S_{\mathrm{dc}\leftrightarrow z} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$—the two opposite compositions generate the two Frobenius-conjugate roots M_χ and $M_\chi^2 = M_\chi^{-1}$:

$$C_{\mathrm{dc}\to z}\, S_{\mathrm{dc}\leftrightarrow z} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = M_\chi, \qquad S_{\mathrm{dc}\leftrightarrow z}\, C_{\mathrm{dc}\to z} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = M_\chi^2. \tag{2}$$

(c) By contrast, the alternative map $\chi'(a,b) = (a, a+b)$ rejected in [Sch26c, Rem. 6.9] equals $C_{\mathrm{dc}\to z}$ alone, which has order 2 with reducible characteristic polynomial $\lambda^2 + 1 = (\lambda+1)^2 \pmod 2$, generating the dual-number ring with a nonzero nilpotent element $\mathbb{F}_2[M_{\chi'}] \cong \mathbb{F}_2[\varepsilon]/(\varepsilon^2)$, which is not *a field (U1).*

Proof. Direct matrix multiplication over $\mathbb{F}_2$ gives $M_\chi^2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) = M_\chi + I_2$, whence $M_\chi^2 + M_\chi + I_2 = 0$ and $M_\chi^3 = M_\chi(M_\chi + I_2) = M_\chi^2 + M_\chi = I_2$. For (b), $C_{\mathrm{dc}\to z}(S_{\mathrm{dc}\leftrightarrow z}(a,b)) = C_{\mathrm{dc}\to z}(b,a) = (b, b+a) = \chi(a,b)$, giving the first line of (2), while reversing the order gives $S_{\mathrm{dc}\leftrightarrow z}(C_{\mathrm{dc}\to z}(a,b)) = (a+b, a) = M_\chi^2(a,b)^T$. For (c), $M_{\chi'}^2 = I_2$, so $(M_{\chi'} + I_2)^2 = 0$ (U1). □

Theorem 3.2 (The Dual-Cipher GF(4) Field Theorem; Discharge of Article 40, Theorem 41)**. [A]** *Let $\mathcal{R}_\chi := \mathbb{F}_2[M_\chi] = \{0, I_2, M_\chi, M_\chi^2\} \subset M_2(\mathbb{F}_2)$ be the commutative $\mathbb{F}_2$-subalgebra generated by χ.*

(a) Because $P(\lambda) = \lambda^2 + \lambda + 1$ is irreducible over $\mathbb{F}_2$, $\mathcal{R}_\chi \cong \mathbb{F}_2[\lambda]/(\lambda^2+\lambda+1) \cong \mathrm{GF}(4)$ is a field of 4 elements.

(b) Evaluating each operator $T \in \mathcal{R}_\chi$ on the pure dual-cipher basis element $1_{\mathrm{GF}(4)} := s_{1,0} = (1,0)^T$ defines an $\mathbb{F}_2$-linear bijection $\Psi_\chi : \mathcal{R}_\chi \xrightarrow{\sim} \Sigma_{\mathfrak{C}}$, $T \mapsto T(s_{1,0})$:

$$0 \longmapsto s_{0,0} = (0,0), \qquad 1 := I_2 \longmapsto s_{1,0} = (1,0), \qquad \omega := M_\chi \longmapsto s_{0,1} = (0,1), \quad \omega^2 := M_\chi^2 \longmapsto s_{1,1} = (1,1). \tag{3}$$

(c) Transporting operator composition in $\mathcal{R}_\chi$ through Ψ_χ defines the structurally induced multiplication $\boxtimes_\chi : \Sigma_{\mathfrak{C}} \times \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$, $x \boxtimes_\chi y := \Psi_\chi(\Psi_\chi^{-1}(x)\Psi_\chi^{-1}(y))$, given in coordinates by

$$s_{\epsilon,\eta} \boxtimes_\chi s_{\epsilon',\eta'} = (\epsilon\epsilon' + \eta\eta',\ \epsilon\eta' + \eta\epsilon' + \eta\eta') \pmod 2. \tag{4}$$

Under $(\boxplus, \boxtimes_\chi)$, $\Sigma_{\mathfrak{C}}$ is a finite field isomorphic to GF(4) *with additive zero $s_{0,0}$, multiplicative identity $1_{\mathrm{GF}(4)} = s_{1,0}$ (pure cipher state), and primitive generator $\omega = s_{0,1}$ (pure z-registry state) satisfying $\omega \boxtimes_\chi x = \chi(x)$ and $\omega^2 \boxplus \omega \boxplus 1_{\mathrm{GF}(4)} = s_{0,0}$ (U1).*

Proof. For any $s_{\epsilon,\eta} = \epsilon s_{1,0} \boxplus \eta s_{0,1} \in \Sigma_{\mathfrak{C}}$, $\Psi_\chi(\epsilon I_2 + \eta M_\chi) = \epsilon s_{1,0} + \eta M_\chi s_{1,0} = (\epsilon,\eta)^T = s_{\epsilon,\eta}$, so Ψ_χ is an isomorphism of additive groups. Multiplying $(\epsilon I_2 + \eta M_\chi)(\epsilon' I_2 + \eta' M_\chi) = \epsilon\epsilon' I_2 + (\epsilon\eta' + \eta\epsilon')M_\chi + \eta\eta' M_\chi^2$ and substituting $M_\chi^2 = M_\chi + I_2 \pmod 2$ gives $(\epsilon\epsilon' + \eta\eta')I_2 + (\epsilon\eta' + \eta\epsilon' + \eta\eta')M_\chi$, whose image under Ψ_χ is (4). All field axioms across the 64 triples of $\Sigma_{\mathfrak{C}}^3$ follow from $\mathcal{R}_\chi \cong \mathbb{F}_2[\lambda]/(\lambda^2+\lambda+1)$ and are verified exhaustively in block U1. □

Proposition 3.3 (Galois, trace, and norm structure of the RNA dictionary and frame group)**. [A]** *Under the RNA dictionary of [Sch26c, Prop. 6.7] (*FF $\mapsto$ A, FB $\mapsto$ C, BF $\mapsto$ G, BB $\mapsto$ U *with* F $= 0$, B $= 1$*) composed with $\chi : (a,b) \mapsto (b, a+b) = (\epsilon,\eta) \in \Sigma_{\mathfrak{C}} \cong$ GF(4):*

(a) The four RNA bases map to the four elements of GF(4) *as*

$$\mathrm{A} = (0,0) \xmapsto{\chi} s_{0,0} = 0, \qquad \mathrm{G} = (1,0) \xmapsto{\chi} s_{0,1} = \omega, \qquad \mathrm{C} = (0,1) \xmapsto{\chi} s_{1,1} = \omega^2, \qquad \mathrm{U} = (1,1) \xmapsto{\chi} s_{1,0} = 1. \tag{5}$$

(b) The purine/pyrimidine coordinate $\epsilon = b$ of [Sch26c, Prop. 6.8] splits the purine line $\{0,\omega\} = \{\mathrm{A},\mathrm{G}\}$ ($\epsilon = 0$) from its affine coset $\{1,\omega^2\} = \{\mathrm{U},\mathrm{C}\}$ ($\epsilon = 1$). The prime subfield is instead $\{0,1\} = \{\mathrm{A},\mathrm{U}\}$; the alignment/hydrogen-bonding coordinate $\eta = a + b$ is identically the finite-field trace $\mathrm{Tr}_{\mathrm{GF}(4)/\mathbb{F}_2}(x) := x \boxplus x^2$; and the indicator of the 3 non-zero units $\mathrm{GF}(4)^\times = \{1,\omega,\omega^2\}$ is the finite-field norm $\mathrm{N}_{\mathrm{GF}(4)/\mathbb{F}_2}(x) := x \boxtimes_\chi x^2 = x^3$:

$$\mathrm{Tr}_{\mathrm{GF}(4)/\mathbb{F}_2}(s_{\epsilon,\eta}) = (\epsilon,\eta) \boxplus (\epsilon+\eta,\eta) = \eta \cdot 1_{\mathrm{GF}(4)}, \qquad \mathrm{N}_{\mathrm{GF}(4)/\mathbb{F}_2}(s_{\epsilon,\eta}) = (\epsilon^2 + \epsilon\eta + \eta^2 \bmod 2) \cdot 1_{\mathrm{GF}(4)}. \tag{6}$$

(c) On each register pair (a,b), the intra-register swap $(a,b)\mapsto(b,a)$ induced by end-to-end reversal rev *([Sch26c, Prop. 6.11]) acts on $(\epsilon,\eta)=(b,a+b)$ as $(\epsilon,\eta)\mapsto(\epsilon+\eta,\eta)$, which is* identically the Frobenius Galois automorphism $\Phi_{\mathrm{Frob}}(x)=x^2=x\boxtimes_\chi x\in\mathrm{Gal}(\mathrm{GF}(4)/\mathbb{F}_2)$ *($0\mapsto0, 1\mapsto1, \omega\leftrightarrow\omega^2$), while the global flip $(a,b)\mapsto(a+1,b+1)$ acts on (ϵ,η) as $(\epsilon+1,\eta)$, which is* affine translation by the multiplicative identity*: $x\mapsto x\boxplus 1_{\mathrm{GF}(4)}$ (U2).*

Proof. Evaluating $\chi(a,b)=(b,a+b)$ gives (5). By (4), $s_{\epsilon,\eta}^2=(\epsilon+\eta,\eta)\pmod 2$. Adding $s_{\epsilon,\eta}$ gives the trace in (6), and multiplying $s_{\epsilon,\eta}\boxtimes_\chi(\epsilon+\eta,\eta)=(\epsilon(\epsilon+\eta)+\eta^2,\epsilon\eta+\eta(\epsilon+\eta)+\eta^2)\equiv(\epsilon^2+\epsilon\eta+\eta^2,0)\pmod 2$ gives the norm. Under $(a,b)\mapsto(b,a)$, $\chi(b,a)=(a,b+a)=(\epsilon+\eta,\eta)=s_{\epsilon,\eta}^2$. Under $(a,b)\mapsto(a+1,b+1)$, $\chi(a+1,b+1)=(b+1,a+b)=(\epsilon+1,\eta)=s_{\epsilon,\eta}\boxplus s_{1,0}$ (U2). □

Theorem 3.4 (Galois-Hermitian palindrome derivation of the 20 codon classes)**. [A]** *Embed the* 64 *six-site stack states into* $\mathrm{GF}(4)^3$ *via* $\mathbf{x}=(x_1,x_2,x_3)$ *where* $x_r=\chi(a_r,b_r)\in\Sigma_{\mathfrak{C}}\cong\mathrm{GF}(4)$ *is the state of register* $r\in\{1,2,3\}$*. Then the frame group* $\mathrm{Fr}=\langle\mathrm{flip},\mathrm{rev}\rangle$ *of [Sch26c, Def. 6.3, Prop. 6.11] acts on* $\mathrm{GF}(4)^3$ *by*

$$\begin{aligned}\mathrm{flip}(x_1,x_2,x_3)&=(x_1\boxplus 1,\ x_2\boxplus 1,\ x_3\boxplus 1),\\ \mathrm{rev}(x_1,x_2,x_3)&=(x_3^2,\ x_2^2,\ x_1^2).\end{aligned}\tag{7}$$

In particular: $\mathrm{Fix}(\mathrm{flip})=\emptyset$ *(0 words);* $\mathrm{Fix}(\mathrm{rev})$ *consists of the* Frobenius-Hermitian palindromes (x_1,x_2,x_1^2) *with* $x_1\in\mathrm{GF}(4)$ *and* $x_2=x_2^2\in\mathbb{F}_2=\{0,1\}$ *($4\times2=8$ words); and* $\mathrm{Fix}(\mathrm{flip}\circ\mathrm{rev})$ *consists of the twisted palindromes* $(x_1,x_2,x_1^2\boxplus1)$ *with* $x_1\in\mathrm{GF}(4)$ *and* $x_2\boxplus x_2^2=1$*, i.e.,* $x_2\in\{\omega,\omega^2\}$ *($4\times2=8$ words), recovering the Burnside tuple* $(64,0,8,8)$ *and orbit count* $(64+0+8+8)/4=20$ *purely from the Galois theory of* $\mathrm{GF}(4)$ *(U2).*

Proof. Reversing the six sites reverses the order of the three registers ($1\leftrightarrow3$, $2\mapsto2$) and swaps the two sites within each register ($(a_r,b_r)\mapsto(b_r,a_r)$), which by Proposition 3.3(c) is $x_r\mapsto x_r^2$, proving (7). A triple (x_1,x_2,x_3) is fixed by rev iff $x_3=x_1^2$ (4 choices for x_1) and $x_2=x_2^2\iff\mathrm{Tr}(x_2)=0\iff x_2\in\{0,1\}$ (2 choices), giving $4\times2=8$ fixed points. It is fixed by flip ∘ rev iff $x_3=x_1^2\boxplus1$ (4 choices for x_1) and $x_2=x_2^2\boxplus1\iff\mathrm{Tr}(x_2)=1\iff x_2\in\{\omega,\omega^2\}$ (2 choices), giving $4\times2=8$ fixed points (U2). □

4 Perimeter correspondence and register coordinates

Proposition 4.1 (Commutation of the perimeter shift δ_{dc} with the C_{2h} inversion ι)**. [A]** *On the* 18*-vertex external perimeter cycle $\partial G_{\mathrm{chr}}$ with ordered vertices*

$$(v_1,v_2,v_3,v_4,w_3,w_5,v_5,v_6,w_6,\\ v_7,v_8,v_9,v_{10},w_4,w_2,v_{11},v_{12},w_1)$$

of [Sch26a, eq. (646)], the primary-to-dual shift $\delta_{\mathrm{dc}}:S_{\mathrm{pri}}\to S_{\mathrm{dual}}$ ($3\mapsto2, 6\mapsto5, 9\mapsto8, 12\mapsto11$) is a uniform cyclic step of -1 along $\partial G_{\mathrm{chr}}$ and commutes identically with the C_{2h} inversion $\iota=(1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)$ of [Sch26a, eq. (661)]:

$$\delta_{\mathrm{dc}}(\iota(j))=\iota(\delta_{\mathrm{dc}}(j))\qquad(j\in S_{\mathrm{pri}}=\{3,6,9,12\}).\tag{8}$$

Consequently, ι maps S_{dual} onto itself by $2\leftrightarrow8$ and $5\leftrightarrow11$. The occupancy sets $D'=\{5,11\}$ and $T'=\{2,5,8,11\}$ are invariant. A decorated molecule inherits this invariance only when matched substituents and conformers transform equivariantly; equal interaction energies further require an invariant environment and interaction law (U3).

Proof. Direct check on S_{pri}: $\delta_{\mathrm{dc}}(\iota(3))=\delta_{\mathrm{dc}}(9)=8=\iota(2)=\iota(\delta_{\mathrm{dc}}(3))$, and $\delta_{\mathrm{dc}}(\iota(6))=\delta_{\mathrm{dc}}(12)=11=\iota(5)=\iota(\delta_{\mathrm{dc}}(6))$; by involution, the same holds at 9 and 12. Along $\partial G_{\mathrm{chr}}$ in [Sch26a, eq. (646)], v_2,v_5,v_8,v_{11} immediately precede v_3,v_6,v_9,v_{12} at step -1 (U3). □

Theorem 4.2 (Register coordinate isomorphism and conditional molecular correspondence)**. [A], [D]** *For a pair of orientation bits $(a,b)\in\mathbb{F}_2^2$, the map $\chi(a,b)=(b,a+b)$ is bijective with inverse $(\epsilon,\eta)\mapsto(\epsilon+\eta,\epsilon)$. Its basis permutation*

$$U_\chi|a,b\rangle=|b,a+b\rangle$$

is a unitary from $\mathbb{C}^2\otimes\mathbb{C}^2$ onto $K_{\mathrm{dc}}\otimes K_z$. Thus the dimer register has an exact four-state coordinate model with $\epsilon=b$ and $\eta=a+b$.

The additional hypothesis H_{dimer} asserts a molecular realization retaining these four logical states during any contraction of stacking-bond indices, together with a calibrated identification of the ϵ observable with the primary/dual attachment-frame observable. Under that hypothesis, the diagram between physical states and their four labels commutes. This physical identification is not a consequence of invertibility of M_χ.

Proof. The stated inverse proves bijectivity. A permutation of an orthonormal basis is unitary. The physical conclusion is exactly the additional realization premise; the scalar stacking contraction $C_z^{(\eta)}:V_{z^+}\otimes V_{z^-}\to\mathbb{C}$ acts on bond indices and is not itself a unitary map from four logical states to one scalar. No substitution migration or chemical reaction is inferred from a coordinate transformation. □

Remark 4.3 (Structural scope)**.** The perimeter map is a combinatorial locant correspondence, not a rigid spatial symmetry of the whole chrysene graph or a synthetic reaction. The finite field describes the selected logical encoding, not nucleobase synthesis, reaction yield, or a physically implemented coherent quantum gate. Experimental discrimination of four states and a transition mechanism are needed for such readings.

5 Global filtration and finite real spectral structure

We analyze the commutant involution $J_{\mathrm{dc}}[A] = [A^*]$ in the tracial crossed-product witness model, and distinguish global filtration properties from finite internal real spectral structure. The global target in [Sch26a, Thm. 45] remains open for a different Dirac construction satisfying order one.

Hypothesis 5.1 (Tracial witness assumptions)**.** The local factors are $M_4(\mathbb{C})$ with faithful normalized product trace; translations act by trace-preserving shifts. The Hilbert space is the tracial GNS space of the algebraic crossed product, whose finite-support monomials form a dense core. The support filtration is the span of those monomials. These assumptions govern all infinite-filtration assertions below.

Lemma 5.2 (Support shift of J_{dc} on the uncalibrated filtration $\mathfrak{F}_n$)**. [D]** *In the witness model $\mathfrak{R}_{\mathrm{ex}}$ of [Sch26a, App. B.4–B.6], let $\mathfrak{F}_n = \mathrm{span}\{[Au_m] : A \in \mathfrak{Q}_{\Lambda_n}, \|m\|_\infty \le n\}$ where $\Lambda_n = \{k \in \mathbb{Z}^3 : \|k\|_\infty \le n\}$ ([Sch26a, eqs. (749)–(750)]). For $[Au_m] \in \mathfrak{F}_n$, the involution $J_{\mathrm{dc}}[B] = [B^*]$ acts by*

$$\begin{aligned} J_{\mathrm{dc}}[Au_m] &= [(Au_m)^*] = [\alpha_{-m}(A^*)u_{-m}], \\ \mathrm{supp}(\alpha_{-m}(A^*)) &= \mathrm{supp}(A) - m \subseteq \Lambda_{2n}. \end{aligned} \tag{9}$$

In particular, $J_{\mathrm{dc}}\mathfrak{F}_n \not\subseteq \mathfrak{F}_n$ for $n \ge 1$: an element with $\mathrm{supp}(A) = \{ne_x\}$ and $m = -ne_x$ lies in $\mathfrak{F}_n$, whereas $\mathrm{supp}(\alpha_{-m}(A^*)) = \{2ne_x\} \in \mathfrak{F}_{2n} \setminus \mathfrak{F}_n$ (U4).

Proof. Equation (9) follows from the crossed-product involution law $(Au_m)^* = \alpha_{-m}(A^*)u_{-m}$ of [Sch26a, eq. (177)] and $\alpha_{-m}(\iota_k(X)) = \iota_{k-m}(X)$ of [Sch26a, eq. (164)]. For $k = ne_x$ and $m = -ne_x$, $\|k\|_\infty = n$ and $\|m\|_\infty = n$, but $\|k - m\|_\infty = \|2ne_x\|_\infty = 2n > n$. □

Proposition 5.3 (Order-one obstruction for both support filtrations)**. [D]** *In the tracial tensor-product witness model, choose a traceless self-adjoint unitary a at a single site k with $\|k\|_\infty = n > 0$. For both the original filtration operator $D_{\mathfrak{C}}$ and the symmetric operator D_{sym} defined below,*

$$[[D, L(a)], R(a)]\,[I] = -2n[I] \neq 0. \tag{10}$$

Moreover, the original $D_{\mathfrak{C}}$ does not commute or anticommute with J_{dc}.

Proof. At translation index zero, the trace-product orthogonality gives $D[I] = 0$ and $D[a] = n[a]$: the traceless site-k tensor is orthogonal to all smaller-support subspaces. Since $a^2 = I$, $L(a)[a] = [I] = R(a)[a]$. Hence $[D, L(a)][a] = -n[I]$, whereas $R(a)[D, L(a)][I] = n[I]$, proving the formula. The symmetric filtration coincides with the original one on these three vectors. This also shows that an internal order-one term cannot cancel the obstruction if its double commutator is zero.

For the original filtration, $J_{\mathrm{dc}}P_n \neq P_nJ_{\mathrm{dc}}$ by Lemma 5.2. If $J_{\mathrm{dc}}D = DJ_{\mathrm{dc}}$, the spectral projections P_n would commute with J_{dc}, a contradiction. Anticommutation is impossible because D is nonzero and nonnegative, whereas $J_{\mathrm{dc}}DJ_{\mathrm{dc}} = -D$ would be nonpositive. □

Theorem 5.4 (Symmetric global filtration and finite internal real triple)**. [D]** *(a)* Symmetrized filtration operator (J_{dc}-commutation): *Define the J_{dc}-symmetric filtration subspaces*

$$\begin{aligned} \mathfrak{F}_n^{\mathrm{sym}} := \mathrm{span}\{[Au_m] \in \mathcal{H}_\omega : \|m\|_\infty \le n, \\ \mathrm{supp}(A) \cup (\mathrm{supp}(A) - m) \subseteq \Lambda_n\}, \end{aligned} \tag{11}$$

with orthogonal projections P_n^{sym}, differences $Q_n^{\mathrm{sym}} := P_n^{\mathrm{sym}} - P_{n-1}^{\mathrm{sym}}$, and $D_{\mathrm{sym}} := \sum_{n=0}^{\infty} nQ_n^{\mathrm{sym}}$. Then D_{sym} is self-adjoint with compact resolvent and bounded commutators on $\mathfrak{A}_{\mathfrak{C}}^{\mathrm{alg}}$, and satisfies Conditions 1 and 2 of [Sch26a, Thm. 45] with sign $\epsilon' = +1$:

$$\begin{aligned} J_{\mathrm{dc}}\,\mathrm{Dom}(D_{\mathrm{sym}}) &= \mathrm{Dom}(D_{\mathrm{sym}}), \\ J_{\mathrm{dc}}D_{\mathrm{sym}} &= +D_{\mathrm{sym}}J_{\mathrm{dc}}. \end{aligned} \tag{12}$$

(b) Bimodule-split internal Dirac operator (exact order-one condition): *On the local internal GNS Hilbert space $\mathcal{H}_{\mathrm{int}} = L^2(M_4(\mathbb{C}), \tau) \cong \mathbb{C}^{16}$ of a local cell ($\mathcal{K}_{\mathfrak{C}} \cong \mathbb{C}^4$), where $\pi_\omega(a) = L(a)$ ($L(a)[X] = [aX]$), $J_{\mathrm{dc}}[X] = [X^*]$, and $\pi_{\mathrm{dual}}(b^{\mathrm{op}}) = J_{\mathrm{dc}}L(b^*)J_{\mathrm{dc}}^{-1} = R(b)$ ($R(b)[X] = [Xb]$), let $H_{\mathrm{int}} = H_{\mathrm{int}}^* \in M_4(\mathbb{C})$ be any non-scalar self-adjoint internal generator (e.g., $H_{\mathrm{int}} = \gamma_{\mathrm{dc}}F_{\mathrm{dc}} \otimes I_z + \gamma_z I_{\mathrm{dc}} \otimes F_z$) and define for $\epsilon' \in \{+1, -1\}$:*

$$\begin{aligned} D_{\mathrm{int}}^{\mathrm{real}} &:= L(H_{\mathrm{int}}) + \epsilon'\, J_{\mathrm{dc}}L(H_{\mathrm{int}})J_{\mathrm{dc}}^{-1} \\ &= L(H_{\mathrm{int}}) + \epsilon' R(H_{\mathrm{int}}). \end{aligned} \tag{13}$$

Then $D_{\mathrm{int}}^{\mathrm{real}}$ is self-adjoint and satisfies all three conditions of [Sch26a, Thm. 45] identically:

$$\begin{aligned} J_{\mathrm{dc}}\,\mathrm{Dom}(D_{\mathrm{int}}^{\mathrm{real}}) &= \mathrm{Dom}(D_{\mathrm{int}}^{\mathrm{real}}), \\ J_{\mathrm{dc}}D_{\mathrm{int}}^{\mathrm{real}} &= \epsilon' D_{\mathrm{int}}^{\mathrm{real}}J_{\mathrm{dc}}, \\ [[D_{\mathrm{int}}^{\mathrm{real}}, L(a)],\, R(b)] &= 0 \quad (\forall\, a, b \in M_4(\mathbb{C})) \end{aligned} \tag{14}$$

(0 violations across all 256 basis pairs (E_{ij}, E_{kl}) of $M_4(\mathbb{C})$, whereas a cross-term $L(F_{\mathrm{dc}})R(F_z)$ violates order-one on 256 of the 256 pairs; U5, U6).

Proof. (a) Under $(A, m) \mapsto (\alpha_{-m}(A^*), -m)$, the pair of sets $\{\mathrm{supp}(A), \mathrm{supp}(A) - m\}$ is mapped to $\{\mathrm{supp}(A) - m, \mathrm{supp}(A)\}$, and $\| - m\|_\infty = \|m\|_\infty$. Thus $J_{\mathrm{dc}}\mathfrak{F}_n^{\mathrm{sym}} = \mathfrak{F}_n^{\mathrm{sym}}$ for every $n \ge 0$, whence $J_{\mathrm{dc}}Q_n^{\mathrm{sym}} = Q_n^{\mathrm{sym}}J_{\mathrm{dc}}$ and (12) holds. The subspaces are nested, finite dimensional and exhaust the dense algebraic core, because the local matrix spaces are finite dimensional and the

support and translation bounds are finite. Their orthogonal differences define a self-adjoint diagonal operator on

$$\mathrm{Dom}(D_{\mathrm{sym}}) = \{\xi : \sum_n n^2 \|Q_n^{\mathrm{sym}}\xi\|^2 < \infty\}.$$

Its eigenvalues escape to infinity with finite multiplicity, giving compact resolvent. Commutation of every projection with J_{dc} proves domain invariance and the sign relation.

For an algebraic A and its adjoint there is a finite integer r such that both shift filtration degree by at most r. Indeed for a monomial $A_c u_l$ multiplying Bu_m, the resulting tensor support is contained in $\mathrm{supp}(A_c) \cup (\mathrm{supp}(B) + l)$ and its shifted support in $(\mathrm{supp}(A_c) - l - m) \cup (\mathrm{supp}(B) - m)$; fixed bounds on A_c, l increase the degree by only a fixed amount. The same applies to A^*. Thus $Q_p L(A) Q_q = 0$ if $|p - q| > r$. Each block diagonal $T_d = \sum_q Q_{q+d} L(A) Q_q$ has norm at most $\|L(A)\|$, and

$$[D_{\mathrm{sym}}, L(A)] = \sum_{d=-r}^{r} dT_d$$

is bounded, with norm at most $r(r+1)\|L(A)\|$. This establishes an ordinary spectral triple and J_{dc} compatibility, not global order one. (b) Since $J_{\mathrm{dc}}^2 = I$ and $J_{\mathrm{dc}} L(H_{\mathrm{int}}) J_{\mathrm{dc}}^{-1} = R(H_{\mathrm{int}})$, conjugating (13) by J_{dc} gives $J_{\mathrm{dc}} D_{\mathrm{int}}^{\mathrm{real}} J_{\mathrm{dc}}^{-1} = R(H_{\mathrm{int}}) + \epsilon' L(H_{\mathrm{int}}) = \epsilon' D_{\mathrm{int}}^{\mathrm{real}}$. Because left and right multiplication commute ($[L(X), R(Y)] = 0$ for all $X, Y \in M_4(\mathbb{C})$), $[D_{\mathrm{int}}^{\mathrm{real}}, L(a)] = [L(H_{\mathrm{int}}), L(a)] = L([H_{\mathrm{int}}, a])$, which commutes with $R(b)$ for every $b \in M_4(\mathbb{C})$, proving (14) (U6). □

Corollary 5.5 (Stability of the real spectral structure under inner metric fluctuations)**.** **[A]** *Let $A = A^* = \sum_j L(a_j)[D_{\mathrm{int}}^{\mathrm{real}}, L(b_j)] \in \Omega_D^1(M_4(\mathbb{C}))$ be any self-adjoint represented internal one-form in the sense of [Sch26a, Defs. 50–51, App. C.7]. Then $A = L(\widetilde{H}_A)$ for the self-adjoint matrix $\widetilde{H}_A = \sum_j a_j [H_{\mathrm{int}}, b_j] \in M_4(\mathbb{C})$, and the real-fluctuated Dirac operator*

$$\begin{aligned} D_{\mathrm{int},A}^{\mathrm{real}} &:= D_{\mathrm{int}}^{\mathrm{real}} + A + \epsilon' J_{\mathrm{dc}} A J_{\mathrm{dc}}^{-1} \\ &= L(H_{\mathrm{int}} + \widetilde{H}_A) + \epsilon' R(H_{\mathrm{int}} + \widetilde{H}_A) \end{aligned} \tag{15}$$

preserves self-adjointness, the sign relation $J_{\mathrm{dc}} D_{\mathrm{int},A}^{\mathrm{real}} = \epsilon' D_{\mathrm{int},A}^{\mathrm{real}} J_{\mathrm{dc}}$, and the exact order-one condition (0 violations across all 256 basis pairs; U6).

Proof. Since $[R(H_{\mathrm{int}}), L(b_j)] = 0$, $[D_{\mathrm{int}}^{\mathrm{real}}, L(b_j)] = L([H_{\mathrm{int}}, b_j])$, so $A = L(\widetilde{H}_A)$ and $J_{\mathrm{dc}} A J_{\mathrm{dc}}^{-1} = R(\widetilde{H}_A^*) = R(\widetilde{H}_A)$. Substituting into (15) reduces $D_{\mathrm{int},A}^{\mathrm{real}}$ to the bimodule-split form (13) with $H_{\mathrm{int}} \mapsto H_{\mathrm{int}} + \widetilde{H}_A$ (U6). □

Theorem 5.6 (Finite even grading and KO-sign realizations)**.** **[A]** *On the doubled internal Hilbert space $\widetilde{\mathcal{H}}_{\mathrm{int}} := \mathcal{H}_{\mathrm{int}} \otimes \mathbb{C}^2 \cong \mathbb{C}^{32}$, represent $a \in M_4(\mathbb{C})$ by $\tilde{\pi}(a) = L(a) \otimes I_2$, equip the space with the grading operator $\Gamma := I_{\mathcal{H}_{\mathrm{int}}} \otimes \sigma_3$ ($\Gamma = \Gamma^\dagger, \Gamma^2 = I, [\Gamma, \tilde{\pi}(a)] = 0$) and the odd Dirac operator $\widetilde{D} := D_{\mathrm{int}}^{\mathrm{real}} \otimes \sigma_1$ ($\Gamma\widetilde{D} = -\widetilde{D}\Gamma$), where $D_{\mathrm{int}}^{\mathrm{real}} = L(H_{\mathrm{int}}) + \epsilon' R(H_{\mathrm{int}})$ with $\epsilon' \in \{+1, -1\}$.*

(a) With the real structure $\widetilde{J}_+ := J_{\mathrm{dc}} \otimes C_0$ (where C_0 is complex conjugation on $\mathbb{C}^2$, so $C_0\sigma_1 = \sigma_1 C_0$ and $C_0\sigma_3 = \sigma_3 C_0$), the Connes sign triple $(\epsilon, \epsilon', \epsilon'')$ defined by $\widetilde{J}^2 = \epsilon I$, $\widetilde{J}\widetilde{D} = \epsilon' \widetilde{D}\widetilde{J}$, and $\widetilde{J}\Gamma = \epsilon'' \Gamma \widetilde{J}$ ([Sch26a, eqs. (543), (550), (552)]) is $(+1, +1, +1)$ for $\epsilon' = +1$, realizing KO-dimension 0 (mod 8)*.*

(b) With $\widetilde{J}_- := J_{\mathrm{dc}} \otimes (\sigma_1 C_0)$ ($\widetilde{J}_-^2 = +I$, $\sigma_1 C_0 \sigma_3 = -\sigma_3 \sigma_1 C_0$) and $\epsilon' = +1$, the sign triple is $(+1, +1, -1)$, realizing KO-dimension 6 (mod 8) *(U7).*

Proof. Since $J_{\mathrm{dc}}^2 = +I$ and $C_0^2 = (\sigma_1 C_0)^2 = +I_2$, $\epsilon = +1$ in both cases. Put $T_+ = C_0$ and $T_- = \sigma_1 C_0$. Each satisfies $T_\pm \sigma_1 = \sigma_1 T_\pm$, so

$$\begin{aligned} \widetilde{J}_\pm \widetilde{D} &= (J_{\mathrm{dc}} D_{\mathrm{int}}^{\mathrm{real}}) \otimes (T_\pm \sigma_1) \\ &= \epsilon' \widetilde{D} \widetilde{J}_\pm. \end{aligned}$$

Finally, $C_0\sigma_3 = +\sigma_3 C_0$ gives $\epsilon'' = +1$ (KO-dimension 0), while $(\sigma_1 C_0)\sigma_3 = -\sigma_3(\sigma_1 C_0)$ gives $\epsilon'' = -1$ (KO-dimension 6; [Con95], U7). □

Remark 5.7 (Finite real spectral convention)**.** The finite statements use the usual self-adjointness, bounded-commutator, compact-resolvent, order-zero, order-one, real-sign and grading conditions; compactness is automatic in finite dimension. Additional reconstruction requirements such as orientability or Poincaré duality are not asserted. On the doubled complex space the real structure is the anti-linear map obtained by entrywise complex conjugation followed by the displayed permutation. Numerical matrix checks include complex Hermitian examples; the algebraic proof holds for arbitrary complex self-adjoint H.

6 Precedence rulings and targets

Ruling 6.1 (Scope of the deferred targets)**.** **[A],[D]** The field construction supplies the algebraic target of Article 40 relative to χ and the chosen cyclic unit. The register map supplies a coordinate bridge for Article 84, with molecular identification retained in H_{dimer}. The finite internal model supplies real spectral and KO-sign examples. The infinite crossed-product order-one target is not discharged: Proposition 5.3 exhibits an obstruction even after support symmetrization.

Target 6.2 (Global real Dirac operator)**.** Construct a densely defined global operator on the crossed-product representation that has compact resolvent, bounded commutators, compatible real structure and order one simultaneously. A tensor-product or sum construction must explicitly verify all four properties; the finite theorem alone does not provide this extension.

Table 1: Established results and remaining scope.

Layer	Result	Scope
Finite field	$\mathbb{F}_2[M_\chi] \cong \mathrm{GF}(4)$; trace, norm, 20 frame orbits	Fixed map χ and selected unit $s_{1,0}$
Perimeter	Uniform locant step and inversion-equivariance	Occupancy correspondence; decorated energy equality is conditional
Register	Four-state permutation unitary U_χ	Physical node identification requires H_{dimer}
Global filtration	Self-adjoint compact-resolvent D_{sym}; bounded commutators; $J_{\mathrm{dc}} D_{\mathrm{sym}} = D_{\mathrm{sym}} J_{\mathrm{dc}}$	Both support filtrations fail order one
Finite internal model	$L(H)+\epsilon' R(H)$ obeys order one; inner-fluctuation stability	Finite $M_4(\mathbb{C})$ bimodule
Even doubling	KO-sign triples $(+,+,+)$ and $(+,+,-)$	KO dimensions 0 and 6 for the stated finite real spectral axioms

Target 6.3 (Steric exclusion volumes of the $2, 5, 8, 11$ dual family)**.** For specified substituents, protonation states, solvent, temperature, relative placement and conformer ensemble, determine by atomistic calculations and structural characterization whether four-arm substitution on $S_{\mathrm{dual}} = \{2, 5, 8, 11\}$ (specifically at the bay-region positions 5 and 11 adjacent to 4 and 10; [Sch26a, Def. 90]) satisfies the disjoint exclusion-volume condition $V_{\mathrm{sub}}(\Psi_{\mathfrak{C}}) \cap V_{\mathrm{sub}}(\Psi'_{\mathfrak{C}}) = \emptyset$ of [Sch26a, Rem. 18, eq. (212)]. A single optimized conformation is a model-specific check, not a universal exclusion-volume proof.

Target 6.4 (Braided coproduct from the CNOT-swap gate U_χ)**.** Determine whether the two-site gate U_χ of Lemma 3.1(b) and Theorem 4.2 induces a noncocommutative coproduct Δ_χ and Yang–Baxter R-matrix discharging [Sch26a, Thm. 48].

7 Conclusion

The register automorphism $\chi(a, b) = (b, a + b)$ generates an explicit four-element field through its irreducible companion matrix. Relative to the selected cyclic unit, the induced multiplication identifies alignment with trace, register reversal with Frobenius, and the simultaneous flip with translation by one. The corrected dictionary maps (A, C, G, U) to $(0, \omega^2, \omega, 1)$; the Burnside count remains 20.

The spectral results separate two scales. Symmetric global support gives an ordinary compact-resolvent spectral triple compatible with J_{dc}, but a local-unitary witness obstructs order one. The finite matrix bimodule has an exact order-one real structure, stable inner fluctuations and even KO-sign realizations. This distinction preserves the constructive internal result and identifies the remaining global problem precisely. The molecular correspondence is a conditional realization program with explicit chemical and operational tests.

A Hypothesis register

Table 2 records the separate algebraic, analytic and realization premises.

B Verification blocks

The following finite verification blocks are implemented in `verify_a92.py` (Python standard library; exact integer arithmetic):

U1 Companion-matrix GF(4) **generation, gate order, and field axioms.**
Verifies $M_\chi^2 + M_\chi + I_2 = 0 \pmod 2$, $\mathrm{ord}(M_\chi) = 3$, $C_{\mathrm{dc}\to z} S_{\mathrm{dc}\leftrightarrow z} = M_\chi$, $S_{\mathrm{dc}\leftrightarrow z} C_{\mathrm{dc}\to z} = M_\chi^2$, and $\mathrm{ord}(M_{\chi'}) = 2$ with $(M_{\chi'} + I_2)^2 = 0$; constructs the 4×4 Cayley tables of $\boxplus$ and $\boxtimes_\chi$ on $\Sigma_{\mathfrak{C}}$ and verifies associativity, commutativity, distributivity, identity $s_{1,0}$, and inverses across all 64 triples (Lemma 3.1, Theorem 3.2). Exact.

U2 Frobenius reversal, field trace, field norm, and $\mathrm{GF}(4)^3$ **Burnside counts.**
Verifies on all 4 elements of $\Sigma_{\mathfrak{C}}$ that $\chi(b, a) = \chi(a, b)^2$ ($\mathrm{rev} = \Phi_{\mathrm{Frob}}$), $\chi(a + 1, b + 1) = \chi(a, b) \boxplus 1$, $\mathrm{Tr}_{\mathrm{GF}(4)/\mathbb{F}_2}(x) = x \boxplus x^2 = \eta$, and $\mathrm{N}_{\mathrm{GF}(4)/\mathbb{F}_2}(x) = x^3 = \epsilon^2 + \epsilon\eta + \eta^2 \pmod 2$ (non-zero on all 3 units of $\mathrm{GF}(4)^\times$); enumerates $\mathrm{GF}(4)^3$ (64 codons) under (7) to confirm fixed-point counts $(64, 0, 8, 8)$ and 20 orbits (Proposition 3.3, Theorem 3.4). Exact.

U3 Perimeter cycle $\partial G_{\mathrm{chr}}$ **and commutation** $\delta_{\mathrm{dc}} \circ \iota = \iota \circ \delta_{\mathrm{dc}}$**.**
Checks the 18-vertex perimeter cycle of [Sch26a, eq. (646)], confirming that each dual locant in $S_{\mathrm{dual}} = \{2, 5, 8, 11\}$ sits at cyclic step -1 from $S_{\mathrm{pri}} = \{3, 6, 9, 12\}$ and satisfies (8) (Proposition 4.1). Exact.

U4 Support shift of J_{dc} **and order-one counterexample on** Q_n**.**
Verifies $\mathrm{supp}(\alpha_{-m}(A^*)) = \mathrm{supp}(A) - m$ and evaluates the

Table 2: Hypotheses of this paper and imported premises, where each is used, and what survives its failure.

Tag	Statement	Used in	What survives its failure
Hypothesis 5.1	Local $M_4(\mathbb{C})$ factors, faithful product trace and exhausting algebraic crossed-product core	Global filtration and order-one obstruction	All finite field and internal matrix identities
[Sch26c, Prop. 6.7–6.8] (imported)	RNA orientation dictionary and register automorphism $\chi(a,b) = (b, a+b)$	Lem. 3.1, Thm. 3.2, Prop. 3.3, Thm. 3.4	The companion-matrix field $\mathbb{F}_2[M_\chi] \cong \mathrm{GF}(4)$ stands unconditionally as an algebraic identity over $\mathbb{F}_2$
H_{dimer} (Thm. 4.2)	Physical coarse-graining preserves four logical labels and realizes the primary/dual attachment-frame observable	Physical clause of Thm. 4.2	The unitary coordinate isomorphism and the single-node locant correspondence and all of Sections 3 and 5
H_{bimod} (Thm. 5.4)	internal Dirac generator acts as $L(H_{\mathrm{int}}) + \epsilon' R(H_{\mathrm{int}})$ without $L(X)R(Y)$ cross-terms	Thm. 5.4(b), Cor. 5.5, Thm. 5.6	Theorem 5.4(a) ($J_{\mathrm{dc}} D_{\mathrm{sym}} = +D_{\mathrm{sym}} J_{\mathrm{dc}}$) and Proposition 5.3

local-unitary witness (10) for both support filtrations (Lemma 5.2, Proposition 5.3). Exact.

U5 Symmetrized filtration invariance $J_{\mathrm{dc}}\mathfrak{F}_n^{\mathrm{sym}} = \mathfrak{F}_n^{\mathrm{sym}}$.
Verifies over finite lattice support sets that $\max(\|k\|_\infty, \|k-m\|_\infty, \|m\|_\infty)$ is invariant under $(k,m) \mapsto (k-m, -m)$, confirming (12) (Theorem 5.4(a)). Exact.

U6 Exact order-one and inner-fluctuation verification on $\mathcal{H}_{\mathrm{int}} \cong \mathbb{C}^{16}$.
Constructs the 16×16 matrices $L(E_{ij})$, $R(E_{kl}) = J_{\mathrm{dc}} L(E_{lk}) J_{\mathrm{dc}}^{-1}$, $D_{\mathrm{int}}^{\mathrm{real}}$, and $D_{\mathrm{int},A}^{\mathrm{real}}$, confirming $J_{\mathrm{dc}} D_{\mathrm{int}}^{\mathrm{real}} = \epsilon' D_{\mathrm{int}}^{\mathrm{real}} J_{\mathrm{dc}}$ and 0 (and 0) violations of $[[D, L(E_{ij})], R(E_{kl})] = 0$ across all 256 basis pairs, against 256 violations for a cross-term $L(F_{\mathrm{dc}})R(F_z)$ (Theorem 5.4(b), Corollary 5.5). Exact.

U7 Even grading Γ and KO-dimension sign table.
Constructs $\widetilde{D}, \Gamma, \widetilde{J}_\pm$ on $\mathbb{C}^{32}$, confirming $\Gamma^2 = I$, $\{\Gamma, \widetilde{D}\} = 0$, and sign triples $(+1,+1,+1)$ (KO-dimension 0) and $(+1,+1,-1)$ (KO-dimension 6; Theorem 5.6). Exact.

C Molecular commentary, isolated

Remark C.1 (Status)**. [A]** Nothing in this appendix enters a numbered statement or proof of the main text.

Remark C.2 (Why the 2, 5, 8, 11 dual positions sit at the K-region and bay-ortho perimeter carbons)**. [E]** In the IUPAC numbering of chrysene ([Sch26a, App. A.5, A.9]), the primary locants {6, 12} sit on the exposed K-region bonds (5, 6) and (11, 12), while {3, 9} sit on the terminal benzo rings. The uniform one-step perimeter shift δ_{dc} ($3 \mapsto 2, 6 \mapsto 5, 9 \mapsto 8, 12 \mapsto 11$) moves {6, 12} to {5, 11} (the other endpoints of the same two K-region bonds, immediately adjacent to the concave bay clefts (4, 5) and (10, 11) of [Sch26a, Def. 90]) and moves {3, 9} to {2, 8} on the terminal rings. Proposition 4.1 shows that this perimeter shift preserves the specified inversion orbits of the locant sets exactly; it does not preserve electronic energies or site reactivities by itself.

Corollary C.3 (Conditional decorated flip invariance)**. [D]** *Suppose inversion carries each substituted arm to an identical arm with an equivariant conformation, or preserves the conformational probability measure. If the interaction/readout kernel and environment are also invariant under that transformation, the corresponding transformed states have equal interaction/readout values, or equal ensemble means.*

Proof. For a fixed state s, covariance gives $E(\iota s) = E(s)$. For an invariant measure μ, change variables $s \mapsto \iota s$ in $\int E(s)\, d\mu(s)$. The locant-set proposition supplies only the occupancy part of these premises. □

Remark C.4 (Why the genetic dictionary's two distinctions generate GF(4))**. [E]** The filing [Sch25] describes its base-four encoding through two binary distinctions on a catalyst–substrate pair: purine vs. pyrimidine (fused-ring class of the nucleobase) and three vs. two hydrogen bonds (canonical Watson–Crick hydrogen-bond count). In Article 84's Proposition 6.8, the adopted assignment of those distinctions to the orientation pair $(a,b) = (\eta_{\mathrm{cat}}, \eta_{\mathrm{sub}})$ selects the coordinate map $\chi(a,b) = (b, a+b)$ rather than the naive map $\chi'(a,b) = (a, a+b)$. Lemma 3.1 and Theorem 3.2 show the algebraic consequence of that encoding choice: χ' has order 2 and generates a degenerate dual-number ring with a nonzero nilpotent element, whereas χ has order 3 and satisfies $\chi^2 + \chi + I = 0$, generating GF(4).

C.1 Chemical observables and empirical closure

The trace classes label the canonical neutral Watson–Crick hydrogen-bond counts: {A, U} versus {C, G}. They do not predict pairing free energy, which also depends on stacking, sequence, salt, solvent, temperature and conformation [YPFK06]. The norm separates zero from nonzero labels; it is not an activity or reactivity index. Primary and dual substitution patterns can denote distinct constitutional isomers, and their relabeling supplies no chemical interconversion mechanism.

A molecular realization program would first prepare and characterize the proposed substituted species and assign locants by suitable spectroscopy or crystallography. Conformer and steric distributions should then be assessed under declared conditions. Independent preparation and readout of both register bits would test the assumed four-state description; an asserted physical gate would require the complete transition truth table, reproducibility and state retention. Finally, measured interaction or spectroscopic observables would need calibration against the proposed operators. Failure to preserve the retained logical labels would falsify the relevant H_{dimer} realization, while asymmetric arm energies would invalidate decorated flip-blindness without affecting the finite field theorem. Neither a permutation unitary nor a KO-sign table alone demonstrates quantum coherence, catalytic action, or a molecular electronic Hamiltonian.

D What was set aside in preparing this paper

Remark D.1 (The nilpotent trap of the naive coordinate $\chi'(a,b) = (a, a+b)$)**. [A]** Article 84's Remark 6.9 noted that the natural first guess for the register coordinate map was $\chi'(a,b) = (a, a+b)$ and rejected it because its first component split {A, C} from {G, U}, matching neither of the filing's distinctions. Block U1 reveals the algebraic counterpart of that rejection: $M_{\chi'} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ satisfies $M_{\chi'}^2 = I_2$, so $M_{\chi'} + I_2$ is nilpotent and $\mathbb{F}_2[M_{\chi'}]$ is not a field. For these two candidate maps, the selected coordinate $\chi(a,b) = (b, a+b)$ couples the two factors non-trivially to produce GF(4).

Remark D.2 (Expecting the uncalibrated filtration Q_n to satisfy the order-one condition)**. [A]** In [Sch26a, App. B.6], the filtration Dirac operator $D_{\mathfrak{C}} = \sum nQ_n$ was constructed to prove existence of a compact-resolvent spectral triple on the infinite $\mathbb{Z}^3$ network, before real structure was imposed. Testing $D_{\mathfrak{C}}$ against J_{dc} in block U4 showed that because $(Au_m)^* = \alpha_{-m}(A^*)u_{-m}$ translates the tensor support by $-m$, Q_n neither commutes with J_{dc} nor satisfies the order-one condition. Symmetrizing the support condition to $\mathrm{supp}(A) \cup (\mathrm{supp}(A) - m) \subseteq \Lambda_n$ in Theorem 5.4(a) and separating left and right internal actions in Theorem 5.4(b) resolves the global J_{dc}-compatibility obstruction and the separate finite internal order-one problem. The global order-one obstruction persists by Proposition 5.3.

Remark D.3 (Opposite gate orders generate the conjugate roots ω and ω^2)**. [A]** An initial statement of Lemma 3.1(b) wrote M_χ as $S_{\mathrm{dc}\leftrightarrow z} \circ C_{\mathrm{dc}\to z}$ without distinguishing the two composition orders. Direct matrix multiplication in block U1 showed that $C_{\mathrm{dc}\to z} S_{\mathrm{dc}\leftrightarrow z} = M_\chi = \omega$ whereas $S_{\mathrm{dc}\leftrightarrow z} C_{\mathrm{dc}\to z} = M_\chi^2 = \omega^2$: non-commutativity of the two order-2 elementary gates in $\mathrm{GL}_2(\mathbb{F}_2)$ is what splits the two conjugate roots of $\lambda^2 + \lambda + 1 = 0$. Both products are now displayed in (2).

References

[Con95] Alain Connes. Noncommutative geometry and reality. *Journal of Mathematical Physics*, 36(11):6194–6231, 1995. https://doi.org/10.1063/1.531241.

[Sch25] Charles D. Schaper. Encoded Design Integrated Synthesis of Nucleic Acids, and Phospholipids, and Related Pharmaceutical Products. U.S. Patent No. 12,195,423 B1, 2025. https://patents.google.com/patent/US12195423B1/en.

[Sch26a] Charles D. Schaper. The Chrysene Formalism Revisited: A Twelve-Axiom Foundation for Molecular Characterization, Tensorial Noncommutative Geometry, and Operator Response. *Annals of the Chrysene Formalism*, 1(2):489–560, 2026. Article 40. https://doi.org/10.68366/5jctze12.

[Sch26b] Charles D. Schaper. The Chrysene-Reachable Material Ledger: Species, Properties, and the Allocation of a Single Pool. *Annals of the Chrysene Formalism*, 1(3):1371–1383, 2026. Article 65. https://doi.org/10.68366/dwqk8m15.

[Sch26c] Charles D. Schaper. The Placed Source: Exogenous Preparation of the Substituted-Chrysene Assembly, and the Equality of Its Reachable Set with the Endogenous One. *Annals of the Chrysene Formalism*, 1(4):1703–1731, 2026. Article 84. https://doi.org/10.68366/a4tjv735.

[Sch26d] Charles D. Schaper. The Source: Substitution-Initiated Self-Separation of the Substituted-Chrysene Stock from a Polyaromatic Bulk, and the Supply It Delivers to the Protocell. *Annals of the Chrysene Formalism*, 1(4):1616–1638, 2026. Article 79. https://doi.org/10.68366/s8sfwq50.

[YPFK06] Peter Yakovchuk, Ekaterina Protozanova, and Maxim D. Frank-Kamenetskii. Base-stacking and base-pairing contributions into thermal stability of the DNA double helix. *Nucleic Acids Research*, 34(2):564–574, 2006. https://doi.org/10.1093/nar/gkj454.